Chapter 5

Uncertainty Analysis

5.1 INTRODUCTION

Whenever we plan a test or later report a test result, we need to know something about the quality of the results. Uncertainty analysis provides a methodical approach to estimating the quality of the results from an anticipated test or from a completed test. This chapter focuses on how to estimate the "± what?" in a planned test or in a stated test result.

Suppose the competent dart thrower of Chapter 1 tossed several practice rounds of darts at a bull's-eye. This would give us a good idea of the thrower's tendencies. Then, let the thrower toss another round. Without looking, can you guess where the darts will hit? Test measurements that include systematic and random error components are much like this. We can calibrate a measurement system to get a good idea of its behavior and accuracy. However, from the calibration we can only estimate how well any measured value might estimate the actual "true" value in a subsequent measurement.

Errors are a property of the measurement. Measurement is the process of assigning a value to a physical variable based on a sampling from the population of that variable. Error causes a difference between the value assigned by measurement and the true value of the population of the variable. Measurement errors are introduced from various elements, for example, the individual instrument calibrations, the data set finite statistics, and the approach used. But because we do not know the true value and we only know the measured values, we do not know the exact values of errors. Instead, we draw from what we do know about the measurement to estimate a range of probable error. This estimate is an assigned value called the *uncertainty*. The uncertainty describes an interval about the measured value within which we suspect that the true value must fall with a stated probability. *Uncertainty analysis* is the process of identifying, quantifying, and combining the errors.

Uncertainty is a property of the result. The outcome of a measurement is a result, and the uncertainty quantifies the quality of that result. Uncertainty analysis provides a powerful design tool for evaluating different measurement systems and methods, designing a test plan, and reporting uncertainty. This chapter presents a systematic approach for identifying, quantifying, and combining the estimates of the errors in a measurement. While the chapter stresses the methodology of analyses, we emphasize the concomitant need for an equal application of critical thinking and professional judgment in applying the analyses. The quality of an uncertainty analysis depends on the engineer's knowledge of the test, the measured variables, the equipment, and the measurement procedures (1).

Errors are effects, and uncertainties are numbers. While errors are the effects that cause a measured value to differ from the true value, the uncertainty is an assigned numerical value that quantifies the probable range of these errors.

This chapter approaches uncertainty analysis as an evolution of information from test design through final data analysis. While the structure of the analysis remains the same at each step, the number of errors identified and their uncertainty values may change as more information becomes available. In fact, the uncertainty in the result may increase. There is no exact answer to an analysis, just the result from a reasonable approach using honest numbers. This is the nature of an uncertainty analysis.

There are two accepted professional documents on uncertainty analysis. The American National Standards Institute/American Society of Mechanical Engineers (ANSI/ASME) Power Test Codes (PTC) 19.1 Test Uncertainty (2) is the United States engineering test standard, and our approach favors that method. The International Organization on Standardization's "Guide to the Expression of Uncertainty in Measurement" (ISO GUM) (1) is an international metrology standard. The two differ in some terminology and how errors are cataloged. For example, PTC 19.1 refers to random and systematic errors, terms that classify errors by how they manifest themselves in the measurement. ISO GUM refers to type A and type B errors, terms that classify errors by how their uncertainties are estimated. These differences are real but they are not significant to the outcome. Once past the classifications, the two methods are quite similar. The important point is that the end outcome of an uncertainty analysis by either method will yield a similar result!

Upon completion of this chapter, the reader will be able to

- explain the relation between an error and an uncertainty,
- execute an appropriate uncertainty analysis regardless of the level and quantity of information available,
- explain the differences between systematic and random errors and treat their assigned uncertainties,
- analyze a test system and test approach from test design through data presentation to assign and propagate uncertainties, and
- propagate uncertainties to understand their impact on the final statement of a result.

5.2 MEASUREMENT ERRORS

In the discussion that follows, errors are grouped into two categories: systematic error and random error. We do not consider measurement blunders that result in obviously fallacious data—such data should be discarded.

Consider the repeated measurement of a variable under conditions that are expected to produce the same value of the measured variable. The relationship between the true value of the population and the measured data set, containing both systematic and random errors, can be illustrated as in Figure 5.1. The total error in a set of measurements obtained under seemingly fixed conditions can be described by the systematic errors and the random errors in those measurements. The systematic errors shift the sample mean away from the true mean by a fixed amount, and within a sample of many measurements, the random errors bring about a distribution of measured values about the sample mean. Even a so-called accurate measurement contains small amounts of systematic and random errors.

Measurement errors enter during all aspects of a test and obscure our ability to ascertain the information that we desire: the true value of the variable measured. If the result depends on more than one measured variable, these errors further propagate to the result. In Chapter 4, we stated that the best estimate of the true value sought in a measurement is provided by its sample mean value and

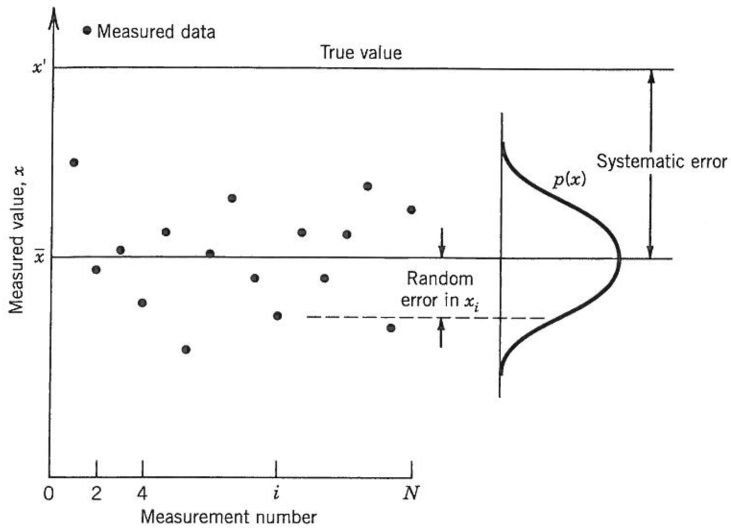

Figure 5.1 Distribution of errors on repeated measurements.

the uncertainty in that value,

$$x' = \overline{x} \pm u_x \quad (P\%) \tag{4.1}$$

But we considered only the random uncertainty due to the statistics of a measured data set. In this chapter, we extend this to uncertainty analysis so that the u_x term contains the uncertainties assigned to all known errors. Certain assumptions are implicit in an uncertainty analysis:

1. The test objectives are known and the measurement itself is a clearly defined process.
2. Any known corrections for systematic error have been applied to the data set, in which case the systematic uncertainty assigned is the uncertainty of the correction.
3. Except where stated otherwise, we assume a normal distribution of errors and reporting of uncertainties.
4. Unless stated otherwise, the errors are assumed to be independent (uncorrelated) of each other. But some errors are correlated, and we discuss how to handle these in Section 5.9.
5. The engineer has some "experience" with the system components.

In regards to item 5, by "experience" we mean that the engineer either has prior knowledge of what to expect from a system or can rely on the manufacturer's performance specifications or on information from the technical literature.

We might begin the design of an engineering test with an idea and some catalogs, and end the project after data have been obtained and analyzed. As with any part of the design process, the uncertainty analysis evolves as the design of the measurement system and process matures. We discuss uncertainty analysis for the following measurement situations: (1) design stage, where tests are planned but information is limited; (2) advanced stage or single measurement, where additional information about process control can be used to improve a design-stage uncertainty estimate; and (3) multiple measurements, where all available test information is combined to assess the uncertainty in a test result. The methods for situation 3 follow current engineering standards.

5.3 DESIGN-STAGE UNCERTAINTY ANALYSIS

Design-stage uncertainty analysis refers to an analysis performed in the formulation stage prior to a test. It provides only an estimate of the minimum uncertainty based on the instruments and method chosen. If this uncertainty value is too large, then alternate approaches will need to be found. So, it is useful for selecting instruments and selecting measurement techniques. At the test design stage, the measurement system and associated procedures may be but a concept. Often little may be known about the instruments, which in many cases might still be just pictures in a catalog. Major facilities may need to be built and equipment ordered with a considerable lead time. Uncertainty analysis at this time is used to assist in selecting equipment and test procedures based on their relative performance. In the design stage, distinguishing between systematic and random errors might be too difficult to be of concern. So for this initial discussion, consider only sources of error and their assigned uncertainty in general. A measurement system usually consists of sensors and instruments, each with their respective contributions to system uncertainty. We first discuss individual contributions to uncertainty.

Even when all errors are otherwise zero, a measured value must be affected by our ability to resolve the information provided by the instrument. This *zero-order uncertainty* of the instrument, u_0, assumes that the variation expected in the measured values will be only that amount due to instrument resolution and that all other aspects of the measurement are perfectly controlled. Essentially, u_0 is an estimate of the expected random uncertainty caused by the data scatter due to instrument resolution.

In lieu of any other information, assign a numerical value to u_0 of one-half of the analog instrument resolution[1] or to equal to its digital least count. This value will reasonably represent the uncertainty interval on either side of the reading with a probability of 95%. Then,

$$u_0 = \frac{1}{2}\text{resolution} = 1\,\text{LSD} \tag{5.1}$$

where LSD refers to the least significant digit of the readout.

Note that because we assume that the error has a normal distribution with its uncertainty applied equally to either side of the reading, we could write this as

$$u_0 = \pm\frac{1}{2}\ \text{resolution}\ (95\%)$$

But unless specifically stated otherwise, the $\pm$ sign for the uncertainty will be assumed for any computed uncertainty value and applied only when writing the final uncertainty interval of a result.

The second piece of information that is usually available is the manufacturer's statement concerning instrument error. We can assign this stated value as the *instrument uncertainty*, u_c. Essentially, u_c is an estimate of the expected systematic uncertainty due to the instrument. If no probability level is provided with such information, a 95% level can be assumed.

Sometimes the instrument errors are delineated into parts, each part due to some contributing factor (Table 1.1). A probable estimate in u_c can be made by combining the uncertainties of known errors in some reasonable manner. An accepted approach of combining uncertainties is termed the *root-sum-squares* (RSS) method.

[1] It is possible to assign a value for u_0 that differs from one-half the scale resolution. Discretion should be used. Instrument resolution is likely described by either a normal or a rectangular distribution, depending on the instrument.

Combining Elemental Errors: RSS Method

Each individual measurement error interacts with other errors to affect the uncertainty of a measurement. This is called *uncertainty propagation*. Each individual error is called an "elemental error." For example, the sensitivity error and linearity error of a transducer are two elemental errors, and the numbers associated with these are their uncertainties. Consider a measurement of x that is subject to some K elements of error, each of uncertainty u_k, where $k = 1, 2, \ldots, K$. A realistic estimate of the uncertainty in the measured variable, u_x, due to these elemental errors can be computed using the RSS *method* to propagate the elemental uncertainties:

$$\begin{aligned} u_x &= \sqrt{u_1^2 + u_2^2 + \cdots + u_k^2} \\ &= \sqrt{\sum_{k=1}^{K} u_k^2} \quad (P\%) \end{aligned} \tag{5.2}$$

The RSS method of combining uncertainties is based on the assumption that the square of an uncertainty is a measure of the variance (i.e., s^2) assigned to an error, and the propagation of these variances yields a probable estimate of the total uncertainty. Note that it is imperative to maintain consistency in the units of each uncertainty in Equation 5.2 and that each uncertainty term be assigned at the same probability level.

In test engineering, it is common to report final uncertainties at a 95% probability level ($P\% = 95\%$), and this is equivalent to assuming the probability covered by two standard deviations. When a probability level equivalent to a spread of one standard deviation is used, this uncertainty is called the "standard" uncertainty (1, 2). For a normal distribution, a standard uncertainty is a 68% probability level. Whatever level is used, consistency is important.

Design-Stage Uncertainty

The *design-stage uncertainty*, u_d, for an instrument or measurement method is an interval found by combining the instrument uncertainty with the zero-order uncertainty,

$$u_d = \sqrt{u_0^2 + u_c^2} \quad (P\%) \tag{5.3}$$

This procedure for estimating the design-stage uncertainty is outlined in Figure 5.2. The design-stage uncertainty for a test system is arrived at by combining each of the design-stage uncertainties for each component in the system using the RSS method while maintaining consistency of units and confidence levels.

Due to the limited information used, a design-stage uncertainty estimate is intended only as a guide for selecting equipment and procedures before a test, and is never used for reporting results. *If additional information about other measurement errors is known at the design stage, then their*

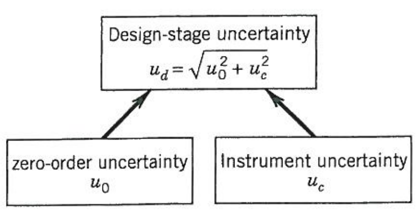

Figure 5.2 Design-stage uncertainty procedure in combining uncertainties.

Table 5.3 Data-Reduction Error Source Group

Element	Error Source[a]
1	Curve fit error
2	Truncation error
3	Modeling error
etc.	

[a]Systematic error or random error in each element.

5.5 SYSTEMATIC AND RANDOM ERRORS

Systematic Error

A systematic error[2] remains constant in repeated measurements under fixed operating conditions. A systematic error may cause either a high or a low offset in the estimate of the true value of the measured variable. Because its effect is constant, it can be difficult to estimate the value of a systematic error or in many cases even recognize its presence. Accordingly, an estimate of the range of systematic error is represented by an interval, defined as $\pm b$. The value b is the estimate of the *systematic standard uncertainty*. Its interval has a confidence level of one standard deviation, equivalent to a probability level of 68% for a normal distribution. The *systematic uncertainty* at any confidence level is given by $t_{\nu,P}b$, or simply tb. The interval defined by the systematic uncertainty at the 95% probability level is written as

$$\pm B = \pm 2b \quad (95\%) \tag{5.4}$$

which assigns a value of $t = 2$. This t value assumes large degrees of freedom in an assigned systematic uncertainty for which $t = 1.96$, which is rounded to 2 for convenience (2).

The reader has probably experienced systematic errors in measurements. Improperly using the floating tang at the end of a metal tape measure will offset the measurement, a systematic error. A more obvious example is reporting the barefoot height of a person based on a measurement taken while the person was wearing high-heeled shoes. In this case this systematic error, a data-acquisition error, is the height of the heels. But these errors are obvious!

Consider a home bathroom scale; does it have a systematic error? How might we assign an uncertainty to its indicated weight? Perhaps we can calibrate the scale using calibrated standard masses, account for local gravitational acceleration, and correct the output, thereby estimating the systematic error of the measurement (i.e., direct calibration against a local standard). Or perhaps we can compare it to a measurement taken in a physician's office or at the gym and compare each reading (i.e., a sort of interlaboratory comparison). Or perhaps we can carefully measure the person's volume displacement in water and compare the results to estimate differences (i.e., concomitant methodology). Or, we can use the specification provided by the manufacturer (i.e., experience). Without any of the above, what value would we assign? Would we even suspect a systematic error?

[2] This error was called a "bias" error in engineering documents prior to the 1990s.

But let us think about this. The insidious aspect of systematic error has been revealed. Why doubt a measurement indication and suspect a systematic error? The mean value of the data set may be offset from some true value that we do not know. Figuratively speaking, there will be no shoe heels staring at us. Experience teaches us to think through each measurement carefully because systematic error is always present at some magnitude. We see that it is difficult to estimate systematic error without comparison, so a good design should include some means to estimate it. Various methodologies can be utilized: (1) calibration, (2) concomitant methodology, (3) interlaboratory comparisons, or (4) judgment/experience. When available, calibration using a suitable standard and method can reduce instrument systematic error to predictable intervals and estimate its associated uncertainty. A quality instrument may come with a certified calibration certificate. Concomitant methodology, which is using different methods of estimating the same thing, allows for comparing the results. Concomitant methods that depend on different physical measurement principles are preferable, as are methods that rely on calibrations that are independent of each other. In this regard, analytical methods could be used for comparison[3] or at least to estimate the range of systematic error due to influential sources such as environmental conditions, instrument response errors, and loading errors. Lastly, an elaborate but good approach is through interlaboratory comparisons of similar measurements, an excellent replication method. This approach introduces different instruments, facilities, and personnel into an otherwise similar measurement procedure. The variations in the results between facilities provide a statistical estimate of the systematic uncertainty (2).

In lieu of the above, a judgment value based on past experience may have to be assigned; these values are usually understood to be made at the 95% confidence level. For example, the value that first came to mind to you in the bathroom scale example above likely covered a 95% interval.

Note that calibration cannot eliminate systematic error, but it may reduce uncertainty. Consider the calibration of a temperature transducer against a National Institute of Standards and Technology (NIST) standard certified to be correct to within 0.01°C. If the calibration data show that the transducer output has a systematic offset of 0.2°C relative to the standard, then we would just correct all the data obtained with this transducer by 0.2°C. Simple enough, we correct it! But the standard itself still has an intrinsic systematic uncertainty of 0.01°C, and this uncertainty remains in the calibrated transducer. We would include any uncertainty in the correction value applied.

Random Error

When repeated measurements are made under fixed operating conditions, random errors manifest themselves as scatter of the measured data. Random error[4] is introduced through the repeatability and resolution of the measurement system components, calibration, and measurement procedure and technique; by the measured variable's own temporal and spatial variations; and by the variations in the process operating and environmental conditions from which measurements are taken.

The estimate of the probable range of a random error is given by its random uncertainty. The *random standard uncertainty*, $s_{\bar{x}}$, is defined by the interval given by $\pm s_{\bar{x}}$, where

$$s_{\bar{x}} = s_x/\sqrt{N} \tag{5.5}$$

[3] Smith and Wenhofer (3) provide examples for determining jet engine thrust, and several complementary measurements are used with an energy balance to estimate the uncertainty assigned to the systematic error.

[4] This error was called a "precision" error in engineering documents prior to the 1990s.

with degrees of freedom $\nu = N - 1$ and assuming the errors are normally distributed.[5] The interval has a confidence level of one standard deviation, equivalent to a probability of 68% for a population of x having a normal distribution. The *random uncertainty* at a desired confidence level is defined by the interval $\pm t_{\nu,P} s_{\overline{x}}$, where t is found from Table 4.4.

5.6 UNCERTAINTY ANALYSIS: ERROR PROPAGATION

Suppose we want to determine how long it would take to fill a swimming pool from a garden hose. One way is to measure the time required to fill a bucket of known volume to estimate the flow rate from the garden hose. Armed with a measurement of the volume of the pool, we can calculate the time to fill the pool. Clearly, small errors in estimating the flow rate from the garden hose would translate into large differences in the time required to fill the pool! Here we are using measured values, the flow rate and volume, to estimate a result, the time required to fill the pool.

Very often in engineering, results are determined through a functional relationship with measured values. For example, we just calculated a flow rate above by measuring time, t, and bucket volume, $\forall$, since $Q = f(t, \forall) = \forall / t$. But how do uncertainties in either measured quantity contribute to uncertainty in flow rate? Is the uncertainty in Q more sensitive to uncertainty in volume or in time? More generally, how are uncertainties in variables propagated to a calculated result? We now explore these questions.

Propagation of Error

A general relationship between some dependent variable y and a measured variable x, that is, $y = f(x)$, is illustrated in Figure 5.3. Now suppose we measure x a number of times at some operating condition so as to establish its sample mean value and the uncertainty due to random error in this mean value, $t_{\nu,P} s_{\overline{x}}$, which for convenience we write simply as $t s_{\overline{x}}$. This implies that, neglecting other random and systematic errors, the true value for x lies somewhere within the interval $\overline{x} \pm t s_{\overline{x}}$. It is reasonable to assume that the true value of y, which is determined from the measured values of x, falls within the interval defined by

$$\overline{y} \pm \delta y = f(\overline{x} \pm t s_{\overline{x}}) \tag{5.6}$$

Expanding this as a Taylor series yields

$$\overline{y} \pm \delta y = f(\overline{x}) \pm \left[\left(\frac{dy}{dx}\right)_{x=\overline{x}} t s_{\overline{x}} + \frac{1}{2}\left(\frac{d^2 y}{dx^2}\right)_{x=\overline{x}} (t s_{\overline{x}})^2 + \cdots \right] \tag{5.7}$$

By inspection, the mean value for y must be $f(\overline{x})$ so that the term in brackets estimates $\pm\delta y$. A linear approximation for δy can be made, which is valid when $t s_{\overline{x}}$ is small and neglects the higher order terms in Equation 5.7, as

$$\delta y \approx \left(\frac{dy}{dx}\right)_{x=\overline{x}} t s_{\overline{x}} \tag{5.8}$$

The derivative term, $(dy/dx)_{x=\overline{x}}$, defines the slope of a line that passes through the point specified by $\overline{x}$. For small deviations from the value of $\overline{x}$, this slope predicts an acceptable, approximate

[5] The estimate of standard uncertainty when estimated from a rectangular distribution (11) is $(b - a)/\sqrt{12}$, where b and a were defined in Table 4.2. The probability is about 58%.

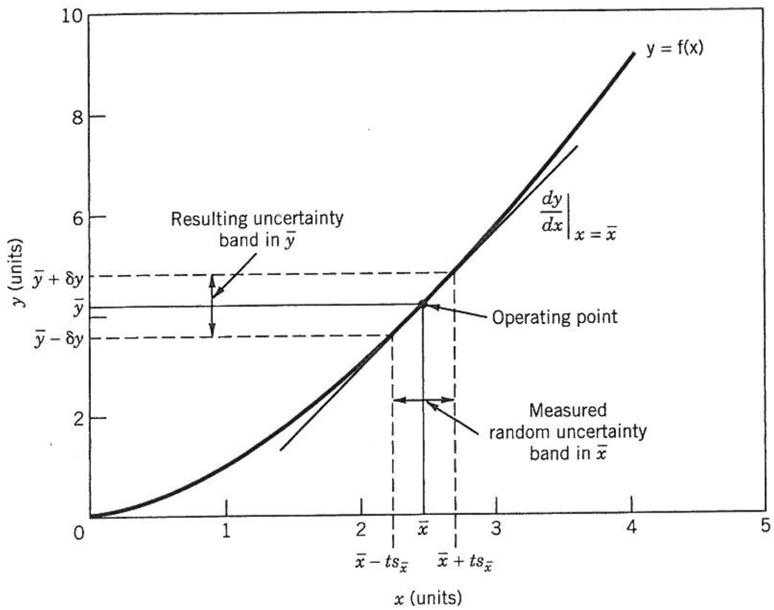

Figure 5.3 Relationship between a measured variable and a resultant calculated using the value of that variable.

relationship between $ts_{\bar{x}}$ and δy. The derivative term is a measure of the sensitivity of y to changes in x. Since the slope of the curve can be different for different values of x, it is important to evaluate the slope using a representative value of x. The width of the interval defined by $\pm ts_{\bar{x}}$ corresponds to $\pm\delta y$, within which we should expect the true value of y to lie. Figure 5.3 illustrates the concept that errors in a measured variable are propagated through to a resultant variable in a predictable way. In general, we apply this analysis to the errors that contribute to the uncertainty in x, written as u_x. The uncertainty in x is related to the uncertainty in the resultant y by

$$u_y = \left(\frac{dy}{dx}\right)_{x=\bar{x}} u_x \tag{5.9}$$

Compare the similarities between Equations 5.8 and 5.9 and in Figure 5.3.

This idea can be extended to multivariable relationships. Consider a result R, which is determined through some functional relationship between independent variables $x_1, x_2, \ldots, x_L$ defined by

$$R = f_1\{x_1, x_2, \ldots, x_L\} \tag{5.10}$$

where L is the number of independent variables involved. Each variable contains some measure of uncertainty that affects the result. The best estimate of the true mean value R' would be stated as

$$R' = \overline{R} \pm u_R \quad (\text{P\%}) \tag{5.11}$$

where the sample mean of R is found from

$$\overline{R} = f_1\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_L\} \tag{5.12}$$

and the uncertainty in $\overline{R}$ is found from

$$u_R = f_1\{u_{\overline{x}_1}, u_{\overline{x}_2}, \ldots, u_{\overline{x}_L}\} \tag{5.13}$$

In Equation 5.13, each $u_{\overline{x}_i}$, $i = 1, 2, \ldots, L$ represents the uncertainty associated with the best estimate of x_1 and so forth through x_L. The value of u_R reflects the contributions of the individual uncertainties as they are propagated through to the result.

A general sensitivity index, θ_i, results from the Taylor series expansion, Equation 5.9, and the functional relation of Equation 5.10 and is given by

$$\theta_i = \frac{\partial R}{\partial x_{i_{x=\overline{x}}}} \quad i = 1, 2, \ldots, L \tag{5.14}$$

The sensitivity index relates how changes in each x_i affect R. Equation 5.14 can also be estimated numerically using finite differencing methods (5), which can be easily done within a spreadsheet or symbolic software package. The index is evaluated using either the mean values or, lacking these estimates, the expected nominal values of the variables.

The contribution of the uncertainty in x to the result R is estimated by the term $\theta_i u_{\overline{x}_i}$. The most probable estimate of u_R is generally accepted as that value given by the second power relation (4), which is the square root of the sum of the squares (RSS). The propagation of uncertainty in the variables to the result is by

$$u_R = \left[\sum_{i=1}^{L} (\theta_i u_{\overline{x}_i})^2\right]^{1/2} \tag{5.15}$$

Sequential Perturbation

A numerical approach can also be used to estimate the propagation of uncertainty through to a result that circumvents the direct differentiation of the functional relations (6). The approach is handy to reduce data already stored in discrete form.

The method uses a finite difference method to approximate the derivatives:

1. Based on measurements for the independent variables under some fixed operating condition, calculate a result R_o where $R_o = f(x_1, x_2, \ldots, x_L)$. This value fixes the operating point for the numerical approximation (e.g., see Fig. 5.3).
2. Increase the independent variables by their respective uncertainties and recalculate the result based on each of these new values. Call these values R_i^+. That is,

$$\begin{aligned} R_1^+ &= f(x_1 + u_{x1}, x_2, \ldots, x_L), \\ R_2^+ &= f(x_1, x_2 + u_{x2}, \ldots, x_L), \ldots \\ R_L^+ &= f(x_1, x_2, \ldots, x_L + u_{xL}), \end{aligned} \tag{5.16}$$

3. In a similar manner, decrease the independent variables by their respective uncertainties and recalculate the result based on each of these new values. Call these values R_i^-.
4. Calculate the differences δR_i^+ and δR_i^- for $i = 1, 2, \ldots, L$

$$\begin{aligned} \delta R_i^+ &= R_i^+ - R_o \\ \delta R_i^- &= R_i^- - R_o \end{aligned} \tag{5.17}$$

5. Evaluate the approximation of the uncertainty contribution from each variable,

$$\delta R_i = \frac{\delta R_i^+ - \delta R_i^-}{2} \approx \theta_i u_i \tag{5.18}$$

Then, the uncertainty in the result is

$$u_R = \left[\sum_{i=1}^{L} (\delta R_i)^2 \right]^{1/2} \tag{5.19}$$

Equations 5.15 and 5.19 provide two methods for estimating the propagation of uncertainty to a result. In most cases, each equation yields nearly the identical result and the choice of method is left to the user. The method can also be used to estimate just the sensitivity index of Equation 5.14 (2). In this case, steps 2 and 3 would apply a small deviation value, typically 1% of the nominal value of the variable, used in place of the actual uncertainty to estimate the derivative (5).

We point out that sometimes either method may calculate unreasonable estimates of u_R. When this happens the cause can be traced to a sensitivity index that changes rapidly with small changes in the independent variable x_i coupled with a large value of the uncertainty u_{x_i}. This occurs when the operating point is close to an minima or maxima inflection in the functional relationship. In these situations, the engineer should examine the cause and extent of the variation in sensitivity and use a more accurate approximation for the sensitivity, including using the higher order terms in the Taylor series of Equation 5.7.

In subsequent sections, we develop methods to estimate the uncertainty values from available information.

Example 5.3

For a displacement transducer having the calibration curve, $y = KE$, estimate the uncertainty in displacement y for $E = 5.00$ V, if $K = 10.10$ mm/V with $u_K = \pm 0.10$ mm/V and $u_E = \pm 0.01$ V at 95% confidence.

KNOWN $y = KE$

$E = 5.00$ V $\quad u_E = 0.01$ V

$K = 10.10$ mm/V $\quad u_K = 0.10$ mm/V

FIND u_y

SOLUTION Based on Equations 5.12 and 5.13, respectively,

$$\bar{y} = f(\overline{E}, \overline{K}) \quad \text{and} \quad u_y = f(u_E, u_K)$$

From Equation 5.15, the uncertainty in the displacement at $y = KE$ is

$$u_y = \left[(\theta_E u_E)^2 + (\theta_K u_K)^2 \right]^{1/2}$$

where the sensitivity indices are evaluated from Equation 5.14 as

$$\theta_E = \frac{\partial y}{\partial E} = K \quad \text{and} \quad \theta_K = \frac{\partial y}{\partial K} = E$$