1^{st} Law of Thermodynamics for a System of Fixed Mass

Rate of increase of total stored energy in the system = net rate of heat addition to the system + net rate at which work is done on the system

$$\frac{D}{Dt}\int_{sys} \rho e \, d\forall = \dot{Q}_{in} + \dot{W}_{in} \qquad (1)$$

e = energy per unit mass

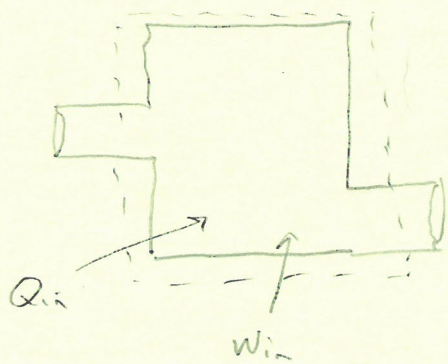
$$= \check{u} + \frac{V^2}{2} + gz \qquad (2)$$

V: magnitude of the velocity

$\check{u}$: internal energy

later we will define specific enthalpy $\check{h} = \check{u} + \frac{p}{\rho}$ (3)

$\check{u}$ & V local quantities that may vary in space



at time t, the system and control volume are coincident so

$$(Q_{in} + W_{in})_{sys} = (Q_{in} + W_{in})_{cv}$$

Apply R.T.T.

$$\frac{D}{Dt}\int_{sys} \rho e \, d\forall = \underbrace{\frac{\partial}{\partial t}\int_{cv} \rho e \, d\forall}_{\text{rate of increase in energy in the control volume}} + \underbrace{\int_{cs} \rho e (V \cdot \hat{n}) \, dA}_{\text{net outflow of energy across control volume surface (positive outward)}}$$

Eqn (1) becomes

$$\frac{\partial}{\partial t}\int_{CV}(\rho e\, d\forall) + \int_{CS}\rho e(\vec{V}\cdot\hat{n})\,dA = \dot{Q}_{in} + \dot{W}_{in} \qquad (4)$$

rate of heat transfer into and work done on the fluid in the control volume

Examine work terms

$$\dot{W}_{in} = \dot{W}_{shaft} + \dot{W}_{pressure} + W_{viscous}$$

generally we neglect

we can write $\quad \dot{W}_{pressure} = -\int_{CS} p(\vec{V}\cdot\hat{n})\,dA$

note

$\vec{V}\cdot\hat{n}\,dA$ is the rate at which a volume of fluid on the system boundary is causing the system boundary to increase

($\vec{V}\cdot\hat{n}$ is velocity across the CV boundary

$V\cdot\hat{n}$ is the velocity of the system boundary surface)

so

$\Rightarrow \quad -p(\vec{V}\cdot\hat{n})\,dA$ = rate at which pressure-volume work ($P\Delta V$) is done on the system boundary

thus

$$\dot{W}_{in} = \dot{W}_{shaft} - \int_{CS} p(\vec{V}\cdot\hat{n})\,dA \qquad (5)$$

substitute into (4)

$$\frac{\partial}{\partial t}\int_{CV}\rho e\,d\forall + \int_{CS}\rho e(\vec{V}\cdot\hat{n})dA = Q_{in} + W_{shaft,in} - \int_{CS} p(\vec{V}\cdot\hat{n})dA$$

rearrange

$$\frac{\partial}{\partial t}\int_{CV}\rho e\,d\forall + \int_{CS}(\rho e + p)(\vec{V}\cdot\hat{n})dA = Q_{in} + W_{s,in}$$

expand $\rho e + p$ term

$$\rho e + p = \rho\left(e + \frac{p}{\rho}\right) = \rho\left(\check{u} + \frac{V^2}{2} + gz + \frac{p}{\rho}\right)$$

but $\check{u} + \frac{p}{\rho} \equiv \check{h}$ so

$$\rho e + p = \rho\left(\check{h} + \frac{V^2}{2} + gz\right)$$

thus.

$$\frac{\partial}{\partial t}\int_{CV}\rho e\,d\forall + \int_{CS}\rho\left(\check{h} + \frac{V^2}{2} + gz\right)(\vec{V}\cdot\hat{n})dA = Q_{in} + W_{s,in} \qquad (6)$$

General energy eqn.

Application:

Steady Flow Energy Eqn - Uniform inlets & outlets

steady flow $\frac{\partial}{\partial t}\int_{CV}\rho e\,d\forall = 0$

uniform inlets/outlets. ρ, e, V are all uniform across control surfaces $(\vec{V}\cdot\hat{n} \neq 0)$

thus

$$\int_{CS} \rho\left(\check{h}+\frac{V^2}{2}+gz\right)(\vec{V}\cdot\hat{n})\,dA = \sum_{out}\dot{m}\left(\check{h}+\frac{V^2}{2}+gz\right) - \sum_{in}\dot{m}\left(\check{h}+\frac{V^2}{2}+gz\right)$$

Steady flow energy eqn w/ uniform inlets & outlets, incompressible flow

$$\boxed{\sum_{out}\dot{m}\left(\check{h}+ke+pe\right) - \sum_{in}\dot{m}\left(\check{h}+ke+pe\right) = \dot{Q}_{in} + \dot{W}_{s,in}} \qquad (7)$$

shouldn't be new

1 inlet / 1 exit $\quad \dot{m}_{OUT} = \dot{m}_{in} = \dot{m}$

$$\left(\check{h}+ke+pe\right)_{OUT} - \left(\check{h}+ke+pe\right)_{in} = \underset{\displaystyle = q_{in}}{\frac{\dot{Q}_{in}}{\dot{m}}} + \underset{\displaystyle w_s}{\frac{\dot{W}_s}{\dot{m}}} \qquad (8)$$

$$= q_{in} + w_s$$

Role of irreversible energy loss

expand $\check{h}$

$$\check{h} = \check{u} + \frac{P}{\rho}$$

$$\left(\check{u}+\frac{P}{\rho}+ke+pe\right)_{OUT} - \left(\check{u}+\frac{P}{\rho}+ke+pe\right)_{in} = q_{in} + w_{s\,in}$$

move internal energy terms to R.H.S & rearrange

$$\left(\frac{P}{\rho}+ke+pe\right)_{OUT} - \left(\frac{P}{\rho}+ke+pe\right)_{in} = w_{s\,in} - \left(\underbrace{\check{u}_{OUT}-\check{u}_{in}-q_{in}}_{\text{define as } q_{loss}}\right)$$

q_{loss} = thermal energy loss due to irreversible effects
- friction (entropy generation)

substitute in q_{loss}
divide through by g

$$\boxed{\left(\frac{P}{\gamma} + \frac{V^2}{2g} + z\right)_{out} - \left(\frac{P}{\gamma} + \frac{V^2}{2g} + z\right)_{in} = h_s - h_L} \qquad (9)$$

h_s = shaft work done on fluid in units of head $= \dfrac{W_{s,in}}{\dot{m} g}$

h_L = head loss (due to friction irreversibilities) = always positive

Use Eqn (9) to analyze pipe flow & other hydraulic systems.

Eqn (9) is the mechanical form of the energy eqn

Use Eqn (8) where thermal energy terms appear explicitly. Eqn (8) is the thermal form of the energy eqn.

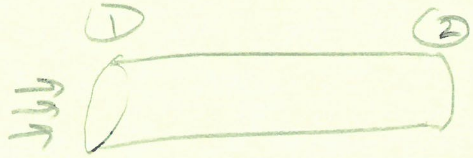
* mechanical form of the energy eqn is used to analyze (most) fluid systems

Note: Bernoulli's Eqn ⇒ restrictive form of energy eqn
h_s & h_L both zero

Implications of head loss term

$$h_L = \frac{q_{loss}}{g}$$

Consider flow in a horizontal pipe



① in
② out

assume steady incompressible flow

$$V_1 A_1 = V_2 A_2 \quad \text{but } A_1 = A_2 \text{ so } V_1 = V_2$$

Apply energy eqn

$$\left(\frac{P}{\rho} + \cancel{\frac{V^2}{2}} + \cancel{gz}\right)_2 - \left(\frac{P}{\rho} + \cancel{\frac{V^2}{2}} + \cancel{gz}\right)_1 = \underbrace{\cancel{w_{s,in}}}_{0} - (\check{u}_2 - \check{u}_1 - q_{in})$$

$$0 \qquad 0$$

thus

$$\frac{P_2 - P_1}{\rho} = -(\check{u}_2 - \check{u}_1 - q_{in})$$

pressure drop in flow direction $P_1 - P_2 > 0$

so $\check{u}_2 - \check{u} - q_{in} > 0$

Case (1)

if Adiabatic pipe flow $q_{in} = 0$

then $\check{u}_2 - \check{u}_1 > 0 \Rightarrow$ internal energy increases in flow direction

Increase is caused by dissipation of mechanical energy case 2. isothermal pipe $T_1 = T_2$

ideal gas / liquid $U(T)$ thus $\check{U}_1 = \check{U}_2$

thus $-q_{in} > 0$ or $q_{in} < 0$

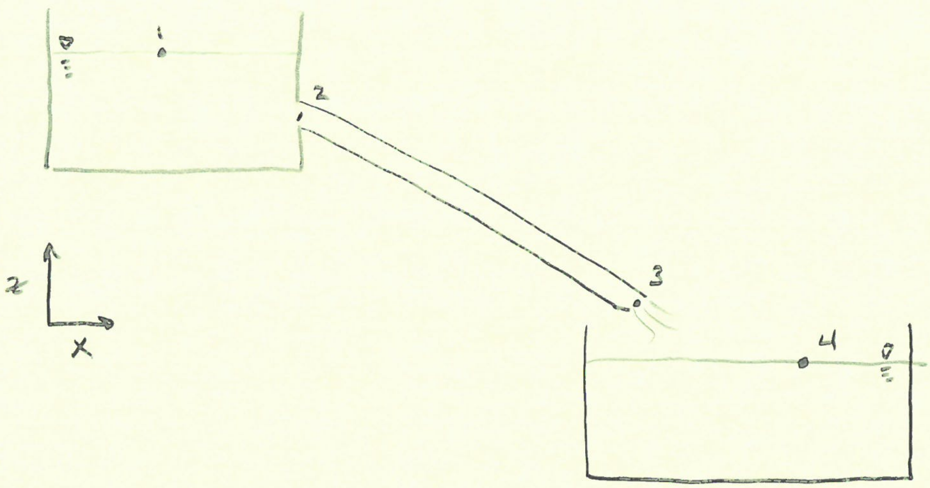
heat is transferred out of the pipe

pressure drop corresponds to a dissipation of energy

If the pipe is isothermal the energy dissipation results in a loss of heat to the environment

Example 2 of energy eqn application

What is the head loss for flow between two reservoirs



Assume: steady, incompressible flow
↑
elevation of free surfaces don't change

Apply steady flow energy eqn between pts 1 & 4

$$\frac{P_4}{\gamma} + \frac{V_4^2}{2g} + z_4 = \frac{P_1}{\gamma} + \frac{V_1^2}{2g} + z_1 + h_s - h_L$$

but

$P_4 = P_1$ because both free surfaces are open to the atmosphere

$V_4 = V_1 = 0$ because flow is steady & reservoirs are large

$h_s = 0$ because there is no turbine or pump between 1 & 4

thus

$$z_4 = z_1 - h_L \quad \text{or} \quad \boxed{h_L = z_1 - z_4}$$

for system shown: all of the potential energy that was available to do work is lost. The energy is dissipated as heat to the environment

- Apply energy eqn between state 1 & 3

$$\frac{P_3}{\gamma} + \frac{V_3^2}{2g} + z_3 = \frac{P_1}{\gamma} + \frac{V_1^2}{2g} + z_1 + h_s - h_{L_{13}}$$

$P_1 = P_3 = 0$ open to atmosphere

$V_1 = 0$, $h_s = 0$

thus

$$\frac{V_3^2}{2g} + z_3 = z_1 - h_{L_{13}} \quad \text{or} \quad h_{L_{13}} = z_1 - z_3 - \frac{V_3^2}{2g}$$

Note kinetic energy is still available to do work. All the energy is not dissipated until the fluid is brough to rest at pt 4

- Addition of a turbine

energy balance between pts 1 & 4 yields

$$h_{L_{14}} = z_1 - z_4 + h_s$$

note: $h_s < 0$ the turbine extracts work from the fluid ie the fluid does work on the environment.

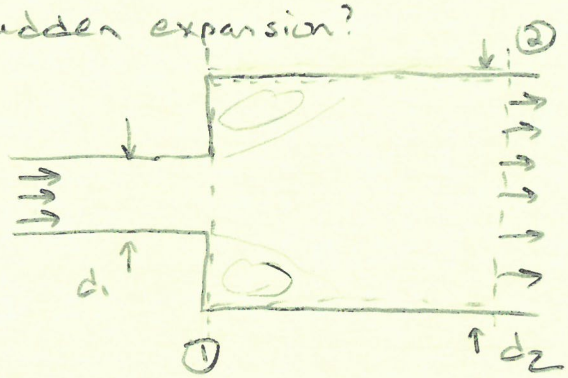
So $h_{L_{14}}$ with the turbine is less than $h_{L_{14}}$ without the turbine. With a turbine $h_s \neq 0$ which means less of the available potential energy is dissipated as heat the fluid still experiences the same change in energy per unit mass $(z_1 - z_4)$ but some of the energy is extracted as work

From a 2nd Law perspective less entropy is generated if there is a turbine in the system

Note the temperature rise of the lower reservoir is less with a turbine

Application of Energy Eqn: Flow through a sudden expansion

What's the loss coefficient for steady flow through a sudden expansion?



Assume: steady flow

velocity profile is uniform at pts 1 & 2

pt. 2 is sufficiently far downstream

expansion is horizontal

Find the loss coefficient:

$$K_L = \frac{h_L}{V_1^2/2g} \qquad (1)$$

need to calculate the head loss.

Steady flow energy eqn

$$\frac{P_2}{\gamma} + \frac{V_2^2}{2g} + \cancel{z_2} = \frac{P_1}{\gamma} + \frac{V_1^2}{2g} + \cancel{z_1} - h_L$$

$$\underbrace{\quad}_{0}$$

solve for h_L

$$h_L = \frac{P_1 - P_2}{\gamma} + \frac{V_1^2 - V_2^2}{2g}$$

or

$$\gamma h_L = P_1 - P_2 + \frac{1}{2}\rho\left(V_1^2 - V_2^2\right) = P_1 - P_2 + \frac{1}{2}\rho V_1^2\left(1 - \frac{V_2^2}{V_1^2}\right)$$

Look at 2nd term: How do we relate V_1 to V_2 ⇒ conservation of mass

$$A_1 V_1 = A_2 V_2 \qquad \text{from} \quad 0 = \frac{\partial}{\partial t}\int_{cv} \rho V \, d\forall + \int_{cs} \rho V \cdot n \, dA$$

thus $\dfrac{V_2}{V_1} = \dfrac{A_1}{A_2}$

so

$$\underbrace{\gamma h_L = P_1 - P_2 + \frac{1}{2}\rho V_1^2}_{\text{looking like loss coefficient}}\left[1 - \left(\frac{A_1}{A_2}\right)^2\right] \qquad (1)$$

What's $P_1 - P_2$? How can we relate this to velocity?

⇒ conservation of momentum

$$\Sigma F_x = \int_{cs} \rho V_x (V \cdot n) \, dA \qquad \rho, V \text{ are uniform}$$

$$P_1 A_2 - P_2 A_2 = \rho V_1(-V_1)A_1 + \rho V_2 (V_2) A_2$$

only pressure forces

Note: pressure at pt 1 acts on A_2, but momentum flows into control volume through A_1

solving for $P_1 - P_2$

$$P_1 - P_2 = \rho^2 V_2^2 - \rho V_1^2 \frac{A_1}{A_2} = \rho V_1^2 \left(\frac{A_1}{A_2}\right)^2 - \rho V_1^2 \frac{A_1}{A_2}$$

(2) $$P_1 - P_2 = \rho V_1^2 \left[\left(\frac{A_1}{A_2}\right)^2 - \frac{A_1}{A_2}\right]$$

Note: $\dfrac{A_1}{A_2} < 1 \Rightarrow \left(\dfrac{A_1}{A_2}\right)^2 < \dfrac{A_1}{A_2}$

$\Rightarrow P_1 - P_2 < 0$ or $P_2 > P_1$

substitute 2 into 1

$$\gamma h_L = \rho V_1^2\left[\left(\frac{A_1}{A_2}\right)^2 - \frac{A_1}{A_2}\right] + \frac{1}{2}\rho V_1^2\left[1 - \left(\frac{A_1}{A_2}\right)^2\right]$$

rearrange

$$\gamma h_L = \rho V_1^2\left[\frac{1}{2}\left(\frac{A_1}{A_2}\right)^2 - \frac{A_1}{A_2} + \frac{1}{2}\right] = \frac{1}{2}\rho V_1^2\left[\left(\frac{A_1}{A_2}\right)^2 - 2\frac{A_1}{A_2} + 1\right]$$

$$= \frac{1}{2}\rho V_1^2\left(\frac{A_1}{A_2} - 1\right)^2$$

divide by γ to get h_L

$$h_L = \frac{1}{2g}V_1^2\left(\frac{A_1}{A_2} - 1\right)^2$$

so

$$K_L = \frac{h_L}{V_1^2/2g} = \left(\frac{A_1}{A_2} - 1\right)^2$$

or since term is squared and $\frac{A_1}{A_2} < 0$ $\left(\frac{A_1}{A_2} - 1\right)^2 = \left(1 - \frac{A_1}{A_2}\right)^2$

so

$$K_L = \left(1 - \frac{A_1}{A_2}\right)^2$$

Differential Analysis:

We've examined the integral control volume approach for conservation of mass and momentum

Conservation of mass

$$\int_{cs} \rho(V \cdot n)\,dA + \frac{\partial}{\partial t}\int_{cv} \rho\,d\forall = 0$$

Conservation of Linear momentum

$$\Sigma F = \int_{cs} \rho V(V \cdot n)\,dA + \frac{\partial}{\partial t}\int_{cv} \rho V\,d\forall$$

Solving these equations requires us to have knowledge of the flow field - what's V.? Do we know this ahead of time?

simplification → V is uniform or we could measure V (easy or hard?)

The above equations represent a global approach.

- what if we want to know local velocities & stresses?

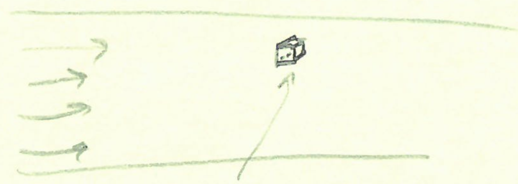
⇓

Differential Approach

- express conservation of mass & momentum as partial differential eqns
  - yields local velocity field
  - flow rates & total stresses can be obtained by integrating velocities and local stresses.

Conservation of mass

consider a small region of space in a moving fluid

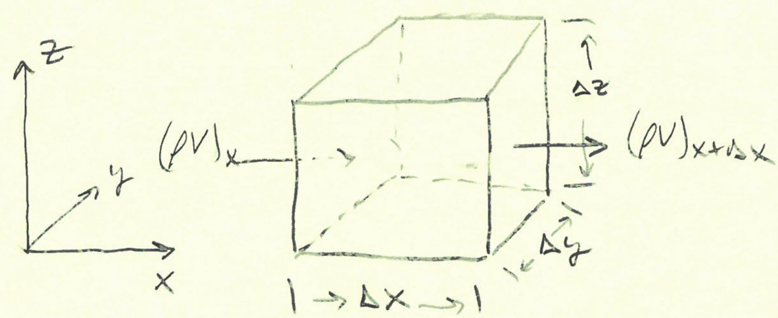


stationary differential element

conservation of mass states

rate of mass accumulation = rate of mass in − rate of mass out

differential element



Consider the face ⊥ to the x-axis at location x

mass crossing the $\Delta y \Delta z$ surface at x $= (\rho V)_x \, \Delta y \Delta z$ mass in

Mass crossing the $\Delta y \Delta z$ surface at $x+\Delta x$ $= (\rho V)_{x+\Delta x} \, \Delta y \Delta z$

net flow of mass along x axis $= (\rho V)_x \Delta y \Delta z - (\rho V)_{x+\Delta x} \Delta y \Delta z$

$$= \left[(\rho V)_x - (\rho V)_{x+\Delta x} \right] \Delta y \Delta z$$

similarly

Net flow of mass in along y-axis $= \left[(\rho V)_y - (\rho V)_{y+\Delta y} \right] \Delta x \Delta z$

Net flow of mass in along z axis $= \left[(\rho V)_z - (\rho V)_{z+\Delta z} \right] \Delta y \Delta x$

the rate of mass accumulation:

$$\left(\frac{\partial \rho}{\partial t}\right) \Delta x \Delta y \Delta z$$

thus

$$\left(\frac{\partial \rho}{\partial t}\right) \Delta x \Delta y \Delta z = \left[(\rho V)_x - (\rho V)_{x+\Delta x} \right] \Delta y \Delta z + \left[(\rho V)_y - (\rho V)_{y+\Delta y} \right] \Delta x \Delta z$$
$$+ \left[(\rho V)_z - (\rho V)_{z+\Delta z} \right] \Delta y \Delta x$$

divide through by $\Delta x \Delta y \Delta z$

$$\frac{\partial \rho}{\partial t} = \frac{\left[(\rho V)_x - (\rho V)_{x+\Delta x} \right]}{\Delta x} + \frac{\left[(\rho V)_y - (\rho V)_{y+\Delta y} \right]}{\Delta y} + \frac{\left[(\rho V)_z - (\rho V)_{z+\Delta z} \right]}{\Delta z}$$

take limit as $\Delta x \to 0$ $\Delta y \to 0$ $\Delta z \to 0$

$$\frac{\partial \rho}{\partial t} = -\left[\frac{\partial}{\partial x}(\rho V_x) + \frac{\partial}{\partial y}(\rho V_y) + \frac{\partial}{\partial z}(\rho V_z) \right]$$

or

$$\boxed{\frac{\partial \rho}{\partial t} = -\underbrace{(\nabla \cdot \rho V)}_{\text{divergence of } \rho V}}$$ Continuity Eqn

We can expand the RHS & collect all ρ terms on left side

$$\frac{\partial \rho}{\partial t} + v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} = -\rho\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)$$

shorthand

$$\frac{D\rho}{Dt} = -\rho(\nabla \cdot v)$$ valid for any fluid

substantial derivative

For incompressible fluid – ρ is constant

thus $\boxed{\nabla \cdot V = 0}$ or $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$

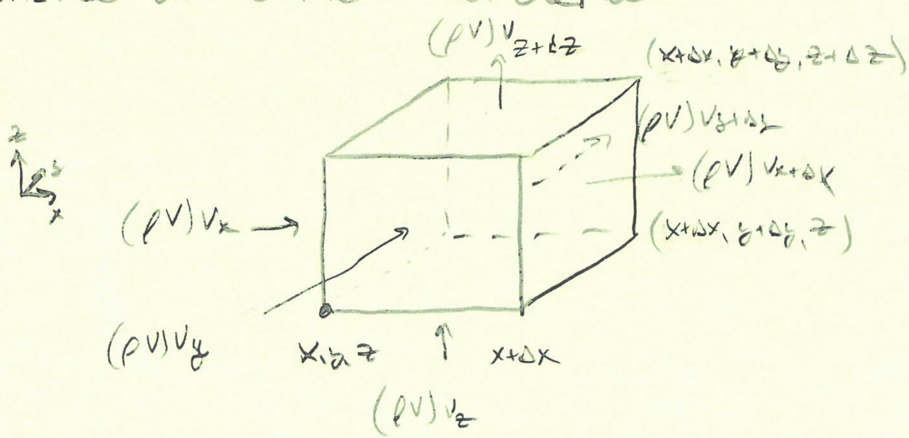
divergence of velocity is zero.

The Equation of Motion

Conservation of momentum:

rate of momentum accumulation = rate of momentum in − rate of momentum out + sum of forces acting on system

consider our differential element



rate of momentum accumulation $\frac{\partial}{\partial t}(\rho V)\, \Delta x \Delta y \Delta z$

if we consider just the x componet of momentum (ρV_x)

x componet accumulation $= \frac{\partial}{\partial t}(\rho V_x)\, \Delta x \Delta y \Delta z$ $\underline{E1}$

rate at which momentum enters the face at x by convection is (due to bulk flow)

$(\rho V) V_x \Delta y \Delta z$

rate at which momentum leaves the face at $x+\Delta x$ by convection

$(\rho V) V_{x+\Delta x} \Delta y \Delta z$

thus net rate of momentum entering along x axis due to convection

$$(\rho V) V_x \Delta y \Delta z - (\rho V) V_{x+\Delta x} \Delta y \Delta z$$

similarly for the y & z axes

rate of momentum entering by convection

$$(\rho V) V_y \Delta x \Delta z - (\rho V) V_{y+\Delta y} \Delta x \Delta z$$

&

$$(\rho V) V_z \Delta y \Delta x - (\rho V) V_{z+\Delta z} \Delta y \Delta x$$

net convective flow of momentum

Consider just the x component of momentum

Net convective flow of x momentum (bulk flow)

$\underline{E2}$

$$\left[(\rho V_x) V_x - (\rho V_x) V_{x+\Delta x}\right] \Delta y \Delta z + \left[(\rho V_x) V_y - (\rho V_x) V_{y+\Delta y}\right] \Delta x \Delta z + \left[(\rho V_x) V_z - (\rho V_x) V_{z+\Delta z}\right] \Delta y \Delta x$$

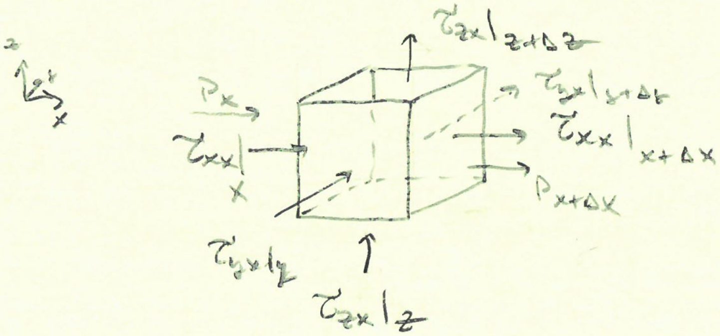
So we have accumulation ad convective

What are our forces on the differential element

$$\Sigma F = \Sigma F_{surface} + \Sigma F_{body}$$

Consider only forces acting in the x direction

if we examine forces acting on the surface of our differential element



So we have normal stresses acting on the x & x+Δx

are shear stresses acting on all other faces.

shear stress
τ_{yx} = flux of x momentum through a face ⊥ to y axis
τ_{zx} = flux of x momentum through a face ⊥ to z axis

Normal stress
τ_{xx} = flux of x momentum through a face ⊥ to x axis

these can be consider stresses or the rate at which momentum enters from molecular transport - really our viscous stresses

thus net x component surface forces from molecular interactions

E3

$$\left(\tau_{xx}|_x - \tau_{xx}|_{x+\Delta x}\right)\Delta y \Delta z + \left(\tau_{yx}|_y - \tau_{yx}|_{y+\Delta y}\right)\Delta x \Delta z + \left(\tau_{zx}|_z - \tau_{zx}|_{z+\Delta z}\right)\Delta x \Delta y$$

Other surface stresses? ⇒ pressure forces which act normal to surface

Pressure force along x axis

E4

$$\left(P_x - P_{x+\Delta x}\right)\Delta y \Delta z$$

So we have pressure, normal & shear surface forces - what are our body forces

Body forces: force acts on entire volume

primary body ⇒ gravity

$$F = \rho g \Delta x \Delta y \Delta z$$

in terms of the x component

E5

$$F_x = \rho g_x \Delta x \Delta y \Delta z$$

there are other body force terms - magnetics, dielectrophoresis

consider advanced topics ; neglected.

Combining E1 thru E5

x-momentum balance

$$\frac{\partial}{\partial t}(\rho V_x)\,\Delta x \Delta y \Delta z = +\left(\rho V_x V_x|_x - \rho V_x V_x|_{x+\Delta x}\right)\Delta y \Delta z + \left(\rho V_x V_y|_y - \rho V_x V_y|_{y+\Delta y}\right)\Delta x \Delta z$$

$$+\left(\rho V_x V_z|_z - \rho V_x V_z|_{z+\Delta z}\right)\Delta x \Delta y + \left(\tau_{xx}|_x - \tau_{xx}|_{x+\Delta x}\right)\Delta y \Delta z$$

$$+\left(\tau_{yx}|_y - \tau_{yx}|_{y+\Delta y}\right)\Delta x \Delta z + \left(\tau_{zx}|_z - \tau_{zx}|_{z+\Delta z}\right)\Delta y \Delta x$$

$$+\left(P_x - P_{x+\Delta x}\right)\Delta y \Delta z + \rho g_x \Delta x \Delta y \Delta z$$

divide through by $\Delta x \Delta y \Delta z$

take limit as $\Delta x \to 0$ $\Delta y \to 0$ $\Delta z \to 0$

$$\frac{\rho V_x V_x|_x - \rho V_x V_x|_{x+\Delta x}}{\Delta x} \to \frac{-\partial}{\partial x}(\rho V_x V_x)$$

thus.

$$\frac{\partial}{\partial t}\rho V_x = -\left(\frac{\partial}{\partial x}\rho V_x V_x + \frac{\partial}{\partial y}\rho V_y V_x + \frac{\partial}{\partial z}\rho V_z V_x\right) - \left(\frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx}\right)$$

$$-\frac{\partial P}{\partial x} + \rho g_x \qquad \text{x component}$$

Similar eqns can be obtained for y & z components

by replacing x with y or z

In shorthand notation - the three components combined are

E6 $$\underbrace{\frac{\partial}{\partial t}\rho v}_{} = -\underbrace{\nabla\cdot\rho v v}_{} - \underbrace{\nabla P}_{} - \underbrace{\nabla\cdot\tau}_{} + \underbrace{\rho g}_{}$$

rate of increase of momentum per unit volume

rate of momentum convected per unit volume

pressure force on element per unit volume

rate of momentum gain by viscous transfer per unit volume

gravitational force on element per unit volume

τ = stress tensor : nine components

If we expand $\nabla\cdot\rho v v = \rho v\cdot\nabla v + v\nabla\cdot\rho v$ and apply continuity & adding to LHS.

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot\rho v$$

E6 reduces to

E7 $$\boxed{\rho\frac{Dv}{Dt} = -\nabla P - \nabla\cdot\tau + \rho g}$$ Cauchy momentum Eqn

Can't solve this - the system is not closed.

we have 3 eqns but 3 components of velocity and nine components of stress. We need 9 more relationships.

Stresses are generally symmetric

$$\tau_{xy} = \tau_{yx}, \quad \tau_{zx} = \tau_{xz}, \quad \tau_{yz} = \tau_{zy}$$ there's 3

To determine the remaining 6 we need a

Constituitive Relationship. between stress & the velocity field

Newtonian fluids - shear stress is directly proportional to shear rate

$$\tau_{xx} = -2\mu \frac{\partial v_x}{\partial x} + \frac{2}{3}(\mu - K)\nabla \cdot v$$

$$\tau_{yy} = -2\mu \frac{\partial v_y}{\partial y} + \frac{2}{3}(\mu - K)\nabla \cdot v$$

$$\tau_{zz} = -2\mu \frac{\partial v_z}{\partial z} + \frac{2}{3}(\mu - K)\nabla \cdot v$$

$$\tau_{xy} = \tau_{yx} = -\mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)$$

$$\tau_{xz} = \tau_{zx} = -\mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)$$

$$\tau_{yz} = \tau_{zy} = -\mu\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right)$$

K is the bulk viscosity

$K = 0$ for monatic gases

K is small for most dense gases & liquids

generally ignore

we can plug these in to get the x, y, & z components for a Newtonian fluid

in short hand

$$\boxed{\rho \frac{Dv}{Dt} = -\nabla p + \mu \nabla^2 v + \frac{1}{3}\mu \nabla(\nabla \cdot v) + \rho g}$$

Navier Stokes Eqn.

If we assume incompressible flow $\nabla \cdot v = 0$

$$\rho \frac{Dv}{Dt} = -\nabla p + \mu \nabla^2 v + \rho g$$

Navier Stokes Eqn for incompressible flow constant viscosity

Conservation of Linear Momentum

linear momentum = mass x velocity = mV

thus
velocity is momentum per unit mass.

Applying the Reynolds Transport Theorem –

$$\frac{D}{Dt}\int_{sys} \rho b \, d\forall = \frac{\partial}{\partial t}\int_{cv} \rho b \, d\forall + \int_{cs} \rho b \vec{V}\cdot\hat{n} \, dA$$

let $b = V$

$$\underbrace{\frac{D}{Dt}\int_{sys} \rho \vec{V} d\forall}_{\text{time rate of change of linear momentum of a system}} = \underbrace{\frac{\partial}{\partial t}\int_{cv} \rho \vec{V} d\forall}_{\text{time rate of change of linear momentum in the control volume}} + \underbrace{\int_{cs} (\rho \vec{V}) \vec{V}\cdot\hat{n} \, dA}_{\text{net rate of linear momentum through the control surface}}$$

From Newton's 2nd Law of motion the

time rate of change of linear momentum of a system = sum of all external forces acting on the system

$$\frac{D}{Dt}\int_{sys} \rho \vec{V} d\forall = \sum F_{sys} = \sum F_{cv}$$

when the system & control volume are coincident at a instant in time

thus.

$$\boxed{\sum F_{cv} = \frac{\partial}{\partial t}\int_{cv} \rho \vec{V} d\forall + \int_{cs} (\rho \vec{V}) \vec{V}\cdot\hat{n} \, dA}$$

Fixed non-deforming CV

Gravity
EM
Dielectrophoretic – Body forces

Pressure
viscous drag
reaction
...
– Surface forces
Surface tension, Electroosmotic

Conservation of Linear Momentum

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e

Working with the momentum Eqn.

1. Draw system & identify the Control Volume

2. Locate coordinate axis

3. Sum the forces

   - identify forces due to structural supports

     • the control volume must cut through the support to expose the force

   - pressure force in fluid are exposed when the C.V cuts through the fluid : Pressure forces always act inward

4. Momentum is a vector eqn. : Balances must apply in each direction

$$\Sigma F_x = \frac{\partial}{\partial t}\int_{cv} \rho V_x \, d\forall + \int_{cs} \rho V_x (\vec{V}\cdot n)\, dA$$

$$\Sigma F_y = \frac{\partial}{\partial t}\int_{cv} \rho V_y \, d\forall + \int_{cs} \rho V_y (\vec{V}\cdot n)\, dA$$

$$\Sigma F_z = \frac{\partial}{\partial t}\int_{cv} \rho V_z \, d\forall + \int_{cs} \rho V_z (\vec{V}\cdot n)\, dA$$

5. Conservation of mass generally relevant to the analysis

Do example →

Conservation of Linear Momentum

- Moving (uniform velocity), non-deforming control volume

  relative velocity $\vec{W} = \vec{V} - \vec{V}_{cv}$

$$\Sigma \vec{F} = \frac{\partial}{\partial t}\int_{cv} \rho \vec{V} dv + \int_{cs} \rho \vec{V} (\vec{W}\cdot\hat{n})\, dA$$

W represent actual velocity across surface

Moving cv work with $\vec{W}$