ECE317 : Feedback and Control

Lecture :
Routh-Hurwitz stability criterion
Examples

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Course roadmap

Modeling

- Laplace transform
- Transfer function
- Block Diagram
- Linearization
- Models for systems
  - electrical
  - mechanical
  - example system

Analysis

- Stability
  - Pole locations
  - Routh-Hurwitz
- Time response
  - Transient
  - Steady state (error)
- Frequency response
  - Bode plot

Design

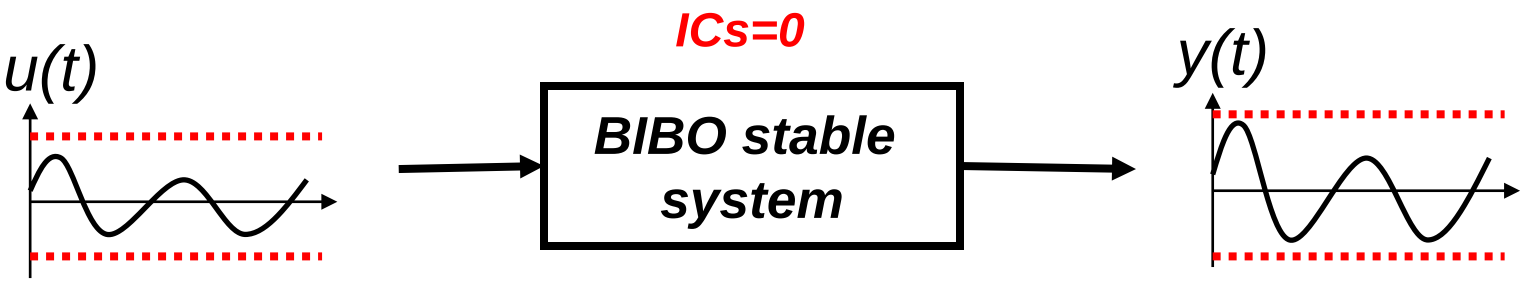
- Design specs
- Frequency domain
- Bode plot
- Compensation
- Design examples

Matlab & PECS simulations & laboratories

Definitions of stability (review)

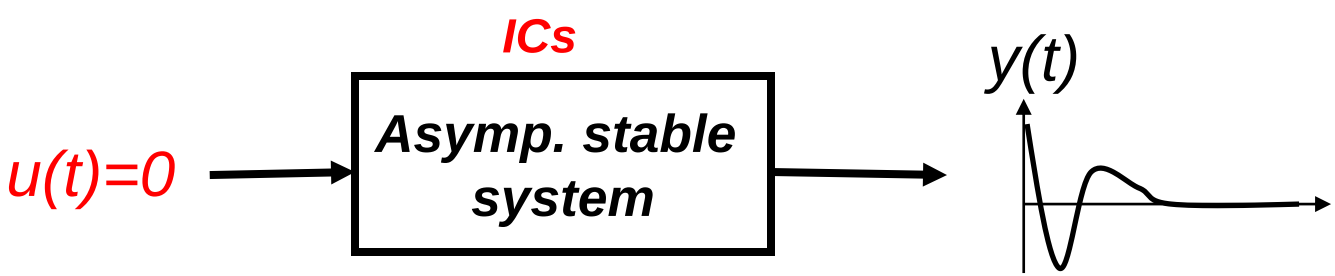
- **BIBO** (Bounded-Input-Bounded-Output) **stability**

  Any bounded input generates a bounded output.



- **Asymptotic stability**

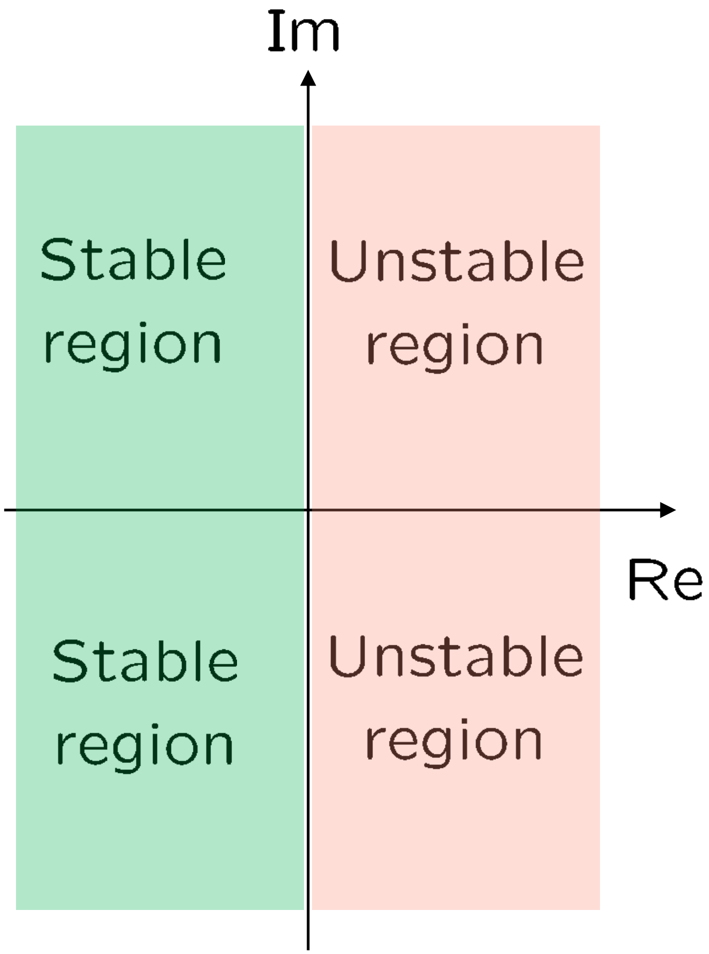
  Any ICs generates y(t) converging to zero.



Stability summary (review)

Let s_i be poles of $G(s)$.
Then, $G(s)$ is …

- (BIBO, asymptotically) stable if $Re(s_i)<0$ for all i.
- marginally stable if
  - $Re(s_i)<=0$ for all i, and
  - simple pole for $Re(s_i)=0$
- unstable if it is neither stable nor marginally stable.



Routh-Hurwitz criterion (review)

- This is for LTI systems with a *polynomial* denominator (without sin, cos, exponential etc.)
- It determines if all the roots of a polynomial
  - lie in the open LHP (left half-plane),
  - or equivalently, have negative real parts.
- It also determines the number of roots of a polynomial in the open RHP (right half-plane).
- It does NOT explicitly compute the roots.
- No proof is provided in any control textbook.

Routh array (review)

$$
\begin{array}{c|ccccc}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & & & \\
s^2 & k_1 & k_2 & & & \\
s^1 & l_1 & & & & \\
s^0 & m_1 & & & &
\end{array}
$$

From the given polynomial (first two rows)

$$Q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

Routh array
(How to compute the third row)

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	$\cdots$
s^{n-2}	b_1	b_2	b_3	b_4	$\cdots$
s^{n-3}	c_1	c_2	c_3	c_4	$\cdots$
$\vdots$	$\vdots$	$\vdots$			
s^2	k_1	k_2			
s^1	l_1				
s^0	m_1				

$$b_1 = \frac{a_{n-2}a_{n-1} - a_n a_{n-3}}{a_{n-1}}$$

$$b_2 = \frac{a_{n-4}a_{n-1} - a_n a_{n-5}}{a_{n-1}}$$

$$\vdots$$

Routh array
(How to compute the fourth row)

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	$\cdots$
s^{n-2}	b_1	b_2	b_3	b_4	$\cdots$
s^{n-3}	c_1	c_2	c_3	c_4	$\cdots$
$\vdots$	$\vdots$	$\vdots$			
s^2	k_1	k_2			
s^1	l_1				
s^0	m_1				

$$c_1 = \frac{a_{n-3}b_1 - a_{n-1}b_2}{b_1}$$

$$c_2 = \frac{a_{n-5}b_1 - a_{n-1}b_3}{b_1}$$

$$\vdots$$

Routh-Hurwitz criterion

$$
\begin{array}{l|lllll}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & & & \\
s^2 & k_1 & k_2 & & & \\
s^1 & l_1 & & & & \\
s^0 & m_1 & & & &
\end{array}
$$

The number of roots in the open right half-plane is equal to the number of sign changes in the first column of Routh array.

Example 1

$$Q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$$

Routh array

s^5	1	2	11
s^4	2	4	10
s^3	~~0~~ ε	6	
s^2	$\frac{4\varepsilon-12}{\varepsilon}$	10	
s^1	≈ 6		
s^0	10		

If 0 appears in the first column of a nonzero row in Routh array, replace it with a small positive number. In this case, *Q* has some roots in RHP.

Two sign changes in the first column ➡ Two roots in RHP

$$\varepsilon \rightarrow \underbrace{\frac{4\varepsilon - 12}{\varepsilon}}_{<0} \rightarrow 6$$

Example 2

$$Q(s) = s^4 + s^3 + 3s^2 + 2s + 2$$

Routh array

s^4	1	3	2
s^3	1	2	
s^2	1	2	
s^1	~~0~~ 2		
s^0	2		

If zero row appears in Routh array, *Q* has roots either on the imaginary axis or in RHP.

No sign changes in the first column ⇒ No roots in RHP

But some roots are on imag. axis.

Take derivative of an *auxiliary polynomial* (which is a factor of Q(s)) $s^2 + 2$

Example 3

$$Q(s) = s^3 + s^2 + s + 1 \ (= (s+1)(s^2+1))$$

Routh array

s^3	1	1
s^2	1	1
s^1	~~0~~ **2**	
s^0	1	

Derivative of auxiliary poly.

$$(s^2+1)' = 2s$$

(Auxiliary poly. is a factor of Q(s).)

No sign changes in the first column → No root in OPEN(!) RHP

Example 4

$$Q(s) = s^5 + s^4 + 2s^3 + 2s^2 + s + 1 \; (= (s+1)(s^2+1)^2)$$

Routh array

s^5	1	2	1
s^4	1	2	1
s^3	~~0~~ **4**	~~0~~ **4**	
s^2	1	1	
s^1	~~0~~ **2**		
s^0	1		

Derivative of auxiliary poly.

$$(s^4 + 2s^2 + 1)' = 4s^3 + 4s$$

$$(s^2 + 1)' = 2s$$

No sign changes
in the first column

→ No root in OPEN(!) RHP

Example 5

$$Q(s) = s^4 - 1 \ (= (s+1)(s-1)(s^2+1))$$

Routh array

s^4	1	0	-1
s^3	~~0~~ **4**	~~0~~ **0**	
s^2	~~0~~ ε	-1	
s^1	$4/\varepsilon$		
s^0	-1		

Derivative of auxiliary poly.

$$(s^4 - 1)' = 4s^3$$

One sign changes
in the first column → One root in OPEN(!) RHP

Notes on Routh-Hurwitz criterion

- Advantages
  - No need to explicitly compute roots of the polynomial.
    - High order *Q(s)* can be handled by hand calculations.
  - Polynomials including undetermined parameters (plant and/or controller parameters in feedback systems) can be dealt with.
    - Root computation does not work in such cases!
- Disadvantage
  - Exponential functions (delay) cannot be dealt with.
    - Example: $Q(s) = e^{-s} + s^2 + s + 1$

Example 6

$$Q(s) = s^3 + 3Ks^2 + (K+2)s + 4$$

Find the range of K s.t. Q(s) has all roots in the left half plane. (Here, K is a design parameter.)

Routh array

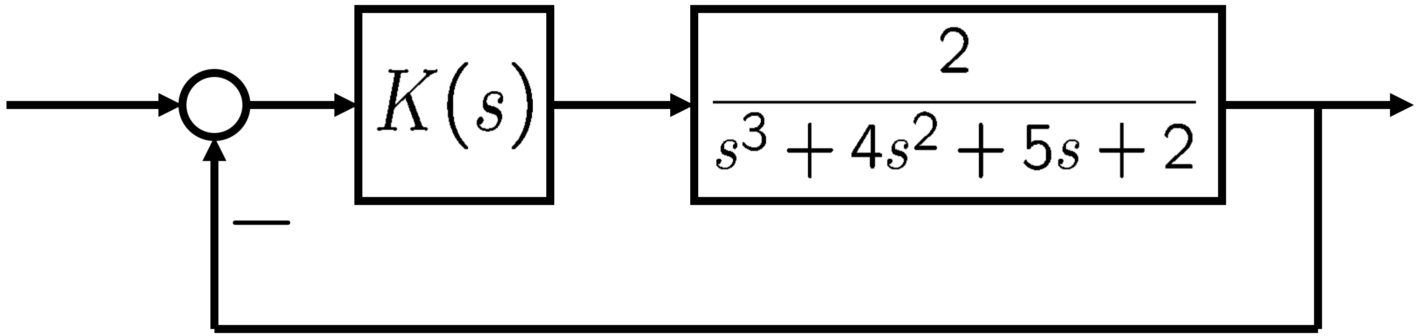
s^3	1	$K+2$
s^2	$3K$	4
s^1	$\frac{3K(K+2)-4}{3K}$	
s^0	4	

No sign changes in the first column

$$\Rightarrow \begin{cases} 3K > 0 \\ 3K(K+2) - 4 > 0 \end{cases}$$

$$\Rightarrow K > -1 + \frac{\sqrt{21}}{3}$$

Example 7



- Design *K(s)* that stabilizes the closed-loop system for the following cases.
  - *K(s) = K* (constant, P controller)
  - $K(s) = K_P + K_I/s$ (PI (Proportional-Integral) controller)

Example 7: *K(s)=K*

- Characteristic equation

$$1 + K\frac{2}{s^3 + 4s^2 + 5s + 2} = 0$$

$$\Rightarrow \quad s^3 + 4s^2 + 5s + 2 + 2K = 0$$

- Routh array

$$\begin{array}{c|ll} s^3 & 1 & 5 \\ s^2 & 4 & 2 + 2K \\ s^1 & \frac{18-2K}{4} & \\ s^0 & 2 + 2K & \end{array} \quad \Rightarrow \quad -1 < K < 9$$

Example 7: $K(s)=K_P+K_I/s$

- Characteristic equation

$$1+\left(K_P+\frac{K_I}{s}\right)\frac{2}{s^3+4s^2+5s+2}=0$$

$$\Rightarrow\quad s^4+4s^3+5s^2+(2+2K_P)s+2K_I=0$$

- Routh array

s^4	1	5	$2K_I$
s^3	4	$2+2K_P$	
s^2	$\frac{18-2K_P}{4}$	$2K_I$	
s^1	$(*)$		
s^0	$2K_I$		

$\frac{18-2K_P}{4} > 0 \rightarrow K_P < 9$

$2K_I > 0 \rightarrow K_I > 0$

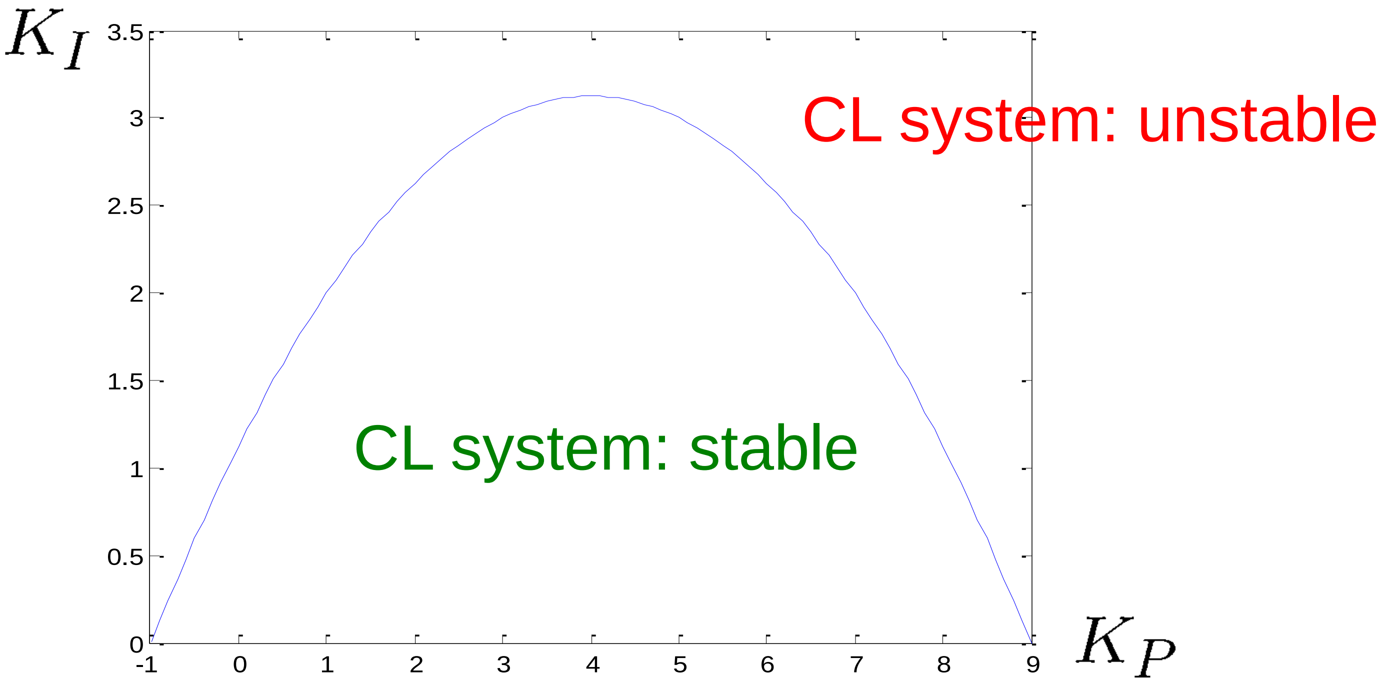
Example 7: Range of (K_P,K_I)

- From Routh array,

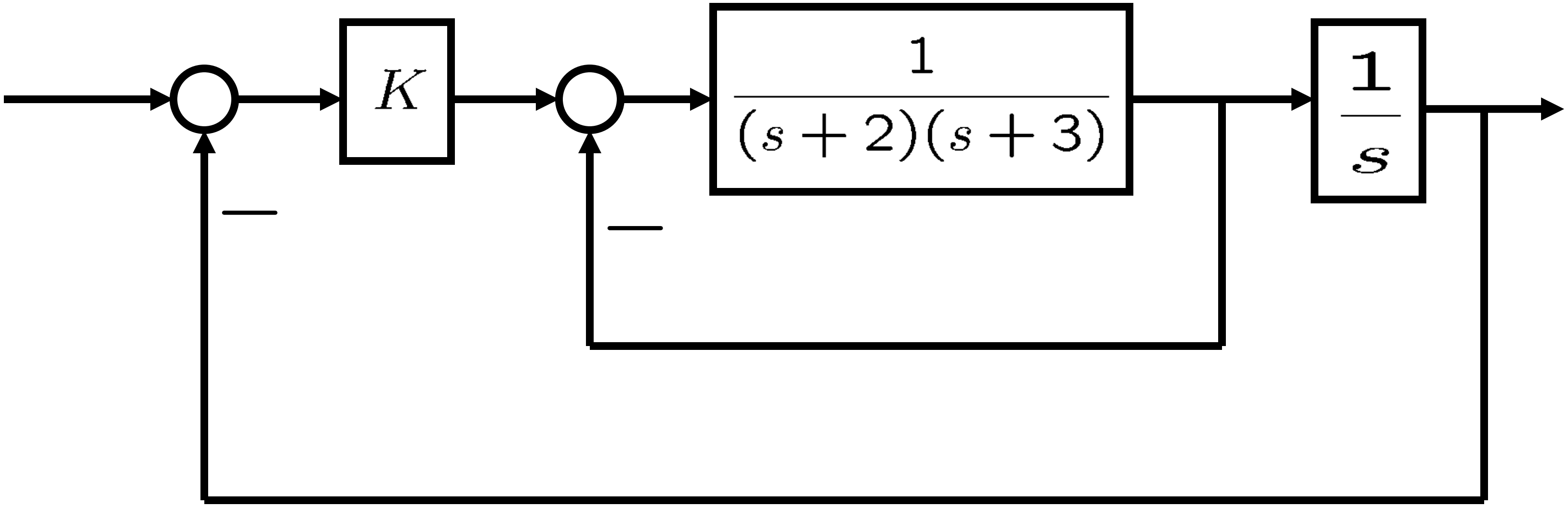
$$K_P < 9$$

$$K_I > 0$$

$$(*) \Leftrightarrow (1 + K_P)(9 - K_P) - 8K_I > 0$$

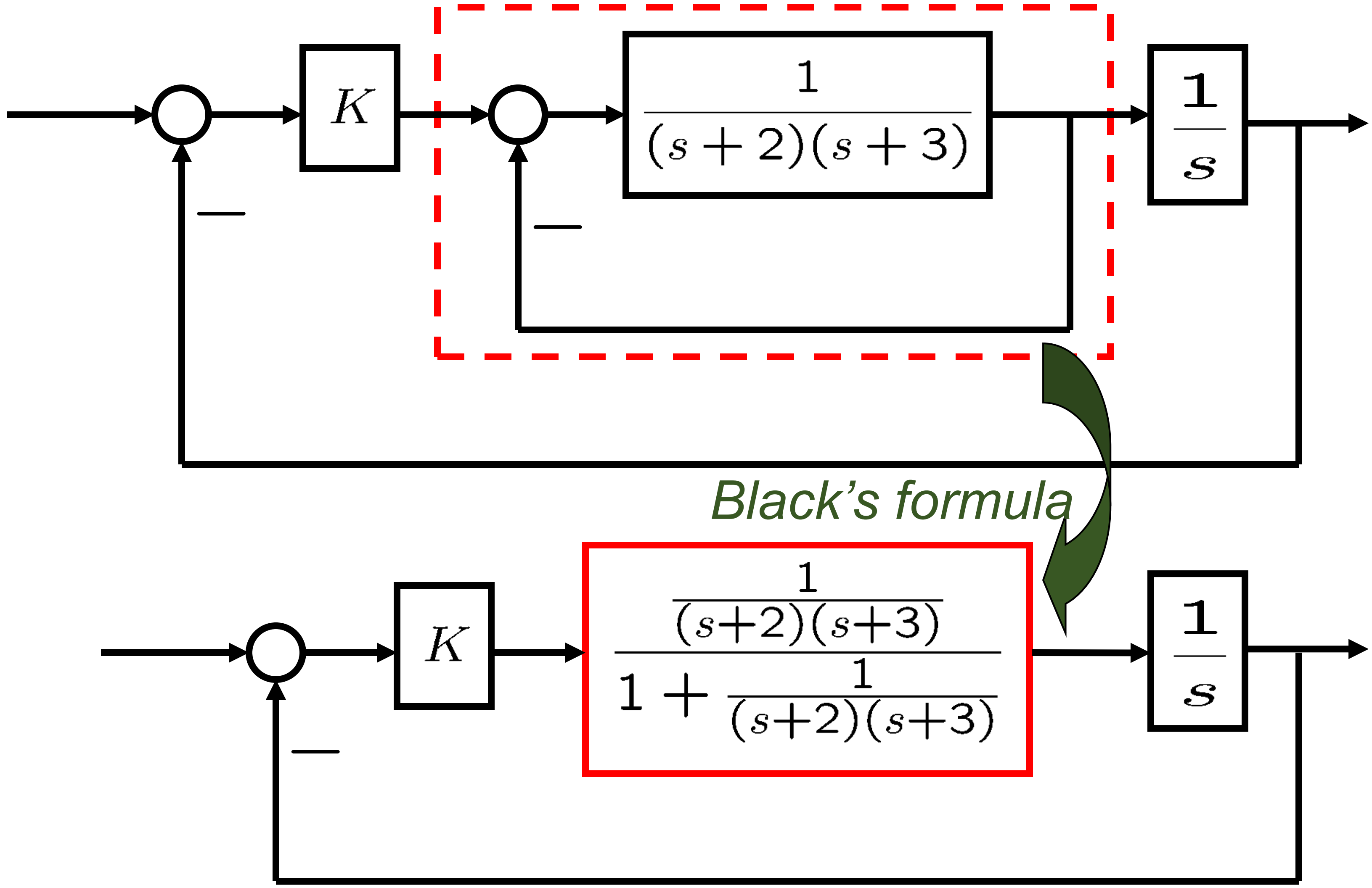


Example 8



- Determine the range of K that stabilize the closed-loop system.

Example 8 (cont'd)



Example 8 (cont'd)

- Characteristic equation

$$1 + K\frac{\frac{1}{(s+2)(s+3)}}{1 + \frac{1}{(s+2)(s+3)}} \cdot \frac{1}{s} = 0$$

$$\Rightarrow \quad 1 + K \cdot \frac{1}{s(s+2)(s+3)+s} = 0$$

$$\Rightarrow \quad s(s+2)(s+3) + s + K = 0$$

$$\Rightarrow \quad s^3 + 5s^2 + 7s + K = 0$$

Example 8 (cont'd)

- Routh array $s^3 + 5s^2 + 7s + K = 0$

s^3	1	7
s^2	5	K
s^1	$\frac{35-K}{5}$	
s^0	K	

$\longrightarrow \quad 0 < K < 35$

- If *K=35*, the closed-loop system is marginally stable. Output signal will oscillate with frequency corresponding to

$$\frac{1}{5s^2+35} = \frac{1}{5} \cdot \frac{1}{s^2+7} = \frac{1}{5} \cdot \frac{1}{s^2+(\sqrt{7})^2}$$

Summary

- Examples for Routh-Hurwitz criterion
  - Cases when zeros appear in Routh array
  - P controller gain range for stability
  - PI controller gain range for stability
- Next
  - Frequency response