Numerical Integration of Ordinary Differential Equations for Initial Value Problems

Gerald Recktenwald
Portland State University
Department of Mechanical Engineering
gerry@pdx.edu

Overview

- Motivation: ODE's arise as models of many applications
- An example of an exact solution
- Direct substitution as a method of verifying the solutions to ODEs

Application: Newton's Law of Motion

Newton's Law of Motion is

$$F = ma$$

Application: Newton's Law of Motion

Newton's Law of Motion is

$$F = ma$$

Acceleration is the time derivative of velocity, so

$$\frac{dv}{dt} = a$$

Application: Newton's Law of Motion

Newton's Law of Motion is

$$F = ma$$

Acceleration is the time derivative of velocity, so

$$\frac{dv}{dt} = a$$

and

$$\frac{dv}{dt} = \frac{F}{m}$$

Application: Newton's Law of Motion

Newton's Law of Motion is

$$F = ma$$

Acceleration is the time derivative of velocity, so

$$\frac{dv}{dt} = a$$

and

$$\frac{dv}{dt} = \frac{F}{m}$$

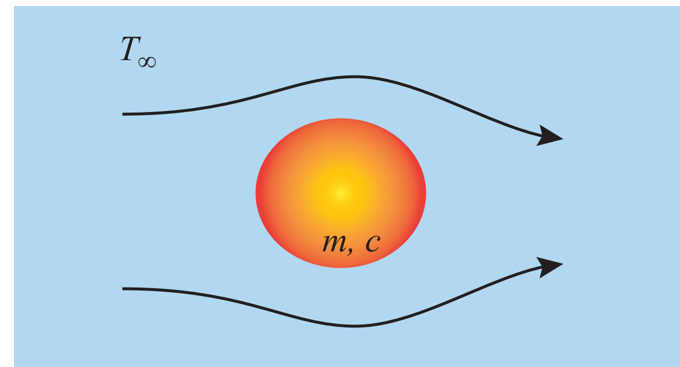
If $F(t)$ and $v(0)$ are known, we can (at least in principle) integrate the preceding equation to find $v(t)$

Application: Newton's Law of Cooling

The cooling rate of an object immersed in a flowing fluid is

$$Q = hA(T_s - T_\infty)$$

where Q is the heat transfer rate, h is the heat transfer coefficient, A is the surface area, T_s is the surface temperature, and T_∞ is the temperature of the fluid.



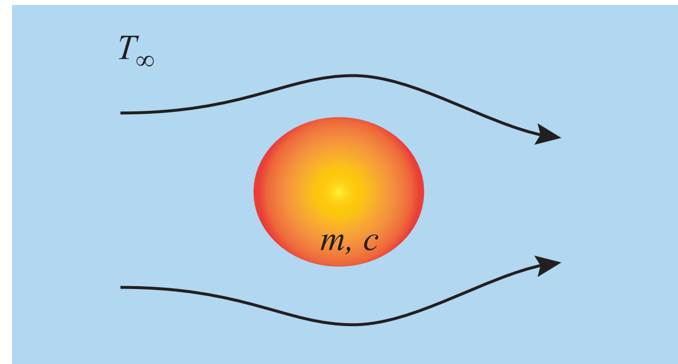
Application: Newton's Law of Cooling

The cooling rate of an object immersed in a flowing fluid is

$$Q = hA(T_s - T_\infty)$$

where Q is the heat transfer rate, h is the heat transfer coefficient, A is the surface area, T_s is the surface temperature, and T_∞ is the temperature of the fluid.

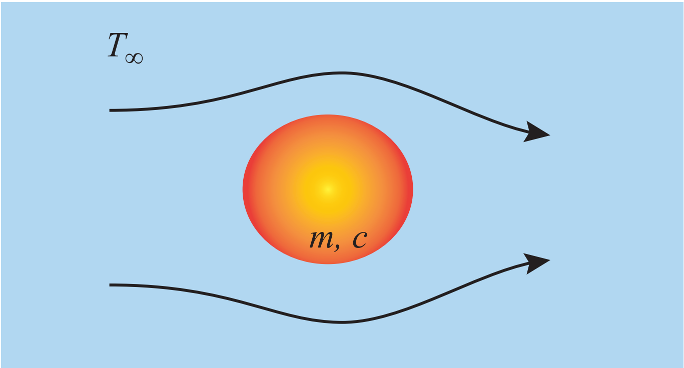
When the cooling rate is primarily controlled by the convection from the surface, the variation of the object's temperature with is described by an ODE.



Newton's Law of Cooling

Apply an energy balance

$$mc\frac{dT}{dt} = -Q = -hA(T_s - T_\infty)$$



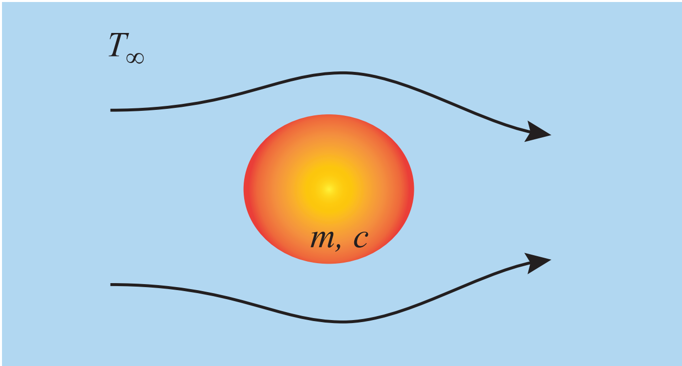
Newton's Law of Cooling

Apply an energy balance

$$mc\frac{dT}{dt} = -Q = -hA(T_s - T_\infty)$$

Assume material is highly conductive $\Rightarrow T_s = T$

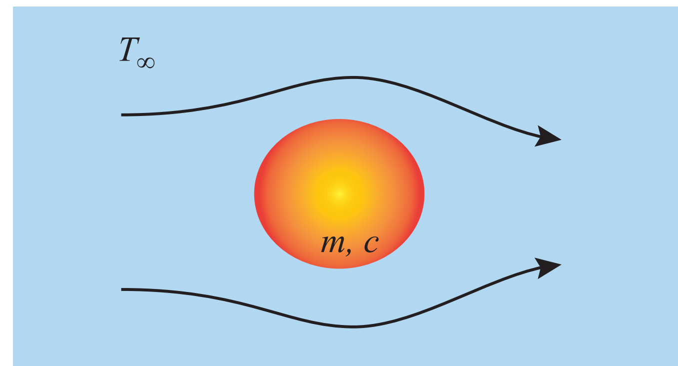
$$mc\frac{dT}{dt} = -hA(T - T_\infty)$$



Newton's Law of Cooling

Apply an energy balance

$$mc\frac{dT}{dt} = -Q = -hA(T_s - T_\infty)$$



Assume material is highly conductive $\Rightarrow T_s = T$

$$mc\frac{dT}{dt} = -hA(T - T_\infty)$$

or

$$\frac{dT}{dt} = -\frac{hA}{mc}(T - T_\infty) \qquad (1)$$

Example: Analytical Solution

Equation (1) has an analytical solution

Let $\theta = T - T_\infty$, so that

$$\frac{d\theta}{dt} = \frac{dT}{dt}$$

and Equation (1) becomes

$$\frac{d\theta}{dt} = -\frac{hA}{mc}\theta$$

Example: Analytical Solution

Equation (1) has an analytical solution

Let $\theta = T - T_\infty$, so that

$$\frac{d\theta}{dt} = \frac{dT}{dt}$$

and Equation (1) becomes

$$\frac{d\theta}{dt} = -\frac{hA}{mc}\theta$$

For convenience, define $\tau = \dfrac{mc}{hA}$ and rearrange

$$\frac{d\theta}{\theta} = -\frac{1}{\tau}dt \qquad (2)$$

Example: Analytical Solution

Equation (1) has an analytical solution

Let $\theta = T - T_\infty$, so that

$$\frac{d\theta}{dt} = \frac{dT}{dt}$$

and Equation (1) becomes

$$\frac{d\theta}{dt} = -\frac{hA}{mc}\theta$$

For convenience, define $\tau = \dfrac{mc}{hA}$ and rearrange

$$\frac{d\theta}{\theta} = -\frac{1}{\tau}dt \qquad (2)$$

Equation (2) can be integrated directly:

$$\ln\theta = -\frac{t}{\tau} + C$$

$$\ln\theta - \ln C_2 = -\frac{t}{\tau}$$

$$\ln\frac{\theta}{C_2} = -\frac{t}{\tau}$$

$$\theta = C_2 e^{-t/\tau}$$

Example: Analytical Solution

Equation (1) has an analytical solution

Let $\theta = T - T_\infty$, so that

$$\frac{d\theta}{dt} = \frac{dT}{dt}$$

and Equation (1) becomes

$$\frac{d\theta}{dt} = -\frac{hA}{mc}\theta$$

For convenience, define $\tau = \dfrac{mc}{hA}$ and rearrange

$$\frac{d\theta}{\theta} = -\frac{1}{\tau}dt \qquad (2)$$

Equation (2) can be integrated directly:

$$\ln\theta = -\frac{t}{\tau} + C$$

$$\ln\theta - \ln C_2 = -\frac{t}{\tau}$$

$$\ln\frac{\theta}{C_2} = -\frac{t}{\tau}$$

$$\theta = C_2 e^{-t/\tau}$$

Applying the initial condition $T(t = 0) = T_0$ or $\theta(t = 0) = T_0 - T_\infty$ gives

$$\theta = \theta_0 e^{-t/\tau}$$

Example: Analytical Solution

Equation (1) has an analytical solution

Let $\theta = T - T_\infty$, so that

$$\frac{d\theta}{dt} = \frac{dT}{dt}$$

and Equation (1) becomes

$$\frac{d\theta}{dt} = -\frac{hA}{mc}\theta$$

For convenience, define $\tau = \dfrac{mc}{hA}$ and rearrange

$$\frac{d\theta}{\theta} = -\frac{1}{\tau}dt \qquad (1)$$

Equation (2) can be integrated directly:

$$\ln\theta = -\frac{t}{\tau} + C$$

$$\ln\theta - \ln C_2 = -\frac{t}{\tau}$$

$$\ln\frac{\theta}{C_2} = -\frac{t}{\tau}$$

$$\theta = C_2 e^{-t/\tau}$$

Applying the initial condition $T(t = 0) = T_0$ or $\theta(t = 0) = T_0 - T_\infty$ gives

$$\theta = \theta_0 e^{-t/\tau} \quad \text{or} \quad \boxed{T = T_\infty + (T_0 - T_\infty)e^{-t/\tau}}.$$

Verifying Analytical Solutions

Given a trial solution to an ODE, test the solution against the ODE by direct subsitution.

Verifying Analytical Solutions

Given a trial solution to an ODE, test the solution against the ODE by direct subsitution.

Example: Determine whether $y = (3t + 1)^{1/3}$ is a solution to $\frac{dy}{dt} = \frac{1}{y^2}$ with $y(0) = 1$.

Verifying Analytical Solutions

Example: Determine whether $y = (3t+1)^{1/3}$ is a solution to $\dfrac{dy}{dt} = \dfrac{1}{y^2}$ with $y(0) = 1$.

First, compute $\dfrac{dy}{dt}$

$$\frac{d}{dt}(3t+1)^{1/3} = \frac{1}{3}(3t+1)^{-2/3}(3) = (3t+1)^{-2/3} \qquad (\star)$$

Verifying Analytical Solutions

Example: Determine whether $y = (3t+1)^{1/3}$ is a solution to $\dfrac{dy}{dt} = \dfrac{1}{y^2}$ with $y(0) = 1$.

First, compute $\dfrac{dy}{dt}$

$$\frac{d}{dt}(3t+1)^{1/3} = \frac{1}{3}(3t+1)^{-2/3}(3) = (3t+1)^{-2/3} \qquad (\star)$$

Second, substitute the suspected solution into the right hand side of the differential equation

$$\frac{1}{y^2} = \frac{1}{\left[(3t+1)^{1/3}\right]^2} = \frac{1}{(3t+1)^{2/3}} = (3t+1)^{-2/3} \qquad (\star\star)$$

Verifying Analytical Solutions

Example: Determine whether $y = (3t+1)^{1/3}$ is a solution to $\dfrac{dy}{dt} = \dfrac{1}{y^2}$ with $y(0) = 1$.

First, compute $\dfrac{dy}{dt}$

$$\frac{d}{dt}(3t+1)^{1/3} = \frac{1}{3}(3t+1)^{-2/3}(3) = (3t+1)^{-2/3} \qquad (\star)$$

Second, substitute the suspected solution into the right hand side of the differential equation

$$\frac{1}{y^2} = \frac{1}{\left[(3t+1)^{1/3}\right]^2} = \frac{1}{(3t+1)^{2/3}} = (3t+1)^{-2/3} \qquad (\star\star)$$

Therefore the left hand side dy/dt reduces to Equation ($\star$), and the right hand side $1/y^2$ reduces to Equation ($\star\star$). Since those two equations are equal, the proposed solution satisfies the differential equation.

Verifying Analytical Solutions

Example: Determine whether $y = (3t+1)^{1/3}$ is a solution to $\dfrac{dy}{dt} = \dfrac{1}{y^2}$ with $y(0) = 1$.

First, compute $\dfrac{dy}{dt}$

$$\frac{d}{dt}(3t+1)^{1/3} = \frac{1}{3}(3t+1)^{-2/3}(3) = (3t+1)^{-2/3} \qquad (\star)$$

Second, substitute the suspected solution into the right hand side of the differential equation

$$\frac{1}{y^2} = \frac{1}{\left[(3t+1)^{1/3}\right]^2} = \frac{1}{(3t+1)^{2/3}} = (3t+1)^{-2/3} \qquad (\star\star)$$

Therefore the left hand side dy/dt reduces to Equation ($\star$), and the right hand side $1/y^2$ reduces to Equation ($\star\star$). Since those two equations are equal, the proposed solution does satisfy the differential equation.

Next, verify the initial condition: y$(0) = (3(0)+1)^{1/3} = 1^{1/3} = 1$.

Verifying Analytical Solutions

Example: Determine whether $y = (3t+1)^{1/3}$ is a solution to $\dfrac{dy}{dt} = \dfrac{1}{y^2}$ with $y(0) = 1$.

First, compute $\dfrac{dy}{dt}$

$$\frac{d}{dt}(3t+1)^{1/3} = \frac{1}{3}(3t+1)^{-2/3}(3) = (3t+1)^{-2/3} \qquad (\star)$$

Second, substitute the suspected solution into the right hand side of the differential equation

$$\frac{1}{y^2} = \frac{1}{\left[(3t+1)^{1/3}\right]^2} = \frac{1}{(3t+1)^{2/3}} = (3t+1)^{-2/3} \qquad (\star\star)$$

Therefore the left hand side dy/dt reduces to Equation ($\star$), and the right hand side $1/y^2$ reduces to Equation ($\star\star$). Since those two equations are equal, the proposed solution does satisfy the differential equation.

Next, verify the initial condition: y$(0) = (3(0)+1)^{1/3} = 1^{1/3} = 1$.

Therefore, since $y = (3t+1)^{1/3}$ satisfies the differential equation *and* the initial condition, it is a solution.

Next Steps

- Introduce nomenclature for numerical solution of ODEs
- Derive Euler's method
- Demonstrate Euler's method