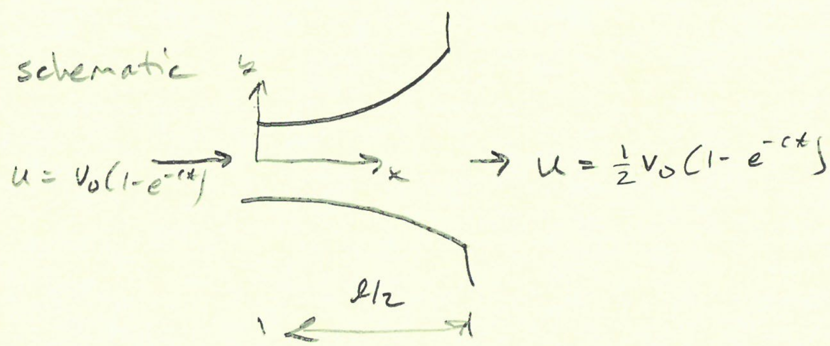
The velocity along the centerline of a gas diffuser has a velocity along it's centerline given by

$$\vec{V} = \hat{\imath}\, V_0 \left(1 - e^{-ct}\right)\left(1 - \frac{x}{\ell}\right)$$

schematic



If $V_0 = 10$ ft/s and $\ell = 5$ ft determine what value of c (other than $c=0$) is needed to make the acceleration zero for any x at $t = 1$ s. Find expression for c

need to calculate fluid acceleration

In general

$$\vec{V} = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$$

comparing terms

$$u = V_0\left(1 - e^{-ct}\right)\left(1 - x/\ell\right)$$

$$v = 0$$

$$w = 0$$

substantial derivative

$$a = \frac{dV}{dt} \quad \text{but } V = (x, y, z, t)$$

apply chain rule

$$= \frac{\partial \vec{V}}{\partial t}\frac{dt}{dt} + \frac{\partial \vec{V}}{\partial x}\frac{dx}{dt} + \frac{\partial \vec{V}}{\partial y}\frac{dy}{dt} + \frac{\partial \vec{V}}{\partial z}\frac{dz}{dt}$$

but $u = \frac{dx}{dt}$ $v = \frac{dy}{dt}$ $w = \frac{dz}{dt}$

thus,

$$\frac{dV}{dt} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

general acceleration in the x-direction

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

with $w = v = 0$

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}$$

compute terms in a_x

$$\frac{\partial u}{\partial t} = V_0 c e^{-ct}\left(1-\frac{x}{\ell}\right)$$

$$\frac{\partial u}{\partial x} = V_0(1-e^{-ct})\left(-\frac{1}{\ell}\right) \Rightarrow u\frac{\partial u}{\partial x} = V_0(1-e^{-ct})\left(1-\frac{x}{\ell}\right) V_0(1-e^{-ct})\left(-\frac{1}{\ell}\right)$$

$$u\frac{\partial u}{\partial x} = -\frac{V_0^2}{\ell}\left(1-\frac{x}{\ell}\right)(1-e^{-ct})^2$$

therefore

$$a_x = V_0\left(1-\frac{x}{\ell}\right)\left[ce^{-ct} - \frac{V_0}{\ell}(1-e^{-ct})^2\right]$$

at $t = 1s$ we need $a_x = 0$

thus

$$0 = V_0\left(1-\frac{x}{\ell}\right)\left[ce^{-c} - \frac{V_0}{\ell}(1-e^{-c})^2\right]$$

or

$$ce^{-c} - \frac{V_0}{\ell}(1-e^{-c})^2 = 0$$ expression for c

need to find the root for an actual numerical example

use calculator, excel, matlab.
(goal seek)

plugging in #'s & evaluating

$$c = 0.490110396965 7$$

Note: the flow rate is increasing with time yet the acceleration is zero. time derivative $\frac{\partial u}{\partial t} > 0$ but the spatial derivative $u\frac{\partial u}{\partial x}$ is equal in magnitude but opposite in sign

To calculate the acceleration we made use of the substantial (or material) derivative

in general we can write

$$\frac{D}{Dt}(\ ) = \frac{\partial(\ )}{\partial t} + u\frac{\partial(\ )}{\partial x} + v\frac{\partial(\ )}{\partial y} + w\frac{\partial(\ )}{\partial z}$$

or

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + \vec{V}\cdot\nabla(\ )$$

substantial derivative relates the change experienced in a Langrangian framework (perspective) to a Eulerian framework.

It's a differential relationship.

— What's the integral (finite size) equivalent of the material / substantial derivative?

⇒ Reynold Transport Theorem

- Control Volume Analysis

1st some definitions

thermo
open
closed

Control volume – a region fixed in space : Eulerian description

System – is a collection of fixed mass : Lagrangian

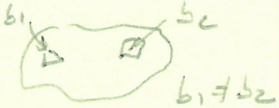
Define system properties

B = extensive property — depends on extent/amount of mass

b = intensivive property amount of B per unit mass of system

$$b = \frac{B}{m}$$

can vary in space



you should be well aware of these definitions from Thermo.

Has everyone either taken thermo already or taking it currently

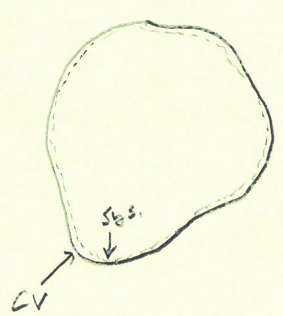
- we can skip a few things if that's the case.

examples?

So how do we relate the Lagrangian (system) to Eulerian (control volume) perspective on an integral basis

System properties — Control volume properties

Consider a control volume & system coincident at some time t



if they are coincident then

$$B_{sys}(t) = B_{cv}(t)$$

Note

$\frac{dB_{sys}}{dt}$ does not have to equal $\frac{dB_{cv}}{dt}$

even if $B_{sys}(t) = B_{cv}(t)$

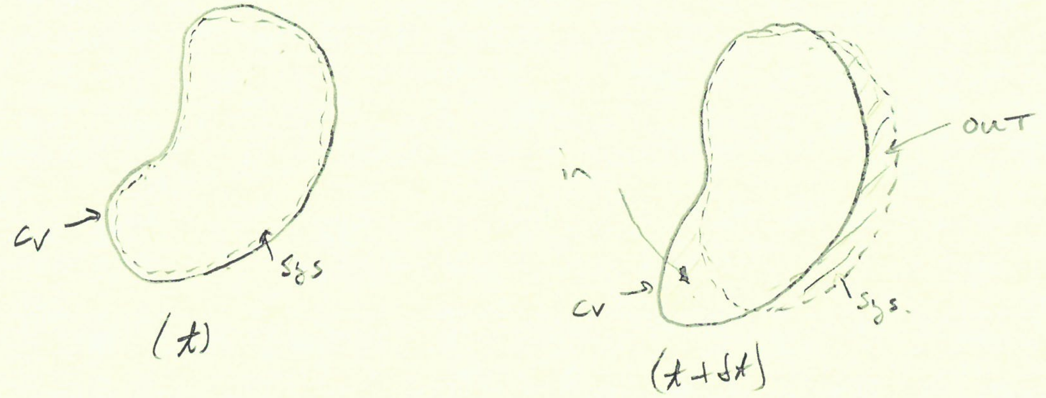
In terms of intensive properties

$$B_{sys} = \int_{sys} \rho b \, d\forall$$

Note b can vary in space $\int$ over all system space covers the possible variation

$$B_{cv} = \int_{cv} \rho b \, d\forall$$

If we look at control volume & system at some later time δt



at t $\quad B_{sys} = B_{cv}$

at $t + \delta t$

$$\underbrace{B_{cv}(t+\delta t)}_{\text{amount of } B \text{ in the control volume}} = \underbrace{B_{sys}(t+\delta t)}_{\text{amount of } B \text{ is system}} - \underbrace{B_{out}}_{\text{Amount out of cv}} + \underbrace{B_{in}}_{\text{amount into control volume}}$$

Conservation of B

subtract $B_{sys}(t)$ to each side

$$B_{cv}(t+\delta t) - B_{sys}(t) = B_{sys}(t+\delta t) - B_{sys}(t) - B_{out} + B_{in}$$

but $B_{sys}(t) = B_{cv}(t)$

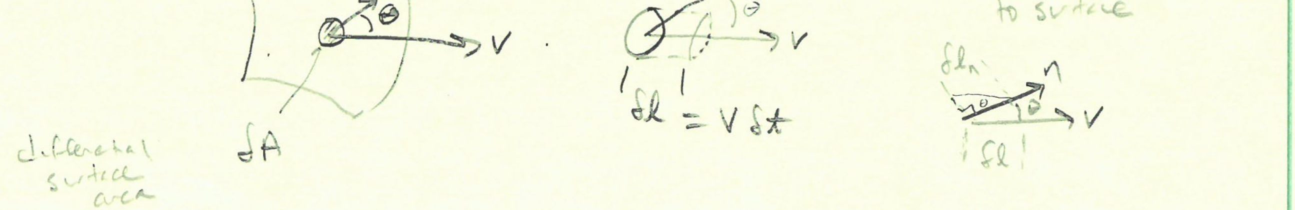
thus

$$B_{cv}(t+\delta t) - B_{cv}(t) = B_{sys}(t+\delta t) - B_{sys}(t) - B_{out} + B_{in}$$

but whats B_{out} & B_{in}

Outflow: examine the surface of the control volume

concerned about amount normal to surface



the amount of B that leave

$$\delta B_{out} = \rho b \underbrace{\delta \forall}_{\substack{\text{differential} \\ \text{volume that leaves}}}$$

relate δV to the velocity of the surface flow

$$\delta \forall = \delta l_n \, \delta A$$

but

$$\delta l_n = \delta l \cos\theta = V \delta t \cos\theta$$

thus the differential volume is

$$\delta \forall = V \delta t \cos\theta \, \delta A$$

$$\delta B_{out} = \rho b V \delta t \cos\theta \, \delta A$$

integrate

$$\int_{CS,out} \delta B_{out} = \int_{CS out} \rho b V \delta t \cos\theta \, \delta A$$

$$B_{out} = \int_{CS,out} \rho b V \delta t \cos\theta \, dA = \delta t \int_{CS,out} \rho b V \cos\theta \, dA$$

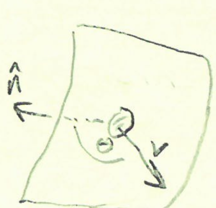
by definition of the dot product $V\cos\theta = V \cdot n$ so

$$B_{out} = \delta t \int_{CS,out} \rho b \, \vec{V} \cdot n \, dA$$

what's

B_{in} — following the same procedure as above

everything is the same except $\hat{n}$ is now opposite the velocity direction



$$B_{in} = -\int_{CS in} \rho b V \cos\theta \, dA \, \delta t = -\delta t \int_{CS in} \rho b (\vec{V} \cdot n) \, dA$$

δl_n is in negative n direction thus

$$B_{cv}(t+\delta t) - B_{cv}(t) = B_{sys}(t+\delta t) - B_{sys}(t) - \delta t \int_{CS,OUT} \rho b(\vec{V}\cdot n)\,dA - \delta t \int_{CS_{in}} \rho b(\vec{V}\cdot n)\,dA$$

thus

$$B_{cv}(t+\delta t) - B_{cv}(t) = B_{sys}(t+\delta t) - B_{sys}(t) - \delta t \left[\int_{CS,OUT} \rho b(\vec{V}\cdot n)\,dA + \int_{CS,in} \rho b(\vec{V}\cdot n)\,dA \right]$$

but

$$\int_{CS_{OUT}} dA + \int_{CS,in} dA = \int_{CS} dA$$

all ins + all outs equals everything

thus

$$B_{cv}(t+dt) - B_{cv}(t) = B_{sys}(t+dt) - B_{sys}(t) - \delta t \int_{CS} \rho b \vec{V}\cdot n\,dA$$

divide by δt

$$\frac{B_{cv}(t+\delta t) - B_{cv}(t)}{\delta t} = \frac{B_{sys}(t+\delta t) - B_{sys}(t)}{\delta t} - \int_{CS} \rho b \vec{V}\cdot n\,dA$$

take limit as $\delta t \to 0$

$$\lim_{\delta t \to 0} \frac{B_{cv}(t+dt) - B_{cv}(t)}{\delta t} = \frac{\partial B_{cv}}{\partial t}$$ definition of the derivative

likewise $$\lim_{\delta t \to 0} \frac{B_{sys}(t+dt) - B_{sys}(t)}{\delta t} = \frac{D B_{sys}}{Dt}$$ ← substantial derivative since system is moving Lagrangian

thus

$$\frac{\partial B_{cv}}{\partial t} = \frac{D B_{sys}}{Dt} - \int_{CS} \rho b \vec{V}\cdot n\,dA$$

but with $B_{cv} = \int_{CV} \rho b\,d\forall$

RTT

$$\boxed{\frac{\partial}{\partial t}\int_{CV} \rho b\,d\forall + \int_{CS} \rho b \vec{V}\cdot n\,dA = \frac{D B_{sys}}{Dt}}$$

So what does the R.T.T. mean

it is essentially the integral form of the substantial derivative

$$\underbrace{\frac{DB}{Dt}}_{} = \underbrace{\frac{\partial}{\partial t}\int_{CV} \rho b \, d\forall}_{} + \int_{CS} \rho b \vec{V} \cdot n \, dA$$

Link between Lagrangian & Eulerian perspective

time rate of change of an extensive parameter

- mass, momentum, energy

time rate of change of B in the control volume as it flows through it

net flowrate of B across the entire control surface

regions of B carried in $V \cdot n > 0$
regions of B carried out $V \cdot n < 0$
regions of no transport $V \cdot n = 0$

This is for a fixed control volume

If the control volume is moving

$$\frac{DB}{Dt} = \frac{\partial}{\partial t}\int_{CV} \rho b \, d\forall + \int_{CS} \rho b \vec{W} \cdot n \, dA$$

$W = V - V_{cv}$ ← velocity of control volume

relative velocity (W) — absolute velocity (V)

Use the relative velocity

it's how much actually crosses the boundary

R.T.T applied to Conservation of Mass

$B = m \quad b = \frac{m}{m} = 1$

By definition

$$\frac{DM_{sys}}{Dt} = 0$$ system is a fixed amount of mass

R.T.T.

$$\boxed{0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot n \, dA}$$ Continuity Eqn

time rate of change of mass in CV

net rate of flow of mass through the control surface

note $\vec{V} \cdot n < 0$ inflow

$\vec{V} \cdot n > 0$ outflow

Accumulation = In − out

Steady flow: $\frac{\partial}{\partial t} = 0$ so $\int_{CS} \rho \vec{V} \cdot n \, dA = 0$

in = out

Mass flow in = mass flow out − multiple inlets & exits

expressions for mass flow rate

typical people write $\dot{m} = \rho Q = \rho A V$ velocity normal to control surface

but actually $\dot{m} = \int \rho \vec{V} \cdot n \, dA$ so V above is an average or representative velocity average velocity

$$\bar{V} = \frac{\int_A \rho V \cdot n \, dA}{\rho A}$$

for incompressible fluids $\bar{V} = \frac{\int_A \rho V \cdot n \, dA}{\rho A} = \frac{\rho \int_A V \cdot n \, dA}{\rho A} = \frac{\int V \cdot n \, dA}{A}$

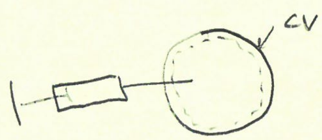
if we know Q

$$\bar{V} = Q/A$$

Note: velocities are not normally uniform

Example:

Pumping up a basketball

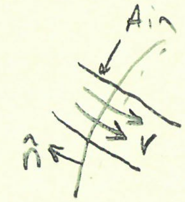


$$\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot n \, dA = 0$$

$$\int \rho \, dV = M_{cv}$$

so $\frac{\partial M_{cv}}{\partial t} + \int_{CS} \rho \vec{V} \cdot n \, dA = 0$

look at surface



$$\int_{CS} \rho \vec{V} \cdot n \, dA = \rho \int -V_{in} \, dA = -\rho V_{in} A_{in}$$

assume constant density

negative because it is in flow $\vec{V}$ & $\hat{n}$ in opposite direction

assuming V_{in} is uniform over the control surface

thus

$$\frac{\partial M_{cv}}{\partial t} - \rho V_{in} A_{in} = 0 \Rightarrow \frac{dM_{cv}}{dt} = \rho V_{in} A_{in}$$