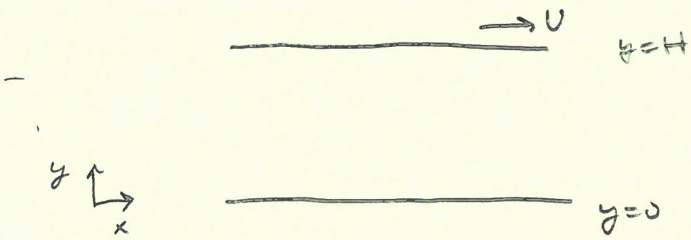
Example

Plane Couette Flow –



previously we've stated that the velocity in this gap is linear

Why is it? we took it as given.

Find the fluid velocity between the gap.

Assume we have a Newtonian incompressible fluid. Steady flow

use
Navier Stokes Eqn & continuity

Continuity (incompressible fluid)

$$\underbrace{\frac{\partial \rho}{\partial t}}_{0} + \rho \frac{\partial}{\partial x} v_x + \rho \frac{\partial v_y}{\partial y} + \rho \frac{\partial v_z}{\partial z} = 0$$

infinite direction no variation in z

steady $\qquad v_y = \text{constant} = 0 \qquad v_z = \text{constant} = 0$

boundary condition

thus $\qquad \dfrac{\partial v_x}{\partial x} = 0$

Navier Stokes

Equivalent pressure
$P = p + \rho g x$

x component

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right)$$

0 s.s $\quad$ 0 continuity $\quad$ $v_y = 0$ $\quad$ $v_z = 0$ $\quad$ no imposed pressure gradient $\quad$ 0 $\frac{\partial v_x}{\partial x} = 0$ $\quad$ no z dependence

thus

$$\frac{\partial^2 v_x}{\partial y^2} = 0$$

y component

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right)$$

entire eqn is zero. If we didn't know $P(y)$ this eqn tells us $P \neq P(y)$

$\frac{dP}{dy} = 0 \Rightarrow P(y) = \text{constant}$

z component → replace v_y with v_z same result entire eqn is zero

thus fluid motion is governed by

$$\frac{\partial^2 v_x}{\partial y^2} = 0 \quad \text{but since } v_x = v_x(y) \quad \frac{d^2 v_x}{dy^2} = 0$$

2nd order ordinary differential eqn.

Need 2 B.Cs. to solve the eqn.

B.C. 1 at $y = 0$ $v_x = 0$

B.C. 2 at $y = H$ $v_x = U$

<u>Solve.</u>

$$\frac{d^2 v_x}{dy^2} = 0 \rightarrow \frac{d}{dy}\left(\frac{dv_x}{dy}\right) = 0$$

derivative of something equals zero something is a constant

separate & integrate

$$\frac{dv_x}{dy} = C_1$$

separate & integrate. $v_x = C_1 y + C_2$

apply B.Cs

$y = 0$ $v_x = 0 \Rightarrow C_2 = 0$

$y = H$ $v_x = U \Rightarrow C_1 = \frac{U}{H}$ thus $\boxed{v_x = \frac{U}{H} y}$

linear velocity

with the velocity profile we can calculate the shear stress

$$\tau_{xy} = \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) = \mu \frac{\partial v_x}{\partial y} = \frac{\mu U}{H}$$

shear rate $\frac{U}{H}$

method to calculate viscosity - plate / plate viscometer

Above example is a simplified case: ie 1-D, infinite parallel plates.

In general N.S. eqns are 2nd order - nonlinear PDE

- analytical solutions of full eqns not possible (rare cases only)
  - 1-D simple problems or neglect terms based on order
- computational solutions theoretically possible but how long do you want to run a simulations - days, weeks, months yrs?

* → Do 1-D examples on pgs 119-124

Need a way to simplify the eqns

<u>Non-dimensional analysis</u>

consider the x component of the N.S. eqn

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right)$$

& continuity

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

What's the order magnitude of each term in each eqn?

→ important terms retained
→ unimportant terms neglected

Need to estimate which terms are important?

- what are the characteristic values?

In our previous example of parallel plates the velocity varies from 0 to U

- zero at the bottom surface
- U at the top surface

So the characteristic velocity would be U

- the velocity varies between 0 and U

What's the characteristic length?

- key dimension the gap between the two plates.
  - other dimensions are infinite

For any problem we can define characteristic lengths, velocity ...
for each direction

what's the characteristic length of this room

If we divide L, V, t by their characteristic values we get dimensionless variables that vary from 0 to 1

Characteristics & Dimensionless Variables

Physical variable	Characteristic	Dimensionless variable
position, x	L	$\tilde{x} = \frac{x}{L}$
velocity, v	$\mathcal{V}$ $\left(\frac{L}{s}\right)$	$\tilde{v} = \frac{v}{\mathcal{V}}$
gradient operator ∇ $\left(\frac{\partial}{\partial x}\right)$	$\frac{1}{L}$	$\tilde{\nabla} = \nabla L$
time	T	$\tilde{t} = \frac{t}{T}$
pressure	Π	$\tilde{P} = P/\Pi$

Continuity eqn:

$$\nabla \cdot V = 0$$

dimensionless (insert $V = \tilde{V}\,\mathcal{V}$ & $\nabla = \frac{\tilde{\nabla}}{L}$)

$$\tilde{\nabla} \cdot \tilde{V} = 0$$ continuity unchanged by non-dimensionalization
no parameters appear

N.S Eqn.

$$\rho \frac{\partial V}{\partial t} + \rho V \cdot \nabla V = -\nabla p + \mu \nabla^2 V$$

substitute in dimensionless variables $V = \tilde{V}\,\mathcal{V}$ $x = \tilde{x} L$ $p = \tilde{p}\,\Pi$ $\nabla = \frac{\tilde{\nabla}}{L}$.

$t = \tilde{t}\,T$

$$\frac{\rho \mathcal{V}}{T} \frac{\partial \tilde{V}}{\partial \tilde{t}} + \left(\frac{\rho \mathcal{V}^2}{L}\right) \tilde{V} \cdot \tilde{\nabla} \tilde{V} = -\left(\frac{\Pi}{L}\right) \tilde{\nabla} \tilde{p} + \left(\frac{\mu \mathcal{V}}{L^2}\right) \tilde{\nabla}^2 \tilde{V}$$

note
$\nabla^2 V = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} V\right)$

divide by $\frac{\mu \mathcal{V}}{L^2}$

$$\left(\frac{\rho L^2}{\mu T}\right) \frac{\partial \tilde{V}}{\partial \tilde{t}} + \left(\frac{\rho \mathcal{V} L}{\mu}\right) \tilde{V} \cdot \tilde{\nabla} \tilde{V} = -\left(\frac{\Pi L}{\mu \mathcal{V}}\right) \tilde{\nabla} \tilde{p} + \tilde{\nabla}^2 \tilde{V}$$

these terms are of order 1

therefore magnitude of each term is governed by the coefficient.

key coefficient.

$$Re = \frac{\rho \mathcal{V} L}{\mu} = \text{ratio of } \frac{\text{inertial forces}}{\text{viscous forces}}$$

with this definition

$$Re\left(\frac{L}{\mathcal{V} T} \frac{\partial \tilde{V}}{\partial \tilde{t}} + \tilde{V} \cdot \tilde{\nabla} \tilde{V}\right) = -\left(\frac{\Pi L}{\mu \mathcal{V}}\right) \tilde{\nabla} \tilde{p} + \tilde{\nabla}^2 \tilde{V}$$

Let Sr = Strouhal # $\equiv \left(\frac{VT}{L}\right)^{-1}$

thus N.S eqn (incompressible fluid)

$$Re\left(Sr^{-1}\frac{\partial \tilde{V}}{\partial \tilde{t}} + \tilde{V}\cdot\tilde{\nabla}\tilde{V}\right) = -\left(\frac{\Pi L}{\mu V}\right)\tilde{\nabla}\tilde{P} + \tilde{\nabla}^2\tilde{V}$$

What's the characteristic pressure Π?

inertially dominated $\Pi = \rho V^2$

dimensionless N.S inertial

$$Re\left(Sr^{-1}\frac{\partial \tilde{V}}{\partial \tilde{t}} + \tilde{V}\cdot\tilde{\nabla}\tilde{V} + \tilde{\nabla}\tilde{P}\right) = \tilde{\nabla}^2\tilde{V}$$

viscous dominated $\Pi = \mu\frac{V}{L}$ ← more common estimate of order of magnitude

dimensionless N.S. viscous dominated

$$Re\left(Sr^{-1}\frac{\partial \tilde{V}}{\partial \tilde{t}} + \tilde{V}\cdot\tilde{\nabla}\tilde{V}\right) = -\tilde{\nabla}\tilde{P} + \tilde{\nabla}^2\tilde{V}$$ ← as a result most common dimensionless form.

Note if there is not an imposed time then the characteristic time is just $\frac{L}{V}$

in this case $Sr \equiv \frac{L}{VT} = 1$

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Ordering Arguments.

dimensionless forms of eqn allow us to make physical based simplifications

— all derivatives are of order unity

hence every term in dimensionless eqn is of order one
+ multiplied by a dimensionless group

Cases

$Sr \gg 1 \Rightarrow \quad Re\, \tilde{V} \cdot \tilde{\nabla} \tilde{V} = -\tilde{\nabla}\tilde{P} + \tilde{\nabla}^2 \tilde{V}$

$Sr \ll 1 \Rightarrow \quad \frac{Re}{Sr} \frac{\partial \tilde{V}}{\partial \tilde{t}} = -\tilde{\nabla}\tilde{P}\ \tilde{\nabla}^2\tilde{V}$

$Sr \gg 1$ & $Re \ll 1$
or
$Sr \ll 1$ & $Re \ll Sr$

$\Rightarrow \quad \tilde{\nabla}^2 \tilde{V} - \tilde{\nabla}\tilde{P} = 0$ creeping flow equations

if $Re \to \infty \quad \frac{1}{Sr} \frac{\partial \tilde{V}}{\partial \tilde{t}} + \tilde{V} \cdot \tilde{\nabla} \tilde{V} + \nabla P = 0$ inviscid fluid Euler Eqn

Note this eqn is singular
highest derivative term neglected
- no slip BC neglected.

Dimensionless parameters are quite important.
- generally represent ratio of forces / mechanisms.
- common throughout thermal / fluid sciences.

Common dimensionless parameters.

Reynolds #

$$Re = \frac{\rho V L}{\mu} \approx \frac{\text{inertia forces}}{\text{viscous forces}}$$ acting on a typical fluid element

ρ = density V = characteristic velocity L = characteristic length
μ = dynamic viscosity

make appropriate choice

— Strouhal # $= \dfrac{l}{TV} \approx \dfrac{\text{inertial forces due to flow unsteadiness}}{\text{inertial forces due to changes in velocity from pt to pt.}}$

$\approx \dfrac{\text{local acceleration}}{\text{convective acceleration}}$ ← imposed time scale

— Mach #

$$Ma = \frac{V}{c} \sim \frac{\text{inertia forces}}{\text{elastic forces}}$$ measure of the effects of compressibility

c = speed of sound

if $Ma < .3$ then compressibility effects are negligible

— Froude #

$$Fr = \frac{V}{\sqrt{gL}} \text{ or } \frac{V^2}{gL} \sim \frac{\text{inertia forces}}{\text{gravity forces}}$$

Application ⇒ open channel flow : magnitude of Fr determines if hydraulic jump occurs

— Weber #

$$We = \frac{\rho V^2 L}{\sigma} \sim \frac{\text{inertial forces}}{\text{surface tension forces}}$$

σ = surface tension

surface tension effects neglected for high We (usually for large L)

– Euler #

$$Eu = \frac{\Delta P}{\rho V^2} \sim \frac{\text{pressure difference}}{\text{dynamic pressure in free stream}}$$

– Drag coefficient

$$C_D = \frac{F_d}{\frac{1}{2}\rho V^2 A} \quad \text{dimensionless drag}$$

F_d - drag force A = frontal area

– Lift coefficient

$$C_L = \frac{F_L}{\frac{1}{2}\rho V^2 A} \quad \text{dimensionless lift force } (F_L)$$

– Friction Factor f

defined by Darcy-Weisbach Eqn (fully developed flow)

$$h_f = f \frac{L}{D} V^2$$

h_f = head loss D = diameter

f = friction factor

Additional non-dimensional #s ⇒ Ca, Bo, Nu, Pe, Pr

→ What good are dimensionless data?

- enable us to simplify experiments - parameters are grouped

- we can correlate seemingly unrelated data

* GERRY'S PPT & sphere near a wall

So if we know the characteristics of a surface & we have a governing eqn with them then we can simplify the gov. eqns & possible enable us to solve them

N.S. Eqn ⇒ Re, St

But what if we don't have governing eqns. or we don't know (are told) what the set of dimensionless parameters are? How do we find dimensionless parameters to correlate our experimental data?

Use definitions of Basic dimensions & dimensional homogeneity

Basic dimensions

dimensions of any parameter/variable can be expressed as a combination of basic dimensions

systems M, L, T, Θ or F, L, T, Θ

Dimensional Homogeneity

each additive term in an eqn must have the same dimensions

ex. Bernoulli Eqn $\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + z_2$

$$\left[\frac{P}{\gamma}\right] = \frac{F/L^2}{F/L^3} = L$$

$$\left[\frac{V^2}{2g}\right] = \frac{L^2/T^2}{L/T^2} = L$$

$$[z] = L$$

So to non-dimensionlize we just need to find groupings of parameters with the same dimensions.

Determination of Dimensionless Groups

1. By inspection or tradition

recognize simple dimensionless groups

$r, D \Rightarrow \frac{r}{D}$

traditional groupings Re, Fr, We (listed previously)

these 2 approaches may not work. → need general method for experiments w/o gov. eqns.

2. Method of undetermined coefficients

Example of method:

what combination of ρ, V, l & σ is dimensionless

σ = surface tension = $\frac{F}{L} = \frac{ML/T^2}{L} = \frac{M}{T^2}$

procedure:

(1) select one variable to isolate (will be known as the non-repeating variable when we expand technique to finding sets of dimensionless groups)

select σ

(2) Form the product of the selected variable and all of the other variables raised to an undetermined power

$$\rho^a v^b \ell^c \sigma = \text{a dimensionless } \#$$

↑ σ not raised to a power

need to determine a, b, c.

$\rho^a v^b \ell^c \sigma$ assured to be dimensionless thus.

$$[\rho^a v^b \ell^c \sigma] = M^0 L^0 T^0$$

or

$$\left(\frac{M}{L^3}\right)^a \left(\frac{L}{T}\right)^b (L)^c \left(\frac{M}{T^2}\right) = M^0 L^0 T^0$$

combine M, L, T terms

$$M^{a+1} L^{-3a+b+c} T^{-b-2} = M^0 L^0 T^0$$

the powers of M L T must be the same on both sides

3 Eqns
3 unknowns

$$M: \quad a+1=0 \Rightarrow a=-1$$

$$L: \quad -3a+b+c=0$$

$$T: \quad -b-2=0 \Rightarrow b=-2$$

with $a=-1$ $b=-2$ $c=-1$

thus

$$\rho^{-1} v^{-2} \ell^{-1} \sigma = \frac{\sigma}{\rho v^2 \ell} \text{ is dimensionless}$$

what's this? We^{-1}

We or We^{-1} are both dimensionless - either one works We by tradition

Determining dimensionless groups from a set of variables?

Buckingham Pi & the method of undetermined coefficients.

Buckingham Pi → k variables may be reduced to k−r dimensionless parameters where r is the # of reference dimensions required to describe the variables

Let

$$f = F(\rho, V, D, \mu) \qquad k = 5 \Rightarrow 5 \text{ dimensional variables } f, \rho, V, D, \mu$$

frequency

f $\frac{1}{T}$

V L/T

D L

ρ M/L^3

μ $\frac{M}{LT}$

$$\mu = \frac{\tau}{\frac{du}{dy}} = \frac{Pa}{L/T/L} = Pa \cdot s = \frac{N}{m^2} s$$

$$= \frac{kg\, m}{s^2 m^2} s = \frac{kg}{m s}$$

thus dimensions $L, M, T \Rightarrow r = 3$

# of dimensionless groups $k - r = 5 - 3 = 2$

We can write $f = F(\rho, V, D, \mu)$ in Pi terms

$$\Pi_1 = F(\Pi_2, \ldots \Pi_{k-r})$$

thus

$$\Pi_1 = F(\Pi_2)$$

Select repeating variables $\Rightarrow$ 3 of ρ, V, D, μ don't choose f

say ρ, V, D

variable of interest

Π_1 : f

$$f \rho^a V^b D^c = M^0 L^0 T^0$$

$$\left(\frac{1}{T}\right)\left(\frac{M}{L^3}\right)^a \left(\frac{L}{T}\right)^b (L)^c = M^0 L^0 T^0$$

M: $a = 0$

L: $-3a + b + c = 0 \Rightarrow b + c = 0 \Rightarrow c = -b$

T: $-1 - b = 0 \Rightarrow b = -1$

$\Rightarrow c = 1$

thus

$$\Pi_1 = f V^{-1} D$$
$$= \frac{fD}{V}$$

check $\frac{(1/s)(m)}{(m/s)} = 1$

Π_2 : μ

$$\mu \rho^a V^b D^c = M^0 L^0 T^0$$

$$\left(\frac{M}{LT}\right)\left(\frac{M}{L^3}\right)^a \left(\frac{L}{T}\right)^b (L)^c = M^0 L^0 T^0$$

M: $1 + a = 0 \Rightarrow a = -1$

L: $-1 - 3a + b + c = -1 + 3 + b + c = 0 \Rightarrow b + c = -2$

T: $-1 - b = 0 \Rightarrow b = -1$

$\Rightarrow c = -1$

thus

$$\Pi_2 = \mu \rho^{-1} V^{-1} D^{-1} = \frac{\mu}{\rho V D}$$

but this looks familiar

$$\frac{\mu}{\rho V D} = Re^{-1}$$

so

$$\Pi_2 = Re$$

Π_2 can be Re^{-1} or Re both are dimensionless

tradition defines $Re = \frac{\rho V D}{\mu}$ use.

with our dimensionless groups determined

we know

$$\Pi_1 = F(\Pi_2)$$

so experimentally we should look for

$\frac{fD}{V}$ to correlate with Re

if we plot

$\frac{fD}{V}$ vs Re all data should collapse to a single curve

Gerry's PPT

Note: data collapses if all significant parameters accounted for

→ if data doesn't collapse then we need to look for another parameter that we overlooked or neglected.

→ multiple length and time scales may exist. ⇒ yields additional parameters

Modeling & Similitude

Dimensionless parameters define physical regimes

$$Re = \frac{\rho V L}{\mu} = \frac{\text{inertial forces}}{\text{viscous forces}}$$

$$Ca = \frac{\mu V}{\sigma} = \frac{\text{viscous forces}}{\text{interfacial forces}}$$

If we want to predict the behavior of a physical system we dont necessarily have to study the actual system

- in fact we would like to know how a system will behave before we build it
  - you don't want to spend millions of dollar building a dam, boat, skyscraper & then see how it performs

With dimensionless similarity we can explore an actual system and examine its behavior over a wide range of parameters with a model

If we have a pipe that's 1mm or 10m in diameter the flow characteristics will be the same for similar values of Re

$$Re = \frac{\rho V D}{\mu}$$ (regardless of the fluid)

to have dimensionless similarity if $\frac{D_1}{D_2} = \frac{10\,m}{10^{-3}\,m} = 10^4\,m$

then $$\frac{\left(\frac{\rho V}{\mu}\right)_1}{\left(\frac{\rho V}{\mu}\right)_2} = 10^{-4}$$

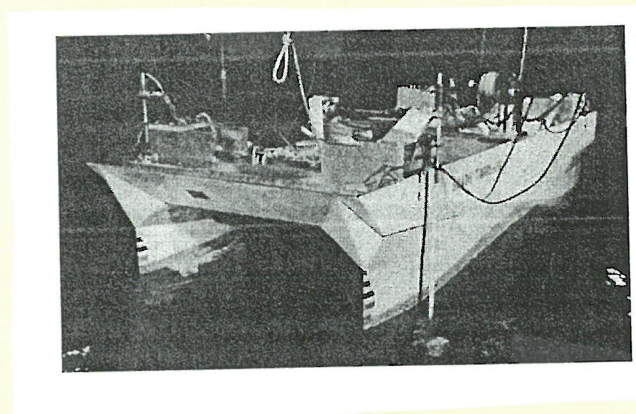
We adjust variables which can be easily manipulated

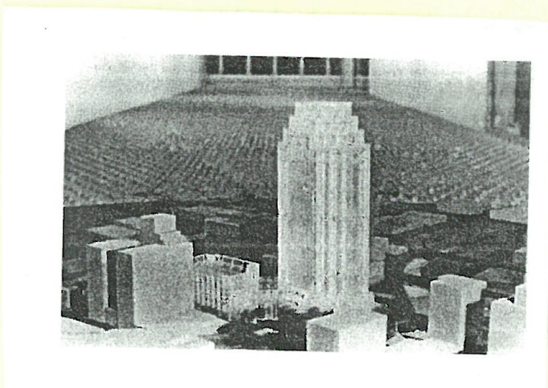
in above example velocity is probably the easiest

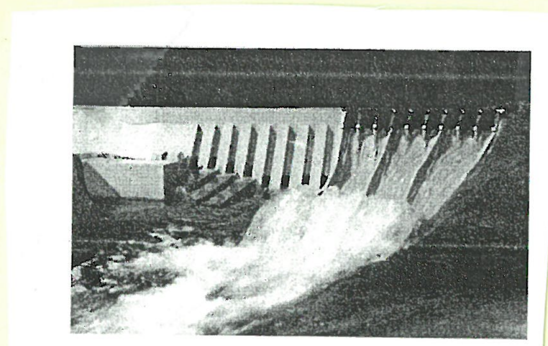
although we can change fluid properties as well

the book shows a # of models

- city Fig 7.6
- dam spillway Fig 7.8
- boat Fig 7.9







Models generally employed when we can't realistically examine the real system

(too big or too small)

Note: use of physical models are being replaced by computational models

- however current computational ability still can't model everything
  - free surfaces, mixed regimes (multiple forces equivalent)

so modeling still has its place. - OSU Tsunami pool ← systems we can't explore in real life.

PSU Wind Tunnel

Dimensionless parameters can be found by combining variables into dimensionless groups. # of groups set by the Buckingham Pi theorem (# of groups = # variable - # dimensions)

- empirical approach?

Deviated

alternately,

dimensionless groups arise from non-dimensionalizing the governing eqns of a problem.

- the applicable Eqn or Eqns may be different for different problems.
- this was our starting pt for discussing dimensionless groups.

Let go back & further examine the Equations of motion.

Navier's Stokes Eqn

$$\rho \frac{Dv}{Dt} = -\nabla p + \mu \nabla^2 v + \rho g$$

Cartesian Coordinates X direction

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) + \rho g_x$$

When can we solve this Eqn, analytically?

2 general scenarios

(1) flow problem sufficiently simple such that most terms drop out

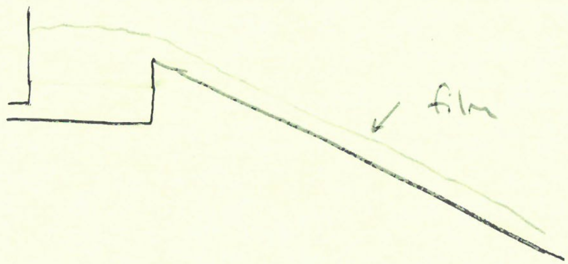
(2) neglect terms based on ordering arguments ⇒ dimensionless parameters

(1) Simple cases (already did an example of this)

generally 1-D steady flows

example.

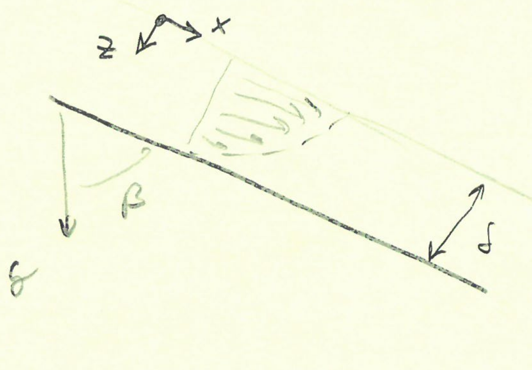
Flow of a falling film - coating applications, gas absorbers evaporators



For a given flow rate how thick (δ) is the film - flow rate from velocity, velocity field should incorporate thickness

assume we are far from the ends so we can neglect edge effects - fully developed

assume we have a steady flow down the slope - incompressible, constant μ



With rectangular coordinates

Continuity

$$\nabla\cdot V = 0 \Rightarrow \frac{\partial v_x}{\partial x} + \underbrace{\frac{\partial v_y}{\partial y}}_{0} + \underbrace{\frac{\partial v_z}{\partial z}}_{0} = 0 \Rightarrow \frac{\partial v_x}{\partial x} = 0$$

1-D flow

- start with y & z components of N.S. eqn
- x component of N.S. eqn.

$$\rho\Big(\underbrace{\frac{\partial v_x}{\partial t}}_{0\ \text{S.S.}} + \underbrace{v_x \frac{\partial v_x}{\partial x}}_{0\ \text{continuity}} + \underbrace{v_y}_{0\ \text{1-D}} \frac{\partial v_x}{\partial y} + \underbrace{v_z}_{0\ \text{1-D}} \frac{\partial v_x}{\partial z}\Big) = \underbrace{-\frac{\partial p}{\partial x}}_{0\ \text{no imposed dp}} + \mu\Big(\underbrace{\frac{\partial^2 v_x}{\partial x^2}}_{0\ \text{continuity}} + \underbrace{\frac{\partial^2 v_x}{\partial y^2}}_{0\ \text{infinite in y}} + \frac{\partial^2 v_x}{\partial z^2}\Big) + \rho g_x$$

thus

$$0 = \mu \frac{\partial^2 V_x}{\partial z^2} + \rho g_x \qquad g_x = g\cos\beta$$

since V_x only depends on z

$$\frac{d^2 V_x}{dz^2} = -\frac{\rho g \cos\beta}{\mu}$$

integrate $\quad \frac{dV_x}{dz} = \left(\frac{-\rho g \cos\beta}{\mu}\right) z + C_1$

separate & integrated

$$V_x = \left(\frac{-\rho g \cos\beta}{\mu}\right)\frac{z^2}{2} + C_1 z + C_2$$

to get the full velocity profile we need to evaluate constants of integration

We need 2 Boundary conditions.

B.C. 1 $\quad$ at $z = \delta \quad V_x = 0$ $\quad$ No-slip condition: fluid velocity matches the velocity at an interface

B.C 2 $\quad$ at $z = 0 \quad \tau_{xz} = 0$

no shear stress at the air-water interface

air essentially inviscid ⇒ stress free condition

$$\tau_{xz} = \mu\left(\frac{\partial V_x}{\partial z} + \frac{\partial V_z}{\partial x}\right) = \mu \frac{\partial V_x}{\partial z} \qquad \tau_{xz_{H_2O}} = \tau_{xz_{air}}$$

so

B.C 2 $\quad$ at $z = 0 \quad \frac{dV_x}{dz} = 0$

apply B.C 2 $\quad \frac{dV_x}{dz} = (-\rho g \cos\beta) z + C_1 \Rightarrow C_1 = 0$

generally good to apply $z=0$ BC 1st

thus
$$V_x = \left(\frac{-\rho g \cos\beta}{\mu}\right)\frac{z^2}{2} + C_2$$

B.C #1 at $z=\delta$ $V_x=0$

$$0 = \frac{-\rho g \cos\beta}{2\mu}\delta^2 + C_2$$

thus $C_2 = \frac{\rho g \cos\beta}{2\mu}\delta^2$

our velocity becomes

$$V_x = \left(\frac{-\rho g \cos\beta}{2\mu}\right)z^2 + \frac{\rho g(\cos\beta)}{2\mu}\delta^2$$

combining / rearranging.

$$V_x = \frac{\rho g \cos\beta}{2\mu}\left[\delta^2 - z^2\right] = \frac{\rho g(\cos\beta)\delta^2}{2\mu}\left[1-\left(\frac{z}{\delta}\right)^2\right]$$

We have the velocity

velocity depends on δ

How do we calculate the flow rate?

$$Q = \int_A V_x\,dA = \int_0^W\int_0^\delta V_x\,dz\,dy = W\int_0^\delta\left(\frac{\rho g(\cos\beta)\delta^2}{2\mu}\right)\left[1-\left(\frac{z}{\delta}\right)^2\right]dz$$

$$Q = \frac{\rho g W \delta^3 \cos\beta}{3\mu}$$

so

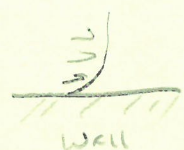
$$\delta^3 = \frac{3\mu Q}{\rho g W \cos\beta} \Rightarrow \delta = \left[\frac{3\mu Q}{\rho g W \cos\beta}\right]^{1/3}$$

thickness of film

To solve any problem we will need Boundary Conditions

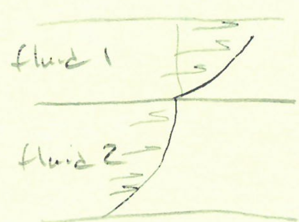
Common Boundary Conditions

(1) No-slip   velocity of fluid matches velocity of the surface its in contact with.



$V_{wall} = 0 \Rightarrow V_{fluid} = 0$

$V_{wall} = u \Rightarrow V_{fluid} = u$



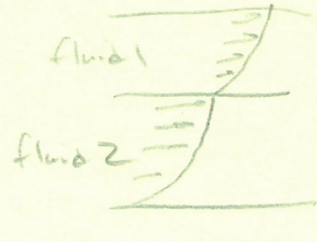
$V_{fluid,1} = V_{fluid,2}$ at interface

velocity is continuous

(2) No stress $\tau_{xy} = 0$ at surface

$\frac{dV}{dy} = 0$ at interface

(3) Stress continuous



$\tau_{xy_1} = \tau_{xy_2}$ at interface

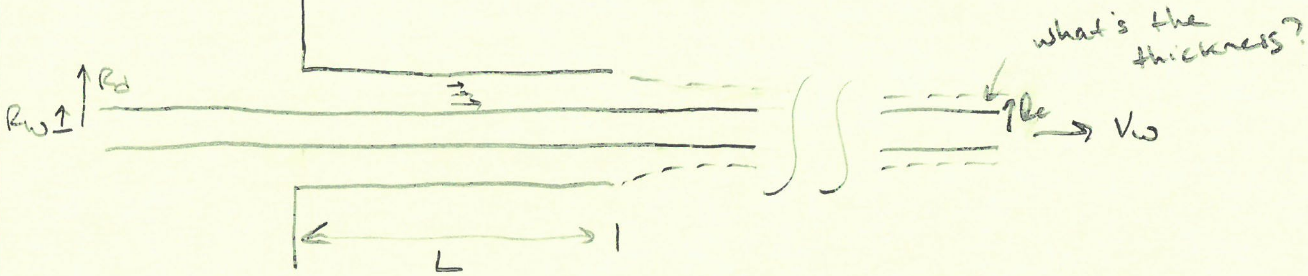
(4) General boundary condition - Navier's B.C. $\Rightarrow$ important for microscale flows

$V_x - V_{wall} = \beta \frac{dV_x}{dy}$ at a surface β = slip length

$\beta = 0$ no slip

Example #2 Wire coating -

wire pulled through liquid bath
thru a die that wipes fluid clean



What the force required to pull the wire through the die?

$$F = \tau A \quad \text{need } \tau_{rz}$$

Assume: steady incompressible flow with constant viscosity $\Rightarrow$ N.S. Eqn

Neglect edge effects - flow is axial only

there is no imposed pressure gradient

- Cylindrical coordinates

Continuity

$$\frac{1}{r}\frac{\partial}{\partial r}(r \underbrace{v_r}_{0}) + \frac{1}{r}\frac{\partial}{\partial \theta}(\underbrace{v_\theta}_{0}) + \frac{\partial v_z}{\partial z} = 0 \Rightarrow \frac{\partial v_z}{\partial z} = 0$$

no velocity or variation for v_r or v_θ +

- Show r & θ components

z component

$$\rho\left(\underbrace{\frac{\partial v_z}{\partial t}}_{0} + \underbrace{v_r}_{0}\frac{\partial v_z}{\partial r} + \underbrace{\frac{v_\theta}{r}}_{0}\frac{\partial v_z}{\partial \theta} + v_z \underbrace{\frac{\partial v_z}{\partial z}}_{0}\right) = -\underbrace{\frac{\partial P}{\partial z}}_{0} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\underbrace{\frac{\partial^2 v_z}{\partial \theta^2}}_{0} + \underbrace{\frac{\partial^2 v_z}{\partial z^2}}_{0}\right]$$

S.S. continuity θ symmetric cont.

so

$$0 = \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) \Rightarrow \frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = 0$$

1st integration

$$r \frac{dV_z}{dr} = C_1$$

separate & integrate $\int dV_z = \int \frac{C_1}{r}\, dr$

$$V_z = C_1 \ln r + C_2$$

B.Cs

at $r = R_w$ $V_z = V_w = C_1 \ln R_w + C_2$

at $r = R_d$ $V_z = 0 = C_1 \ln R_d + C_2$

solve for C_1 & C_2 & rearrange

$$V_z = V_w \frac{\ln(r/R_d)}{\ln(R_w/R_d)}$$

Force to pull wire through die

$$F = \tau_{rz} A = \tau_{rz}(2\pi R_w L)$$

$$\tau_{rz} = \mu\left(\frac{\partial V_z}{\partial r} + \frac{\partial V_r}{\partial z}\right)\Bigg|_{r=R_w} = \mu \frac{dV_z}{dr}\Bigg|_{r=R_w}$$

$$\frac{dV_z}{dr} = \frac{V_w}{\ln\left(\frac{R_w}{R_d}\right)} \cdot \frac{1/R_d}{r/R_d} = \frac{V_w}{r \ln(R_w/R_d)}$$

$$\tau_{rz}\Big|_{r=R_w} = \frac{\mu V_w}{R_w \ln(R_w/R_d)}$$

thus

$$F = \frac{\mu V_w}{R_w \ln(R_w/R_d)} \cdot 2\pi R_w L = \frac{2\pi \mu V_w L}{\ln(R_w/R_d)}$$

What's the thickness of the coating?

Conservation of mass - what's flows through die is deposited on the wire

$$M_{\text{around wire}} = M_{\text{flowing in die}}$$

if density constant $\Rightarrow$ incompressible

$$Q_{\text{wire}} = Q_{\text{die}}$$

Velocity of the wire is a constant so

$$Q_{\text{wire}} = V_w A_{\text{coating}}$$

$$= V_w\left(\pi R_c^2 - \pi R_w^2\right) = \pi V_w\left(R_c^2 - R_w^2\right)$$

Fluid velocity in the die is not a constant

$$Q_{\text{die}} = \int V_z\,dA = \int_{R_w}^{R_d} V_z(2\pi r\,dr)$$

Insert velocity profile

$$Q_{\text{die}} = \int_{R_w}^{R_d} \frac{V_w \ln(r/R_d)}{\ln(R_w/R_d)} \cdot 2\pi r\,dr = \frac{2\pi V_w}{\ln(R_w/R_d)} \int_{R_w}^{R_d} \underbrace{r \ln\left(\frac{r}{R_d}\right) dr}_{\text{can integrate}}$$

from integral tables

$$x \ln(ax)\,dx = \frac{x^2}{2}\ln(ax) - \frac{x^2}{4}$$

evaluating

$$Q_{dc} = \frac{2\pi V_w}{\ln\left(\frac{R_w}{R_d}\right)} \left[\frac{R_d^2}{2} \underbrace{\ln\left(\frac{R_d}{R_d}\right)}_{0} - \frac{R_d^2}{4} - \frac{R_w^2}{2} \ln\left(\frac{R_w}{R_d}\right) + \frac{R_w^2}{4} \right]$$

$$= \frac{2\pi V_w}{\ln\left(\frac{R_w}{R_d}\right)} \left[\frac{R_w^2}{4} - \frac{R_d^2}{4} - \frac{R_w^2}{2} \ln\left(\frac{R_w}{R_d}\right) \right]$$

$$Q_{dc} = Q_{coat}.$$

$$\frac{2\pi \cancel{V_w}}{\ln\left(\frac{R_w}{R_d}\right)} \left[\frac{R_w^2}{4} - \frac{R_d^2}{4} - \frac{R_w^2}{2} \ln\left(\frac{R_w}{R_d}\right) \right] = \cancel{V_w} \pi \left(R_c^2 - R_w^2\right)$$

$$R_c^2 = \frac{2}{\ln\left(\frac{R_w}{R_d}\right)} \left[\frac{R_w^2}{4} - \frac{R_d^2}{4} - \frac{R_w^2}{2} \ln\left(\frac{R_w}{R_d}\right) \right] + R_w^2$$

$$= \frac{R_w^2 - R_d^2}{2 \ln\left(\frac{R_w}{R_d}\right)} - R_w^2 + R_w^2$$

$$= \frac{R_w^2 - R_d^2}{2 \ln\left(R_w / R_d\right)} = \frac{R_d^2 - R_w^2}{2 \ln\left(\frac{R_d}{R_w}\right)}$$

so

$$R_c = \left(\frac{R_d^2 - R_w^2}{2 \ln\left(R_d / R_w\right)} \right)^{1/2}$$

thickness $d = R_c - R_w$

$$d = \left(\frac{R_d^2 - R_w^2}{2 \ln\left(R_d / R_w\right)} \right)^{1/2} - R_w$$

For "simple" problems we can solve the N.S eqns.

→ 1-D, easy geometry, proper assumption (apparent)

2^{nd} Method: Simplify the governing eqns ; solve an approximate eqn.

→ Go Back to pg 103

Dimensionless N.S.

$$Re\left(Sr^{-1}\frac{\partial V}{\partial t} + V\cdot\nabla V\right) = -\left(\frac{\Pi L}{\mu V}\right)\nabla\tilde{P} + \nabla^2 V$$

$\Pi = \rho V^2$ for inertial dominated flow

$\Pi = \frac{\mu V}{L}$ for viscous dominated flow

If $Re \to \infty$ then

$$Sr^{-1}\frac{\partial V}{\partial t} + V\cdot\nabla V + \nabla\tilde{P} = 0$$ Euler's eqn of motion

singular ; non-linear ⇒ non-linearity limits solvability

With further assumptions - irrotational flow we can express the velocity in terms of a potential & we get Laplace's Equation

$$\nabla^2\phi = 0 \qquad \phi = \text{velocity potential}$$

inviscid, incompressible, irrotational flow fields.

Section 6.4-6.7 discuss this & how this can be applied to calculate flows.

Note: Book develops Euler Eqns through Approximations then states its a simplified form of N.S.

If $Re \to 0$ viscous dominated flows.

$$Re\left[Sr^{-1}\frac{\partial V}{\partial t} + V\cdot\nabla V\right] = -\nabla\tilde{p} + \nabla^2 V$$

$\to 0$

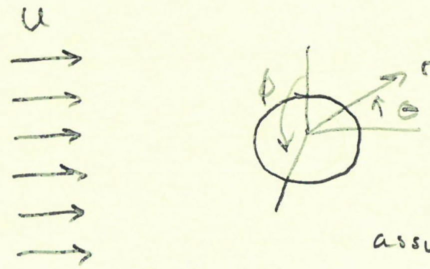
$$0 = -\nabla\tilde{p} + \nabla^2 V$$

Dimensional Eqn.

$$0 = -\nabla p + \rho g + \mu\nabla^2 V$$

nice eqn = linear 2nd order eqn → can be solve for many cases.

Example of low Re problem : Stokes flow



What's the drag on a sphere in uniform flow?

– sphere settling in a fluid

assume infinite fluid domain with uniform velocity u. incompressible, Newtonian fluid

characteristic velocity = u

characteristic length = D diameter of sphere

$Re = \frac{DU\rho}{\mu}$ if $Re \ll 1$ we having creeping flow

Flow is best described in spherical components (we have a sphere)

Assume ϕ direction symmetry $\Rightarrow \frac{\partial}{\partial\phi} = 0$ and $V_\phi = 0$

there are both a V_θ & V_r component of velocity that vary with r & θ

Continuity Eqn.

$$\frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho v_\theta \sin\theta) + \frac{1}{r\sin\theta}\underbrace{\frac{\partial}{\partial\phi}(\rho v_\phi)}_{0} = 0$$

simplifies to

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) = 0$$

r-component of N.S eqn. (show)

when $Re \ll 1$ eqn simplifies to

$$0 = -\frac{\partial p}{\partial r} + \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_r}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right) - \frac{2v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta} - \frac{2}{r^2}v_\theta\cot\theta\right]$$

θ component simplifies to

$$0 = -\frac{1}{r}\frac{\partial p}{\partial\theta} + \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_\theta}{\partial\theta}\right) + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r^2\sin\theta}\right]$$

3 Simplified eqns govern our problem

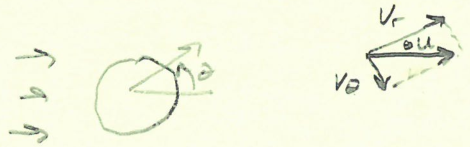
- to solve we need boundary conditions

- No slip at sphere surface $v_r = v_\theta = 0$ at $r = R$ ← radius of sphere

additional 2 needed.

we can note that as $r \to \infty$ the velocity should approach the free stream velocity U

we can resolve U into r and θ components



$$V_r = U\cos\theta$$
$$V_\theta = -U\sin\theta$$

so the B.Cs become

— as $r \to \infty$ $\quad V_r = U\cos\theta \quad V_\theta = U\sin\theta$

So we have 3 PDE's, 3 unknowns P, V_r, V_θ and enough B.C's.

— let solve the eqns.

B.C's generally provide insight into the structure of the solution (solution has to match the boundaries)

Look for solution

$$V_r = A(r)\cos\theta \qquad V_\theta = B(r)\sin\theta$$

at $r = R$ $\quad A(R) = B(R) = 0$

$r \to \infty$ $\quad A(\infty) = U \quad B(\infty) = -U$

If we plug these in we will find that the continuity eqn will relate A to B

$$A(r) + \frac{r}{2}\frac{dA}{dr} + B(r) = 0$$

substituting into r component we get a functional form for P

$$P = P_0 + \mu\Pi(r)\cos\theta$$

and

$$0 = -\frac{d\Pi(r)}{dr} + \frac{1}{r^2}\frac{d}{dr}r^2\frac{dA(r)}{dr} - \frac{4A(r)}{r^2} - \frac{4B(r)}{r^2}$$

the θ component yields

$$0 = \Pi(r) + \frac{1}{r}\frac{d}{dr} r^2 \frac{dB(r)}{dr} - \frac{2B(r)}{r} - \frac{2A(r)}{r}$$

these can be combined to get a single eqn for $A(r)$ where $A(r) = r^n$

solve for A (n components)

with A solve for B by continuity

with A & B solve for $\Pi(r) \Rightarrow P$

the solution becomes

$$V_r = U\left[1 - \frac{3}{2}\frac{R}{r} + \frac{1}{2}\left(\frac{R}{r}\right)^3\right]\cos\theta$$

$$V_\theta = -U\left[1 - \frac{3}{4}\frac{R}{r} - \frac{1}{4}\left(\frac{R}{r}\right)^3\right]\sin\theta$$

$$P = P_0 - \frac{3\mu U}{2R}\left(\frac{R}{r}\right)^2\cos\theta$$

uniform pressure far from sphere

What's the drag on the sphere? Form drag & friction drag.

To calculate the drag we need to calculate the force exerted on the sphere in flow direction

let's assume the flow is horizontal (neglect bouyancy effects, ie gravity unimportant)

pressure exerted on sphere is

$$p = p_0 - \frac{3\mu U}{2R}\left(\frac{R}{r}\right)^2\cos\theta$$

pressure is high on inflow axis

pressure is low on outflow axis

– pushes sphere to right

2^{nd} force are the shearing forces.

$$\tau_{r\theta} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

substitute v_r & v_θ into $\tau_{r\theta}$ and evaluate at $r=R$

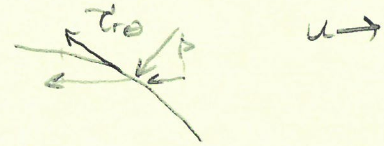
$$\tau_{r\theta} = -\frac{3\mu u}{2R}\sin\theta$$

no shear on inflow axis
no shear on outflow axis
max shear on sides
- pushes sphere to right

to get the total force we need to integrate over entire surface

component of $\tau_{r\theta}$ in flow direction is $-\tau_{r\theta}\sin\theta$

component of pressure in flow direction is $-p\cos\theta$



differential surface area in spherical coordinates = $R^2\sin\theta\, d\theta\, d\phi$

thus

$$F = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{2\pi}\left(-p\cos\theta - \tau_{r}\sin\theta\right)R^2\sin\theta\, d\theta\, d\phi$$

$$= \iint \left[\frac{3\mu u}{2R}\cos\theta + \frac{3\mu u}{2R}\sin\theta\right]R^2\sin\theta\, d\theta\, d\phi$$

integrating.

$$F = \underbrace{2\pi\mu R u}_{\text{form drag}} + \underbrace{4\pi\mu R u}_{\text{friction drag}} = 6\pi\mu R u$$

Stokes Law

if settling sphere
drag = buoyancy
speed can measure viscosity