Where would this be important?

more general - where would the force on a submerged object be important

↓

- Anything under water

specifically for the hatch - any system that requires a hatch

do you want to open it or keep it closed?

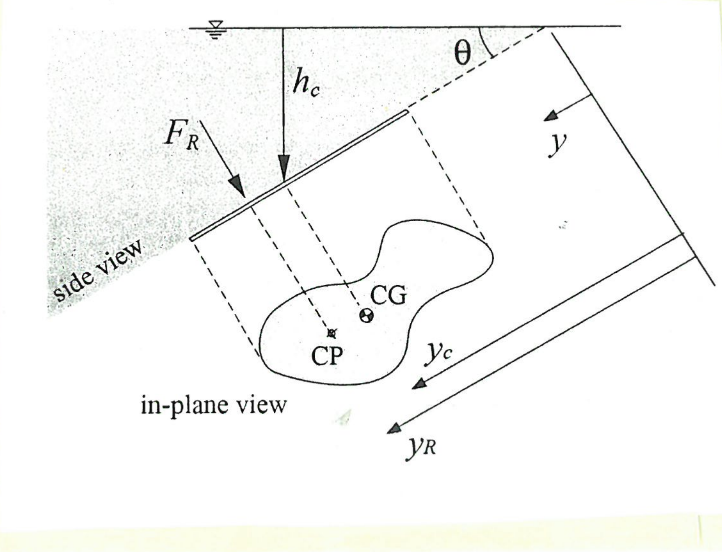
- how much force is required to open a car door if you were to drive you car into a lake/river & sink?
  - could you open the door?
- if you have a truck full of fish & your transporting them around dams you probably don't want the hatch to fail? but you also don't want a hatch so big & over engineered that it affects truck performance.

We probably could think up quite a few applications

So how do we analyze it?

Nomenclature: Fig. 2.17

For a plane surface



Resultant force: $F_R$   essentially we sum the force over the entire area

$$F_R = \int_{A_s} P\,dA = \int (P_{atm} + \gamma h)\,dA = P_{atm} A_s + \gamma \int_{A_s} h\,dA$$

if $\gamma$ is constant
ie. incompressible

need $h = h(A)$ or $A = A(h)$

From geometry   $h = y \sin\theta$

so substituting in RHT

$$\gamma \int_{A_s} h\,dA = \gamma \sin\theta \int_{A_s} y\,dA$$

but from solid mechanics   $y_c$ (location of the centroid measure from the free surface)

$$y_c = \frac{1}{A_s} \int y\,dA$$

so

$$\gamma \sin\theta \int_{A_s} y\,dA = \gamma \sin\theta\, y_c A_s = \gamma h_c A_s = P_c A_s$$

depth of the centroid

pressure at centroid of plane surface 2D

Resultant Force acts through the center of Pressure

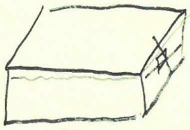
$$y_R = y_c + \frac{I_{xc}}{y_c A_s} \qquad x_R = x_c + \frac{I_{xyc}}{y_c A_s}$$

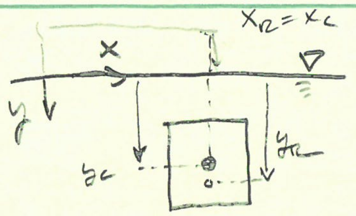
note
$y_R > y_c$ always
$\frac{I_{xc}}{y_c A_s}$ is positive

$x_c, y_c$ coordinates of centroid

$I_{xc}$ = moment of inertia of the surface about the x axis

$I_{xyc}$ = product of inertia of the surface about the x & y axes passing through the surface centroid



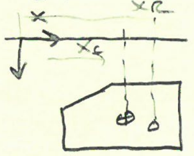


this example

plate symmetric about y axis

so $I_{xyc} = 0$ so $x_R = x_c$

if not symmetric



$I_{xyc} \neq 0$

$x_R > x_c$

For common, geometrically simple shapes $I_{xc}$, $I_{xyc}$ are known

see Fig 2.18

Complex shape - numerically integrate $\int P dA$

In carrying out calculations we should work in gage units

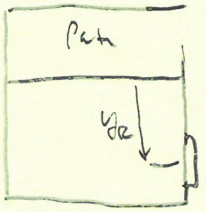
why?

1) For a vented tank the atmospheric pressure on the air side does not contribute to the net force

2) For an pressurized tank gage pressure correctly accounts for the net force

how do we know that

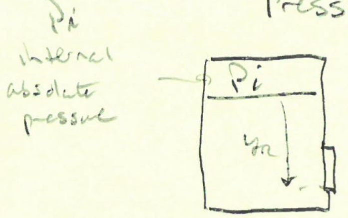
Vented



$P_{atm}$

$F_R = p_c A = \gamma h_c A$

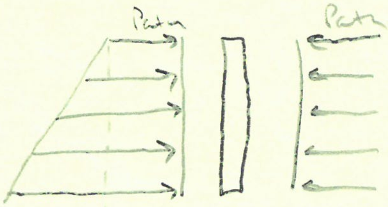
Pressurized



$P_{atm}$

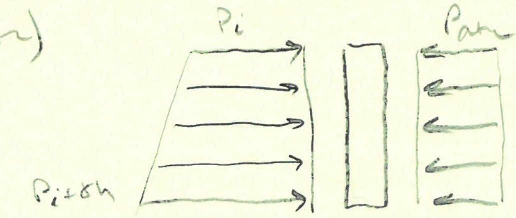
$F_R = P_c A = (P_i + \gamma h_c) A$

Free body diagram (Pressure prism)



$P = \gamma h + P_{atm}$

Net force $F = (P_{atm} + P_c)A - P_{atm} A = P_c A$



Net force $F = (P_i + P_c)A - P_{atm} A = (P_i - P_{atm}) + P_c$

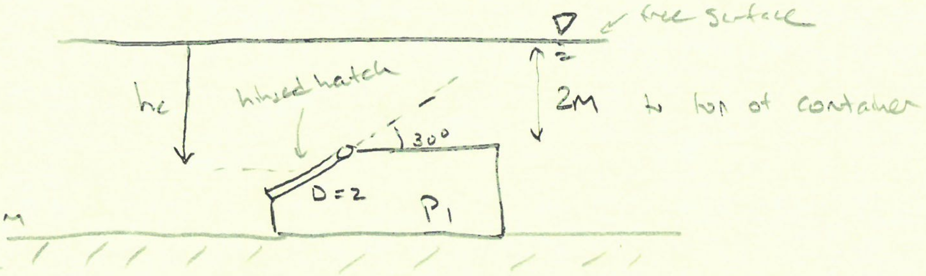
$P_i - P_{atm}$ is the gage pressure in tank

So work in gage units

Example: Consider a submersed container in the ocean

A 2m diameter hinged hatch is located on an inclined wall.

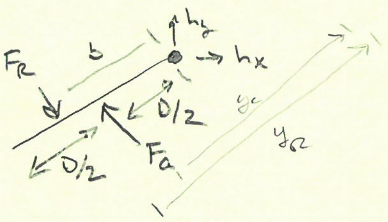
Determine the minimum air pressure within the container to open the hatch.



neglect weight of hatch & friction of hinge

To open hatch we need the pressure inside to balance the hydrostatic pressure

Soln: start with FBD on hatch



Note $\rho$ air is small $\rho g \Delta h$ small
therefore pressure is essentially constant with depth.
Assume exactly constant

Forces on hatch

forces: $F_R = \underbrace{\gamma h_c}_{\text{known or easily calculated}} A$ $\quad F_a = \overset{\text{looking for } P_1 \text{ (gage units)}}{P_1} A$

To open hatch moments about hinge must balance

$$F_R b = F_a D/2$$

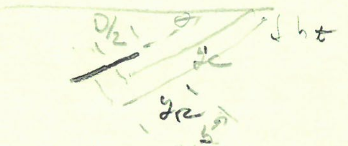
$$F_R b = P_1 A D/2 \Rightarrow P_1 = \frac{2F_R b}{AD} = \frac{2\gamma h_c b}{D}$$

thus we need b and $h_c$ to calculate $F_R$

Locate $F_R$: use moment of inertia formulas

Note: pressure prism doesn't work easily as hatch is round
volume & centroid of prism difficult to compute

formulas



$$y_R = \frac{I_{xc}}{y_c A} + y_c \Rightarrow y_c = \frac{h_t}{\sin\theta} + \frac{D}{2}$$

$$I_{xc} = \frac{\pi}{4} R^4$$ ⟵ all easily computed or known

find b

$$b = \frac{D}{2} + y_R - y_c \Rightarrow y_R - y_c = \frac{I_{xc}}{y_c A}$$

thus

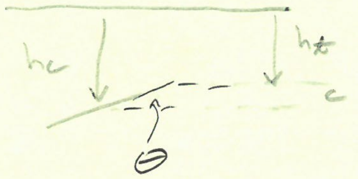
$$b = \frac{D}{2} + \frac{I_{xc}}{y_c A} = \frac{D}{2} + \frac{\pi R^4}{4 y_c A}$$

$A = \pi R^2$

so

$$\frac{\pi R^4}{4 y_c A} = \frac{R^2}{4 y_c}$$

find h_c



$$c = \frac{D}{2}\sin\theta$$

thus

$$h_c = h_t + \frac{D}{2}\sin\theta$$

Everything now known → put into $P_1$ eqn. or calculate individual terms

$$P_1 = \frac{2\gamma h_c b}{D} = \frac{2\gamma\left(h_t + \frac{D}{2}\sin\theta\right)\left(\frac{D}{2} + \frac{R^2}{4 y_c}\right)}{D}$$

now plug & chug $\gamma = 10.1 \times 10^3 \frac{N}{m^3}$

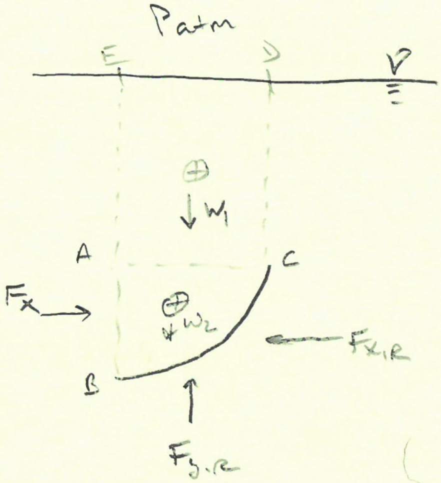
$h_t = 2m$ $D = 2m$ $\theta = 30°$ $R = 1m$

$$y_c = \frac{2m}{\sin\theta} + 1m = 5m$$

plug in Answer: $P_1 = 26.5$ kPa

# Curved Surfaces:

Not all submerged surfaces are planar - what if the tank is spherical => curved surface.



Fx = x direction hydrostatic force on AB

Fx,R is the reaction force by the surface to Fx

in equilibrium Fx = FxR

therefore

horizontal force on curved surface is Fx

$W_1$ is the weight of fluid in ACDE

$W_2$ is the weight of fluid in volume ABC

$F_{y,R}$ is reaction force by the surface to $W_1$ & $W_2$

$$F_{y,R} = W_1 + W_2 + P_{atm} A_{ED}$$

$A_{ED}$ is horizontal projection of curved surface

So

Fx & Fy can be computed

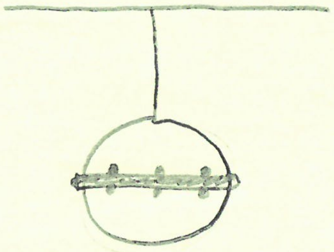
Find line of action by taking a moment balance about a convenient pt

Simple shapes: circular $F_R = \sqrt{F_x^2 + F_y^2}$ along center radial line & proper orientation

Complex shapes require numerical integration of $F = \int P\hat{n}\,dA$

Example

2 hemispherical shells are bolted together



Sphere diameter = 3ft

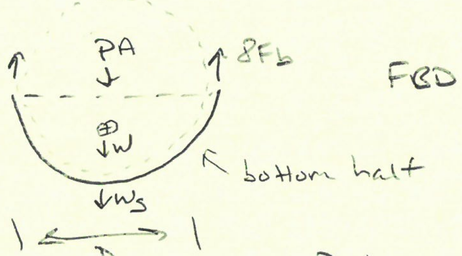
the resulting spherical container weighs 400 lb

Container is filled with mercury & supported by a cable

Container is vented at the top

If eight bolts are symmetrically located around the circumference,

what is the vertical force that each bolt must carry?



$F_b$ = force on one bolt

$W_s$ = weight of shell bottom half

$W_{Hg}$ = weight of Hg in bottom half

at equilibrium

P = hydrostatic pressure at mid plane

A = midplane area

$$\sum F_{vertical} = 0$$

$$8F_b = W_s + W_{Hg} + pA$$

$$P = \gamma h = \gamma_{Hg}\left(\frac{D}{2}\right)$$

$$F_b = \frac{W_s + W_{Hg} + pA}{8}$$

$$= \frac{1/2\,(400\,\text{lb}) + \gamma_{Hg}\left(\frac{1}{2}\right)\left(\frac{\pi}{6}D^3\right) + \gamma_{Hg}\left(\frac{D}{2}\right)\left(\frac{\pi}{4}D^2\right)}{8}$$

$F_b = 1896$ lb

## Buoyancy

1. A body immersed in a fluid experiences a vertical buoyant force equal to the weight of the fluid it displaces
2. A floating body displaces its own weight in the fluid in which it floats.

The buoyant force is the resultant force exerted by a static fluid on a submerged or partially submerged body.

Archimedes (287-212 BC) 1st discovered principles of specific gravity & of the lever - 1st to explain buoyancy 1800 yrs before advent of calculus

Newton 1642-172

it is claimed Archimedes exclaimed "Eureka" [I have found it] when he discovered buoyancy

Buoyant force equal to weight of fluid displaced

$$F_b = \rho g V_b = \gamma V_b$$

If time - show derivation on next page

Buoyant force passes through the center of buoyancy which is at centroid of displaced fluid.

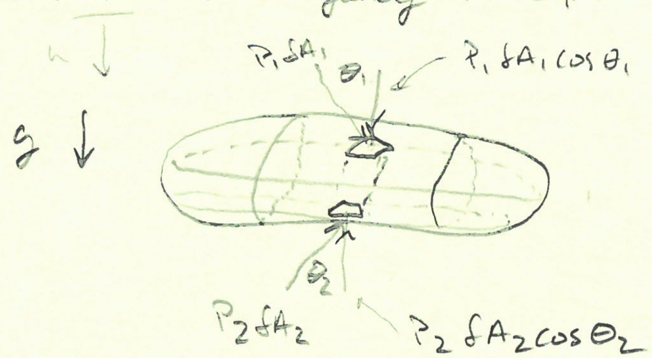
In a force balance on an object, the object weight must be included separately from the buoyancy force. $F_w = \rho_b g V_b = \gamma_b V_b$

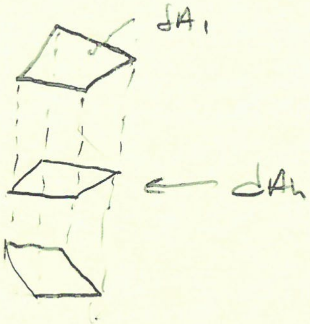
Sometimes these are paired together

Net force on object

$$F_{net} = (\gamma_b - \gamma_{liq}) V_b$$

Derivation of Buoyancy Principle - consider a completely submersed object





force projection parallel to g $\delta A_1 \cos\Theta_1 = dA_h$

$\delta A_2 \cos\Theta_2 = dA_h$

$P_2 \delta A_2$ is the normal force on surface $\delta A_2$

$P_1 \delta A_1$ is the normal force on surface $\delta A_1$

+ upward

$-P_1 \delta A_1 \cos\Theta_1 =$ force in the vertical direction on $\delta A_1$

$P_2 \delta A_2 \cos\Theta_2 =$ force in the vertical direction on $\delta A_2$

Net vertical force $= dF = P_2 \delta A_2 \cos\Theta_2 - P_1 \delta A_1 \cos\Theta_1$

but $\delta A_1 \cos\Theta_1 = \delta A_2 \cos\Theta_2 = dA_h$

so

$$dF = (P_2 - P_1) dA_h$$

using gage pressure $P_1 = \rho g h_1$
$P_2 = \rho g h_2$

with $\rho g h$

$$dF = \rho g (h_2 - h_1) dA_h$$

but the volume of our fluid element is $(h_2 - h_1) dA_h = dV$

thus $dF = \rho g dV$

integrate

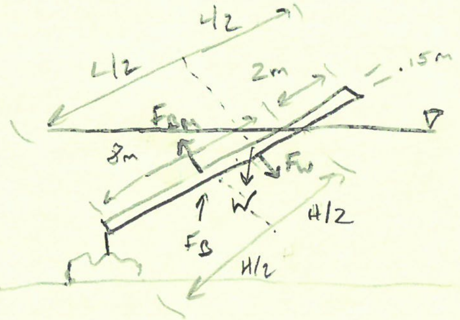
$$F = \int \rho g dV = \rho g V$$

where V is the volume of fluid displaced by the body

+ $\rho g V$ is the weight of liquid displaced

Buoyancy force is the weight of the displaced liquid.

Ex. A board is tethered to the bottom of a pond. [what's the specific weight of the board and] the tension in the tether.



board dimension
.15m x .35m x 10m

At equilibrium $\Sigma M = 0$ moments balance. $F_{Bm}(H/2) = F_w(L/2)$

so $(F_b \cancel{\cos\theta})(H/2) = (W \cancel{\cos\theta})(L/2)$

$$W = \gamma_B V = \gamma(0.15m \times 0.35m \times 10m) = 0.525\gamma_B$$

$$F_B = \gamma_{H_2O} V_{submerged} = \gamma_{H_2O}(0.15m \times 0.35m \times 8m) = 0.420\gamma_{H_2O}$$

thus

$$0.525\gamma_B\left(\frac{10m}{2}\right) = 0.420\gamma_{H_2O}\left(\frac{8m}{2}\right)$$

$$\gamma_B = 6.27\ kN/m^3$$

Tension in rope ⇒ In addition to moments $\Sigma F_{vertical} = 0$

thus

$$T = F_B - W$$

$$= 0.420\gamma_{H_2O} - 0.525\gamma_B$$

$$T = (0.420m^3)(9.8\ \tfrac{kN}{m^3}) - (0.525m^3)(6.27\ \tfrac{kN}{m^3})$$

$$T = 824\ N$$

## Pressure Distribution in Rigid Body Motion:

Rigid Body motion ⇒ distance between any two particles remains fixed ⇒ NO SHEAR

2 case:

A) uniform linear acceleration

B) rigid body rotation

Without shear there are no viscous forces

In absence of $F_v$ Fundamental Eqn of motion for a fluid element is

$$\vec{\nabla} p = \rho(\vec{g} - \vec{a})$$

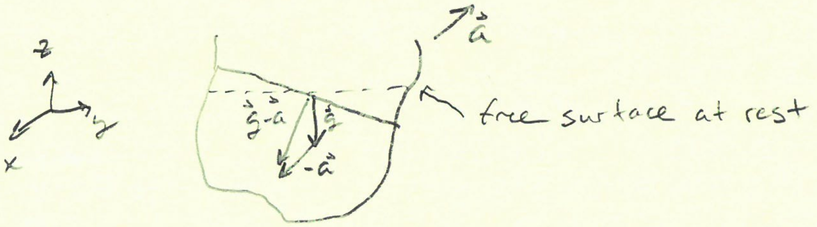
derived in Section 2.2 Eqn. 2.2

Note sign which way is acceleration

component form

$$\frac{\partial p}{\partial x} = \rho a_x \qquad \frac{\partial p}{\partial y} = \rho a_y \qquad \frac{\partial p}{\partial z} = \rho(g - a_z)$$

**Case A** uniform linear acceleration



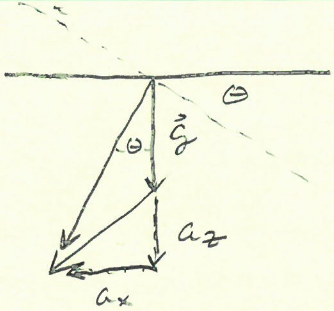
free surface at rest

Fluid acts as if it were under the influence of a new gravity vector $\vec{g}^* = \vec{g} - \vec{a}$

$$\nabla P = \rho \vec{g}^*$$

pressure increases most rapidly in the $\vec{g}^*$ direction

Surfaces of constant pressure are perpendicular to $\vec{g}^*$ direction



θ = angle of inclination of lines of constant pressure

$$\theta = \tan^{-1}\left(\frac{a_x}{g + a_z}\right)$$

since the free surface is a surface of constant pressure the free surface is inclined at an angle θ

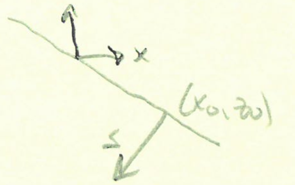
Let $\hat{g}^*$ be a unit vector in the $\vec{g}^*$ direction

$$\hat{g}^* = \frac{\vec{g}^*}{|\vec{g}^*|} = \frac{-\hat{\imath}\, a_x - \hat{k}\,(g + a_z)}{\sqrt{a_x^2 + (g+a_z)^2}}$$

Let s be the coordinate in $g^*$ direction then the pressure distribution is governed by

$$\frac{dP}{ds} = \rho G \qquad \text{where } G = \sqrt{a_x^2 + (g+a_z)^2}$$

for an incompressible liquid $P(s) = P_a + \rho G s$



$$x = x_0 - s\cos\theta \qquad \Rightarrow \qquad s = \frac{x_0 - x}{\cos\theta}$$

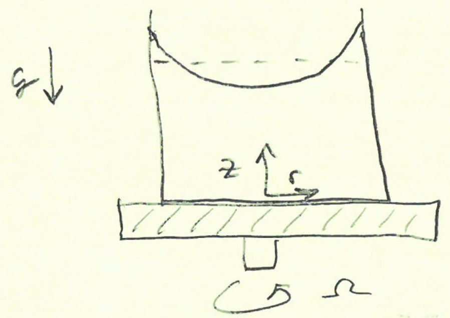
$$z = z_0 - s\sin\theta \qquad\qquad s = \frac{z - z_0}{\sin\theta}$$

$$s = \sqrt{(x-x_0)^2 + (z-z_0)^2}$$

So basically we can reorient ourselves with a new coordinate to describe our system

## Case b) Rigid Body Rotation

Consider a liquid-filled container on a turntable



unit vectors are $\hat{k}$ ; $\hat{i}_r$

$$\vec{a} = -\hat{i}_r r\Omega^2$$

$$\nabla P = \rho(\vec{g} - \vec{a})$$

$$\hat{i}_r \frac{\partial P}{\partial r} + \hat{k}\frac{\partial P}{\partial z} = \rho\left(-\hat{k}g + \hat{i}_r r\Omega^2\right)$$

separating into components

$$\frac{\partial P}{\partial r} = \rho r\Omega^2 \qquad (1)$$

$$\frac{\partial P}{\partial z} = -\rho g \qquad (2)$$

How do we solve this?

"special" solution technique

- integrate eqn (1) with respect to r only

$$P = \int \rho r\Omega^2 dr + f(z) \qquad (3)$$

Eqn (3) satisfies Eqn (1) → plug (3) into (1) to prove

evaluate $\int \rho r\Omega^2 dr$

$$P = \frac{1}{2}\rho r^2\Omega^2 + c_1 + f(z) \qquad (4)$$

take $\frac{\partial}{\partial z}$ of (3)

$$\frac{\partial p}{\partial z} = 0 + f(z)$$

but

$$\frac{\partial p}{\partial z} = -\rho g \qquad \text{so} \quad f(z) = -\rho g$$

integrate w.r.t. z

$$p(z) = -\rho g z + C_2 \qquad (5)$$

Combine Eqs (4) & (5)

$$P = \frac{1}{2}\rho r^2 \Omega^2 - \rho g z + C_3 \qquad \text{where } C_3 = C_1 + C_2$$

let $p = P_0$ at $(r, z) = (0, 0)$

$$\Rightarrow \quad P - P_0 = {}^1\!/_2\, \rho r^2 \Omega^2 - \rho g z \qquad (6)$$

So what does this pressure field look like?

what is the equation for lines of constant pressure

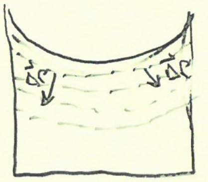
solve Eqn (6) for $z = F(p, r)$

$$\Rightarrow \quad z = \frac{P_0 - P}{\rho g} + \frac{r^2 \Omega^2}{2g}$$

Shape of a constant pressure line say $P = P_1$

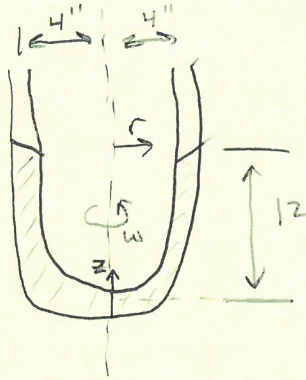
$$z = \frac{P_0 - P_1}{\rho g} + \frac{r^2 \Omega^2}{2g} = a + b r^2$$

lines of constant P have a parabolic shape



Note $\nabla P$ is not a constant

Example: M40 2.98



U tube partially filled with water & rotates around center axis. What's the angular velocity that will cause the water to start to vaporize at the bottom of the tube

the eqn $P = \frac{1}{2}\rho r^2 \omega^2 - \gamma z + C$ applies

fix coordinate system and evaluate C

work in absolute pressure & find ω such that P at r=0 z=0 is equal to the vapor pressure

Find C

$$P_0 = \frac{1}{2}\rho R^2 \omega^2 - \gamma h + C \Rightarrow C = P_0 - \frac{1}{2}\rho R^2 \omega^2 + \gamma h$$

$P_0 = 1$ atm R & h are known

thus

$$P = \frac{1}{2}\rho r^2 \omega^2 - \gamma z + P_0 + \gamma h - \frac{1}{2}\rho R^2 \omega^2$$

at r=0 & z=0 $P = P_V$ vapor pressure

$$P_V = P_0 - \frac{1}{2}\rho R^2 \omega^2 + \gamma h$$

solve for ω

$$\omega = \sqrt{\frac{2(\gamma h + P_0 - P_V)}{\rho R^2}}$$

with $P_V = 0.256$ psia roughly room T

$\omega = 141$ rad/s

## Fluids in motion ⇒ Fluid kinematics

Static fluids do not move because there are no viscous stresses and the pressure gradient is exactly balanced by the acceleration of gravity ($\nabla P = \rho \vec{g}$)

Fluids move in response to viscous stresses & imbalances in normal stresses. (Note: the normal stress is primarily due to pressure)
but not always ⇒ non-Newtonian fluids.

Fluid kinematics is the description of fluid motion without regard to the forces that cause the motion
- its a description of the fluid flow

In fluid motion we should expect
non-uniform velocity fields
non-uniform pressure fields

The velocity at an arbitrary pt in Cartesian coordinates can be written

$$\vec{V} = u(x,y,z,t)\hat{e}_x + v(x,y,z,t)\hat{e}_y + w(x,y,z,t)\hat{e}_z$$

or

$$\vec{V} = u(x,y,z,t)\hat{i} + v(x,y,z,t)\hat{j} + w(x,y,z,t)\hat{k}$$

alternately in component form

$$u = u_x \quad v = u_y \quad w = u_z$$

Magnitude of velocity

$$V = |\vec{V}| = \sqrt{u^2 + v^2 + w^2}$$

The velocity vector is independent of the coordinate system used to define its components

In general velocity & pressure fields can be fully 3-D & unsteady

- consider weather on a windy day

A flow is 3-D when variations in u & P in all 3 coordinate directions affect quantities of engineering significance

- drag

Certain flow situations can be simplified to 2-D or 1-D if variation along a given coordinate are negligible

ME320 primarily focuses on 1-D

↖ Lower D easier to solve

Note: in our viscometer example we assumed/were told the velocity was linear $V = \frac{U}{b}y$. This is a specific simple flow scenario. Velocity fields are rarely linear. Usually highly complex.

So how do we describe ~~fluid~~ fluid motion.

1st remember we can consider or do consider fluids to be a continuum. Any point in a fluid actually consists of many many molecules.

We often refer to motion of fluid particles when discussing fluid velocity & acceleration

So how do we define fluid motion? & other properties for that matter

what framework is best?

- do we assign quantities to a fixed pt in space?

or - do we assign quantities to a fixed parcel of fluid?

2 views of fluid motion - complementary, phsics of fluid phenomena must be identical

(reference frame irrelevant)

- Langragian
  - describes changes that occur as you travel along with a fluid particle
  - description focuses on individual fluid particles
- Eulerian
  - describes changes as they occur at a fixed pt in space
  - description focuses on field equations

to show this

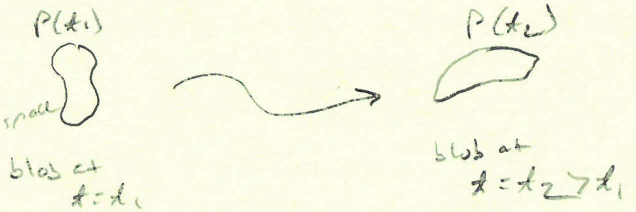
Consider how the pressure of a fluid blob changes as the blob moves

Lagrangian

$x_p, y_p, z_p$ particle position in space

velocity components

$\frac{dx_p}{dt}, \frac{dy_p}{dt}, \frac{dz_p}{dt}$

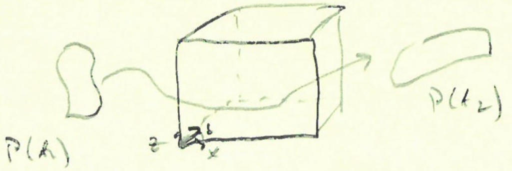


P, T, V, ρ vary with time/path

Note: a blob with fixed mass will deform as it moves
⇓
difficult to analyze

Eulerian

same blob passes through a fixed control volume



in this case the pressure field

$P(x,y,z,t)$ defines the values of pressure the blob experiences as it moves through space

Consider the flow of a river which gets narrower & faster as you move downstream.

Eulerian view : flow is steady at each pt on the river

Langrangian view: flow accelerates as you move with the fluid particles

obvious that these views lead to different mathematical derivatives ( i.e. partial & material derivatives)

- however each view is useful for different applications

- fixed measurement systems are Eulerian - river data / weather
- chemical modeling is Langrangian - how do chemicals evolve down a flow
- fluid dynamics generally use Eulerian - apply velocity field theory
- theoretical mechanics - Langrangian relative motion between particles

Everyone has used these descriptions / views / frameworks in everyday life - determined by natural convenience

- when you travel your iterary uses Langrangian view
- Arrivals & departures on airport monitors are Eulerian
- your friend who comes to the airport to pick you up uses the Eulerian view

Eulerian description is more pratical for solving flow problems

Langrangian used to motivate intuitive understanding of fluid behavior

Eulerian Framework used almost exclusively in engineering analysis of fluids. Conservation laws of mass, energy, & momentum are applied to a fluid entering & leaving a control volume

However,
Langrangian framework is used in M&O to derive the Bernoulli Eqn.

- we will follow this analysis shortly

## Kinematics of Stagnation Point flow

Show cross junction μ-fluidic chip. - describe flow system/use

By introducing opposing flows we generate a stagnation pt

- show flow video ⇒ point out stagnation pt, streamline

we can actually measure the velocity with μ-PIV

- show velocity

in our measurement plane ⇒ flow resembles planar extensional flow

$$V = \dot{\gamma}(\hat{i}x - jy)$$

with definition $V = \hat{i}u + \hat{j}v + \hat{k}w$

$$u = \dot{\gamma}x \qquad v = -\dot{\gamma}y \qquad w = 0 \leftarrow \text{2-D flow no z flow}$$

Note in the X-junct. $\dot{\gamma} = \gamma(z)$

In this flow field, the fluid accelerates near the stagnation point

$$a = \frac{DV}{Dt} = \frac{\partial V}{\partial t} + u\frac{\partial V}{\partial x} + v\frac{\partial V}{\partial y} + w\frac{\partial V}{\partial z}$$ substantial derivative

or in component terms

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + \vec{V}\cdot\nabla(\ )$$

↑ reconciles the Langrangian & Eulerian perspectives

$$a_x = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = \frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

substituting in the velocity $u = u(x)$ $v = v(y)$ $w=0$

the acceleration components for the 2-D extensional flow

$$a_x = u\frac{\partial u}{\partial x} \qquad a_y = v\frac{\partial v}{\partial y} \qquad a_z = 0$$

with $u = \dot{\gamma}x$ ad $v = -\dot{\gamma}y$ evaluate.

$$a_x = \dot{\gamma}^2 x \qquad a_y = \dot{\gamma}^2 y$$

What does this look like

show plot of acceleration field

Observations: Simplified picture of flow only near stagnation pt

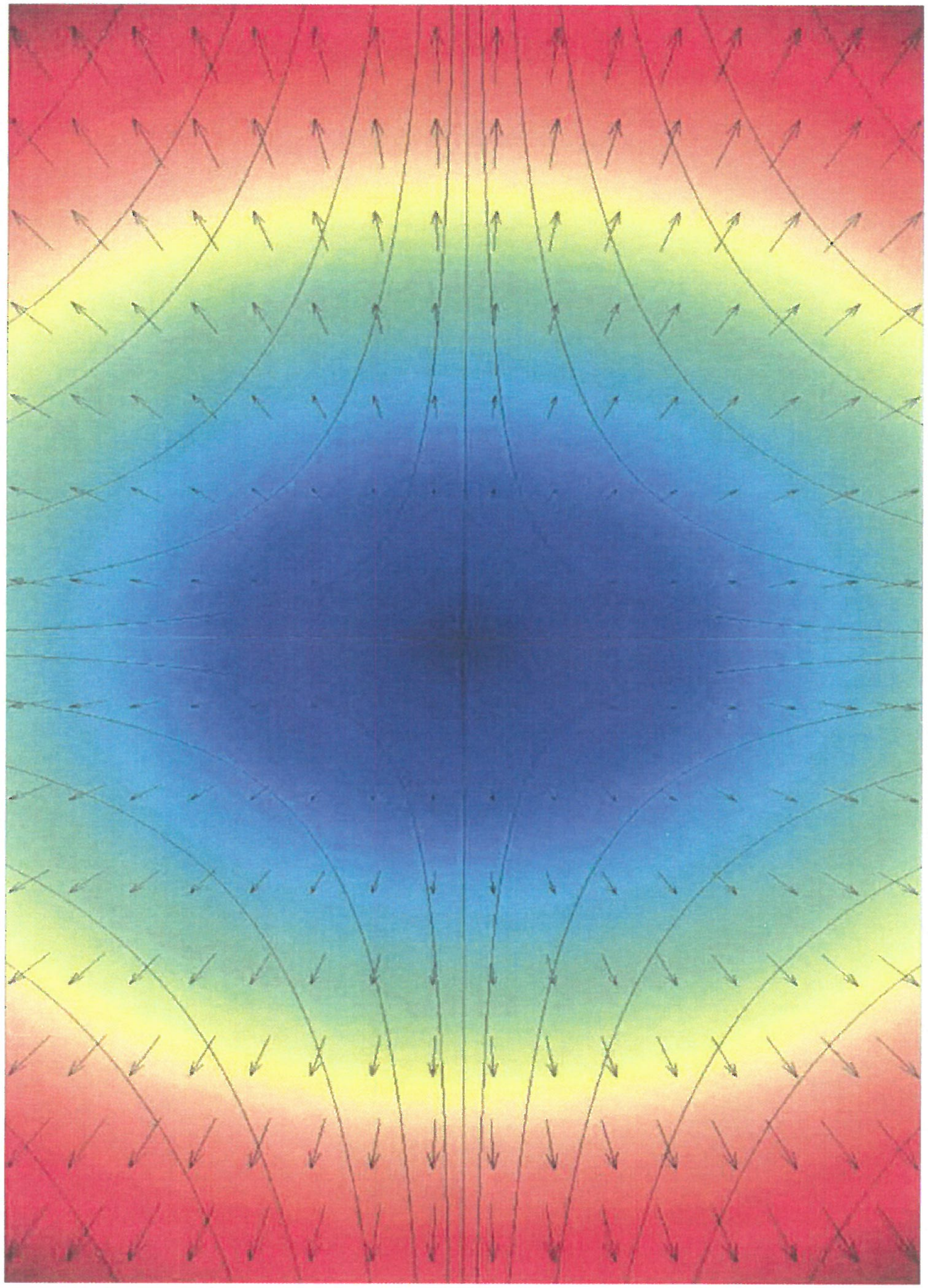
$$|V| = \dot{\gamma}\sqrt{x^2+y^2}$$ velocity increases indefinitely with distance from stag. pt.

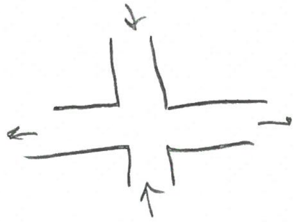
acceleration also increases

$$|a| = \dot{\gamma}^2\sqrt{x^2+y^2}$$

Not realistic at large distances

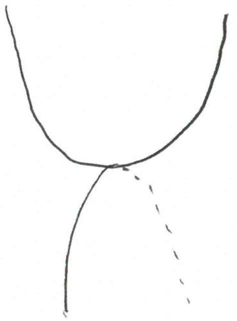
# Acceleration Field

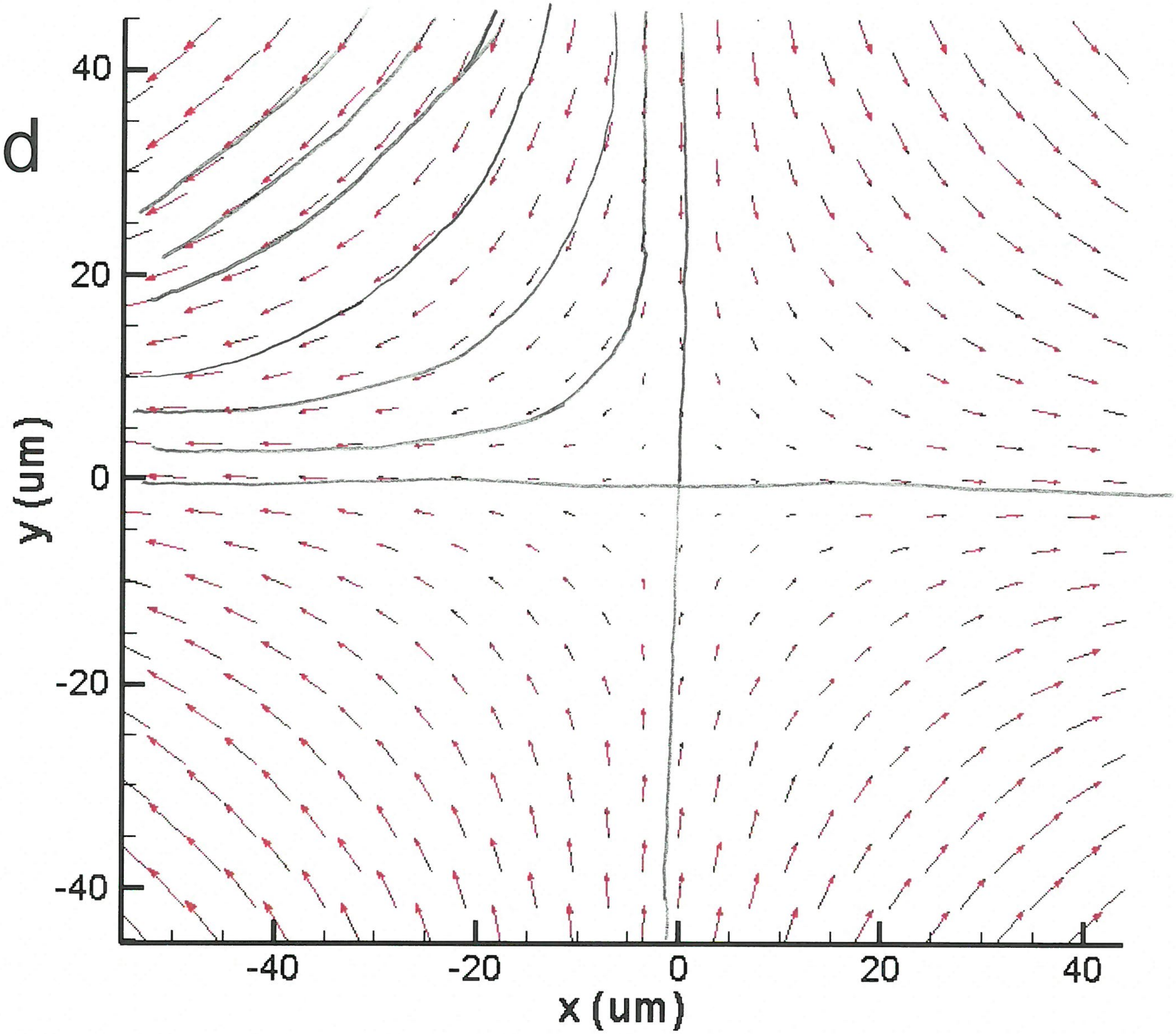




# Velocity Field

lines are streamlines



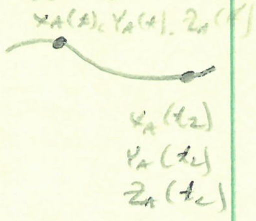


If we construct a curve from local tangents to the velocity vector we generate

Streamlines (draw streamlines on piv plot)

For steady flow $V=V(x,y,z)$ velocity at specific pt does not change

streamlines are the same as pathlines (trajectory of an individual particle)



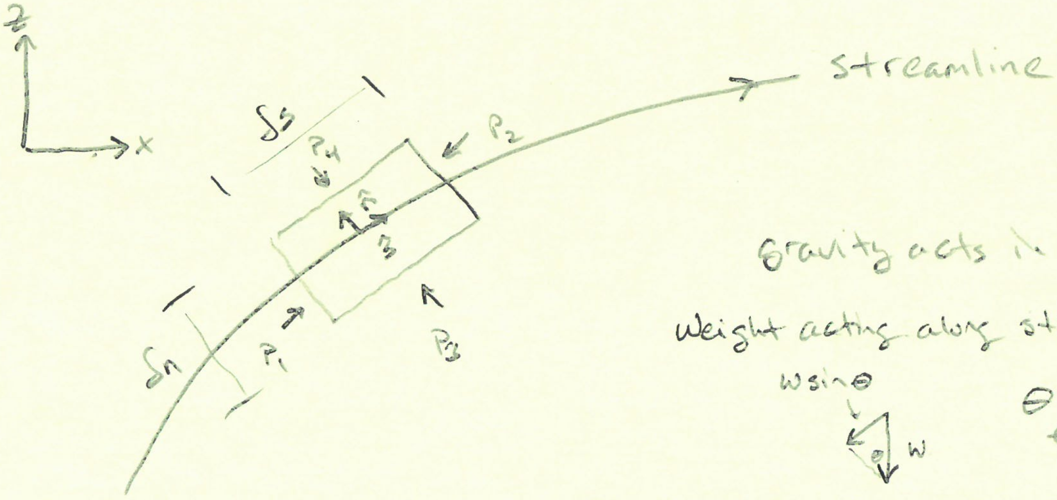
and streakline (collection of particles released at same pt at sequential times)

ex. - smoke streak

Let's focus on streamlines

## Conservation of Momentum Along a Streamline

* Assume no viscous effects, 2-D Flow *



Gravity acts in neg z

Weight acting along streamline

θ angle streamli from vertical

Note pressure subscripts are temporary labels

Force balance along a streamline

(We can take a force balance in any particular direction)

$$\sum \delta F_s = \delta m \, a_s$$

$$\underbrace{P_1 \, \delta n \, \delta y - P_2 \, \delta n \, \delta y}_{\text{pressure}} - \underbrace{\delta w \sin\theta}_{\text{differential weight}} = \delta m \, a_s$$

note $W_s = w \sin\theta$

$$\delta w_s = \delta w \sin\theta$$

but

$$\delta m = \rho \, \delta \forall = \rho \, \delta n \, \delta s \, \delta y$$

$$\delta w = \delta m \, g = \rho \, \delta n \, \delta s \, \delta y \, g$$

substitute.

$$(P_1 - P_2) \, \delta n \, \delta y - \rho \, \delta n \, \delta s \, \delta y \, g \sin\theta = \rho \, \delta n \, \delta s \, \delta y \, a_s$$

divide by $\delta n \, \delta s \, \delta y$

$$\frac{P_1 - P_2}{\delta s} - \rho g \sin\theta = \rho \, a_s \qquad (1)$$

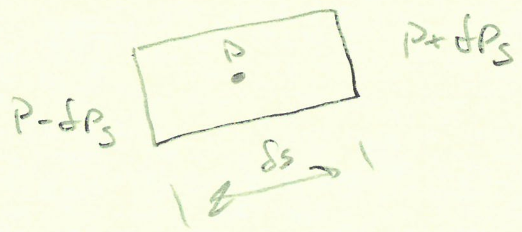
Can we express $P_1 - P_2$ in terms of a pressure gradient?

yes. Use Taylor Series Expansion

Let pressure at center be $P$

pressure change across the element is $2 \, \delta P_s$

[Factor of 2 arises from choice of pressure at cente distance to face $1/2 \, \delta s$]



$$P_1 = P - \delta P_s = P - \frac{\delta s}{2}\frac{\partial P}{\partial s}\bigg|_{center} + \ldots \text{H.O.T}$$

$$P_2 = P + \delta P_s = P + \frac{\delta s}{2}\frac{\partial P}{\partial s}\bigg|_{at\ center} + \text{H.O.T}$$

$$P_1 - P_2 = (P - \delta P_s) - (P + \delta P_s)$$

$$= P - \frac{\delta s}{2}\frac{\partial P}{\partial s}\bigg|_{at\ center} - \left(P + \frac{\delta s}{2}\frac{\partial P}{\partial s}\bigg|_{at\ center}\right)$$

$$= -\delta s \frac{\partial P}{\partial s}\bigg|_{at\ center}$$

Substituting into (1)

$$-\frac{\partial P}{\partial s} - \rho g \sin\theta = \rho a_s \qquad (2)$$

Acceleration along streamline

$$a = V\frac{\partial V}{\partial s}$$

recall

$$dV = \frac{\partial V}{\partial s}\,ds$$

$$a = \frac{dV}{dt} = \frac{\partial V}{\partial s}\frac{ds}{dt} = V\frac{\partial V}{\partial s}$$

substituting.

$$-\frac{\partial P}{\partial s} - \rho g \sin\theta = \rho V \frac{\partial V}{\partial s} \qquad (3)$$

This eqn says: pressure imbalance (pressure gradient) and the fluid weight cause fluid particles to accelerate $V\frac{\partial V}{\partial s}$

Furthermore, at each pt in space (when viscous forces may be neglected) the pressure gradient (component of), the fluid weight and the acceleration are in balance

Eqn (3) is true for any inviscid fluid or more practically any flow in which viscous forces are negligible compared to pressure forces & the gravitational force.

Bernoulli Eqn

named after Daniel Bernoulli

Eqn obtained by integrating (3) along a streamline

Restricting ourselves to measuring finite changes along a streamline leads to several mathematical simplifications

recall

$$dV = \frac{\partial V}{\partial s} ds + \frac{\partial V}{\partial n} dn \qquad dP = \frac{\partial P}{\partial s} ds + \frac{\partial P}{\partial n} dn$$

With integration path restricted to streamline $dn = 0$

so

$$dV = \frac{\partial v}{\partial s} ds \quad ; \quad dp = \frac{\partial p}{\partial s} ds$$

or

$$\left.\frac{dV}{ds} = \frac{\partial V}{\partial s} \quad ; \quad \frac{dp}{ds} = \frac{\partial p}{\partial s}\right\} \text{along streamline}$$

thus (3) becomes

$$-\frac{dp}{ds} - \rho g \sin\theta = \rho V \frac{dV}{ds} \qquad (4)$$

only valid along a streamline

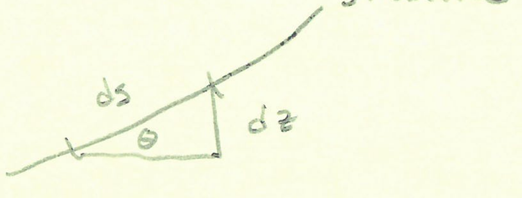
Further manipulation

$$V\frac{dv}{ds} = \frac{1}{2}\frac{d}{ds}\left(v^2\right) \qquad \frac{d}{ds}\left(v^2\right) = 2v\frac{dV}{ds}$$

so

$$-\frac{dp}{ds} - \rho g \sin\theta = \frac{1}{2}\rho\frac{d(v^2)}{ds}$$

also



From geometry

$$\frac{dz}{ds} = \sin\theta$$

substitute into (4)

$$-\frac{dp}{ds} - \rho g\frac{dz}{ds} = \frac{\rho}{2}\frac{d}{ds}\left(v^2\right)$$

multiply by ds ; integrate

$$-\int dp - \int \rho g\,dz = \frac{1}{2}\int \rho\, d(v^2)$$

or

$$-\int \frac{dp}{\rho} - \int g\,dz = \frac{1}{2}\int d(v^2)$$

$$-\int \frac{dp}{\rho} - gz = 1/2\, V^2 + C_1$$

if fluid is incompressible $\rho = \text{constant}$

and

$$\boxed{P + \frac{1}{2}\rho V^2 + \rho g z = C}$$

Bernoulli Eqn

Valid only along a streamline

* for an incompressible, inviscid fluid with no losses. *

Equivalent form

LAX commuter terminal →

$$\underbrace{\frac{P}{\gamma}}_{\text{static pressure head}} + \underbrace{\frac{v^2}{2g}}_{\text{velocity head}} + z = \text{Constant}$$

(different constant from above)

^ dynamic pressure head

elevation head ← hydrostatic pressure head

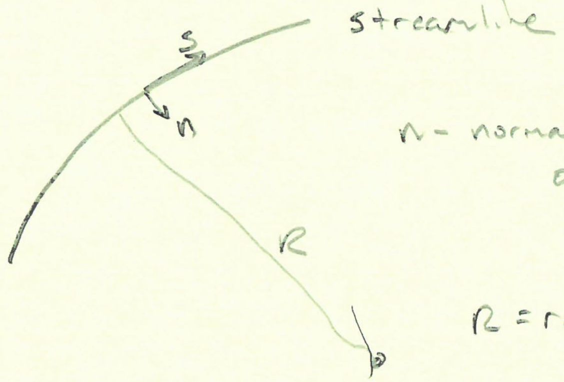
If we examine 2 pts along a streamline

$$P_1 + 1/2\,\rho V_1^2 + gz_1 = P_2 + 1/2\,\rho V_2^2 + gz_2$$

or

$$(P_1 - P_2) + 1/2\,\rho(V_1^2 - V_2^2) + g(z_1 - z_2) = 0$$

# Momentum balance Normal to a streamline



n - normal pointing towards center of curvature

R = radius of curvature

acceleration of fluid particle

$$\vec{a} = \frac{d\vec{V}}{dt} \qquad a = \underset{\substack{\uparrow \\ \text{along streamline}}}{a_s} + \underset{\substack{\uparrow \\ \text{normal to streamline}}}{a_n}$$

$$= \frac{1}{dt}\left[\left(\frac{\partial V}{\partial s}\right)ds + \left(\frac{\partial V}{\partial n}\right)dn\right]$$

$$= \left(\frac{\partial V}{\partial s}\right)\frac{ds}{dt} + \left(\frac{\partial V}{\partial n}\right)\frac{dn}{dt}$$

$$= \underbrace{V\frac{\partial V}{\partial s}}_{\text{along streamline}} + \underbrace{\left(\frac{\partial V}{\partial n}\right)\left(\frac{dn}{dt}\right)}_{\text{normal} = a_n}$$

$$a_s \quad + a_n$$

$a_n$ = acceleration around a curve centripetal acceleration $\quad a_n = \frac{V^2}{R}$

$$\Sigma F_n = m a_n$$

following procedure used last time (see pgs 101-103)

we get

$$\int \frac{dP}{\rho} + \int \frac{V^2\,dn}{R} + gz = \text{constant}$$

for incompressible fluids

$$P + \rho \underbrace{\int \frac{V^2 dn}{R}} + \gamma z = \text{constant in n-direction}$$

to integrate need to know function form

limiting case $R = \infty$

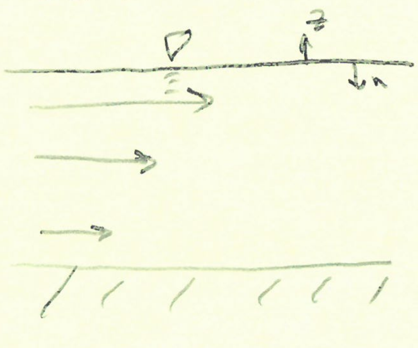
if radius of curvature $R=\infty$ then streamlines are parallel & straightlines

and $\frac{V^2 dn}{R} = 0$ always

so for straight streamlines

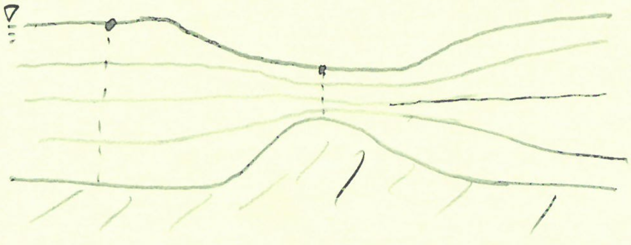
$$\frac{P(n)}{\rho} + gz = \text{constant}$$

Examples: Flow in a river

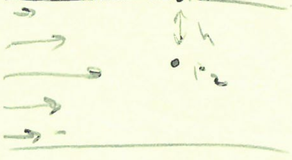


$P(n) = -\gamma z$ constant = 0

good along river
local approximation
-streamlines appear straight



flow in a pipe



$P_2 = P_1 + \gamma h$

So we have 2 Eqns.

$$\frac{P}{\gamma} + \frac{V^2}{2g} + z = \text{constant} \quad \text{along a streamline}$$

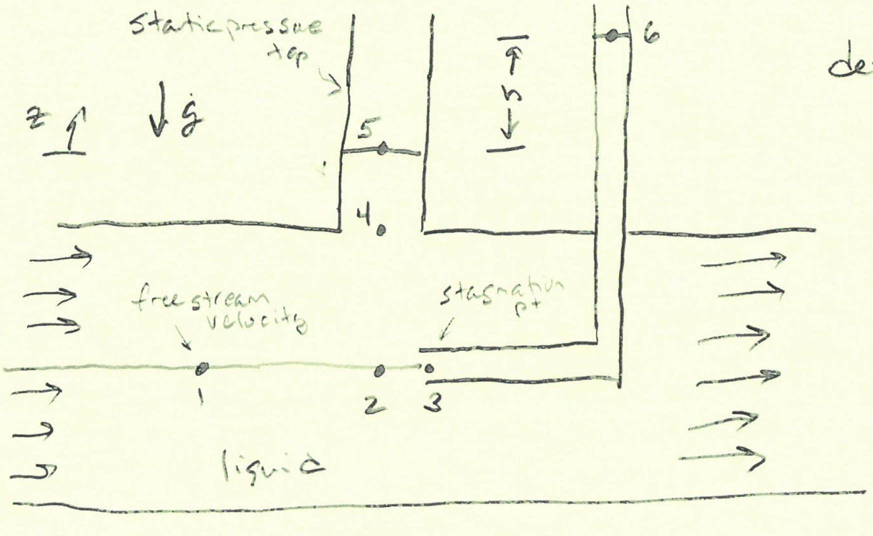
$$\frac{P}{\gamma} + z = \text{constant (different) normal to a streamline}$$

* valid - steady, incompressible, inviscid flow with no losses or work on the fluid. *

Not always valid - improper use can lead to completely erroneous resuts.

Example of use of Bernoulli Eqn.

Velocity measurements in a horizontal duct - Piezometer



describe flow scenario

What's the velocity in the duct

- relate velocity to pressure measurement

Along a streamline

$$P_1 + \frac{\rho V_1^2}{2} + \gamma z_1 = P_2 + \frac{\rho V_2^2}{2} + \gamma z_2 = P_3 + \frac{\rho V_3^2}{2} + \gamma z_3$$

but $V_3 = 0$ ; $z_3 = z_2$

thus $$P_3 = P_2 + \frac{\rho V_2^2}{2} \implies P_3 - P_2 = \frac{\rho V_2^2}{2} \quad (1)$$

Applying momentum equation normal to a streamline

$$P_2 + \gamma z_2 = P_4 + \gamma z_4 = P_5 + \gamma z_5$$
$$\Downarrow$$
$$P_2 = P_5 + \gamma(z_5 - z_2) \quad (2)$$

Applying manometer eqn to piezometer

$$P_3 = P_6 + \gamma(z_6 - z_3) \quad (3)$$

subtract Eqn (2) from Eqn (3)

$$P_3 - P_2 = P_6 - P_5 + \gamma h$$

but $P_5 = P_6$

$$P_3 - P_2 = \gamma h$$

with Eqn (1) & Eqn (4) eliminate $P_3 - P_2$

so $$\frac{\rho V_2^2}{2} = \gamma h$$

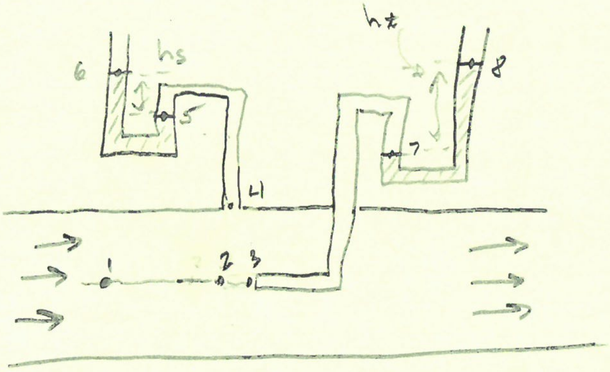
solve for $V_2$

$$V = \sqrt{\frac{2\gamma h}{\rho}} = \sqrt{2gh}$$

since fluid in piezometer & static pressure tap is the same

# Gas flowing horizontally - Basis of a Pitot tube

Pressure measurement is made with U-tube manometers



Apply Bernoulli along streamline $P_3 - P_2 = \rho \frac{V_2^2}{2}$ (7)

Momentum eqn normal to streamline

$$P_2 + \gamma z_2 = P_4 + \gamma z_4 \Rightarrow P_2 = P_4 + \gamma(z_4 - z_1) \quad (8)$$

Apply manometer eqns to U-tube manometer eqns to left side

$$P_4 = P_6 + \gamma_m h_s + \gamma(z_5 - z_4) \quad (9)$$

substitute (9) into (8)

$$P_2 = P_6 + \gamma_m h_s + \gamma(z_5 - z_4) + \gamma(z_4 - z_2)$$

$$= P_6 + \gamma_m h_s + \gamma(z_5 - z_2) \quad (10)$$

apply hydrostatic eqn to the stagnation tube

$$P_3 = P_8 + \gamma_m h_t + \gamma(z_7 - z_3) \quad (11)$$

subtract Eqn (10) from Eqn (11)

$$P_3 - P_2 = \gamma_m(h_t - h_s) + \gamma(z_7 - z_5)$$

with $P_6 = P_8$ $z_2 = z_3$

& assuming

$\gamma(z_7 - z_5) <<< \gamma_m(h_t - h_s)$ as

as $\gamma < \gamma_m$ and $(z_7 - z_5)$ is small

Eliminate $P_3 - P_2$ with Eqn (7) & Eqn (12)

$$\rho \frac{V_2^2}{2} = \gamma_m (h_t - h_s)$$

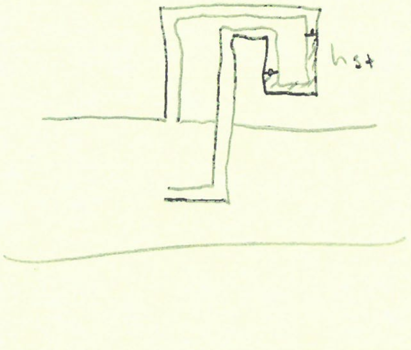
rearranging

$$V_2 = \sqrt{\frac{2\gamma_m (h_t - h_s)}{\rho}}$$

Note $\rho_m \neq \rho$ so we can not cancel $\rho_m$ in numerator with $\rho$ in denominator

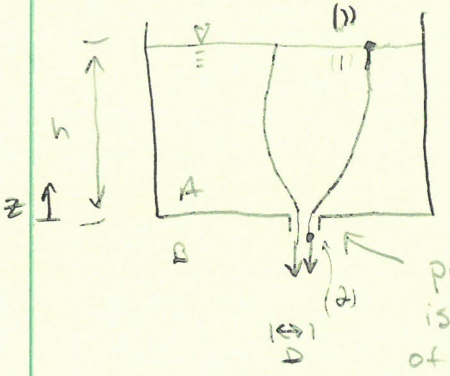
fluid in manometer different than fluid in duct ←

For simplicity we could combine output of static tube with output of stagnation pt.



repeat calculation $V_2 = \sqrt{\frac{2\gamma_m h_{st}}{\rho}}$

# Free jets



what's the velocity of the jet leaving the tank?

pressure at jet exit is equal to pressure of surroundings ⇒ Lack of curvature at jet exit requires pressure in fluid A equal surroundings fluid B

Applying Bernoulli Eqn along streamline (1) → (2)

$$\cancel{\frac{P_1}{\gamma}} + \frac{V_1^2}{2g} + z_1 = \cancel{\frac{P_2}{\gamma}} + \frac{V_2^2}{2g} + z_2$$

but

$P_1 = P_2$ so

$$\frac{V_2^2 - V_1^2}{2g} = z_1 - z_2 = h$$

assume $V_1 << V_2$ thus

diameter of tank > jet diameter

$$V_2 = \sqrt{2gh}$$

How long until the tank drains?

Mass flow out of tank has to match the decrease of mass in the tank (top surface moves downward)

$$\rho V_2 A_0 = \rho A_t \left(-\frac{dh}{dt}\right)$$

cross section area of drain

area of tank

substitute velocity in & rearrange

$$\frac{dh}{dt} = -\frac{A_o}{A_t}\sqrt{2gh}$$

separate and integrate

let $h = h_o$ when $t = 0$

$$\int_{h_o}^{h} \frac{dh}{h^{1/2}} = \int_{0}^{t} -\frac{A_o}{A_t}\sqrt{2g}\; dt$$

$$h^{1/2}(t) = \sqrt{h_o} - \frac{A_o}{A_t}\sqrt{\frac{g}{2}}\, t$$

tank is drained when $h = 0$ so $t_{max} = \frac{A_t}{A_o}\sqrt{\frac{2h_o}{g}}$

· What are the assumptions of this calculation?

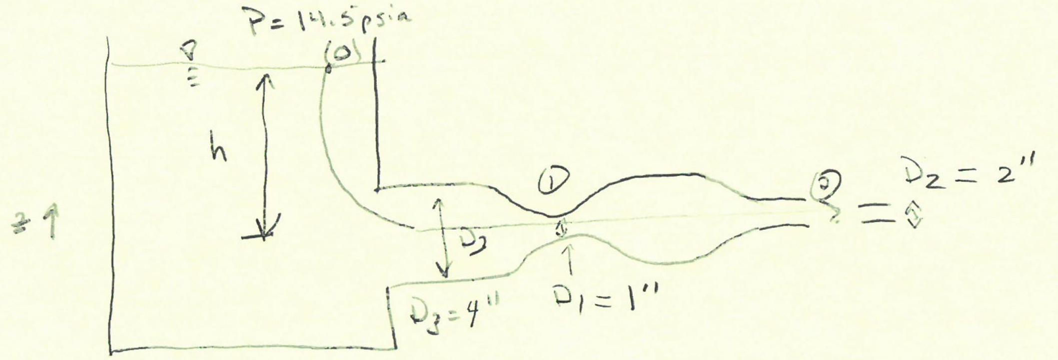
steady, incompressible, inviscid, no losses or work done

note ↑ psuedo-steady height doesn't change too rapidly

Alternate measurement- mass flow rate calculated from pressure

Example:

Water flows from a tank through a variable cross section pipe



if viscous effects are neglected at what height (h) will cavitation occur in the pipe - Vapor pressure of $H_2O$ 1.60 psia when do we have cavitation?

Do we ever expect cavitation? it's possible anytime flow accelerates and V gets high enough

Where should it occur? at pt 1

insert previous page.

V₁ highest thus if $\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = $ const means $P_1$ is lowest.

Assumptions: steady - h doesn't change much with time $V_0 = 0$

incompressible

no viscous effects or other losses/work

Apply Bernoulli eqn. from ⓪, to ②

$$\frac{P_0}{\gamma} + \frac{V_0^2}{2g} + z_0 = \frac{P_1}{\gamma} + \frac{V_1^2}{2g} + z_1$$

solve for h as a function of $P_1$

$$h = \frac{P_1 - P_0}{\gamma} + \frac{V_1^2}{2g} \qquad (1)$$

use absolute P as $P_v$ is always in absolute.

known

what's $V_1$?

we need to find $V_1$ in terms of things we know

Apply Bernoulli between ⓪ and ②

$$\underbrace{\frac{P_0}{\gamma} + \underset{0}{\frac{V_0^2}{2g}} + z_0 = \frac{P_2}{\gamma}}_{0} + \frac{V_2^2}{2g} + z_2$$

$P_2 = P_0 = atm$

thus

$$h = \frac{V_2^2}{2g} \qquad (2)$$

so h can be expressed in terms of $V_1$ & $V_2$

how do we relate $V_1$ to $V_2$? ⟹ Conservation of mass