ECE 510 Lecture 7
Goodness of Fit, Maximum Likelihood

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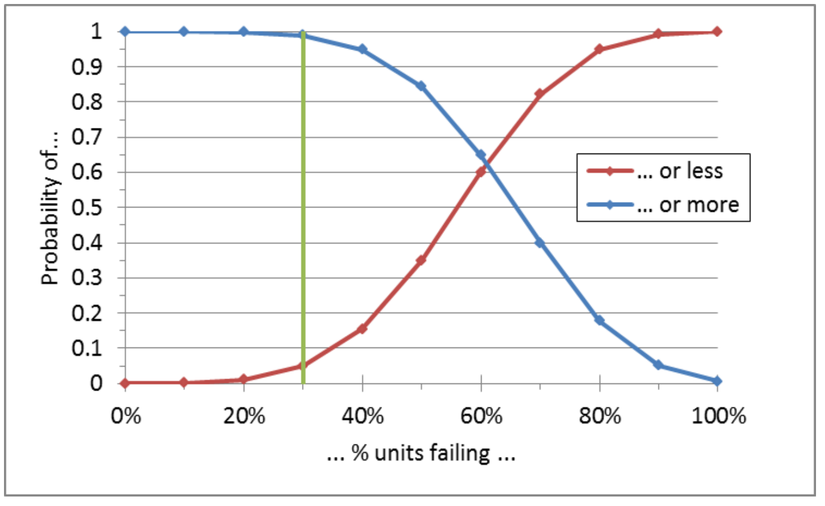
Confidence Limits

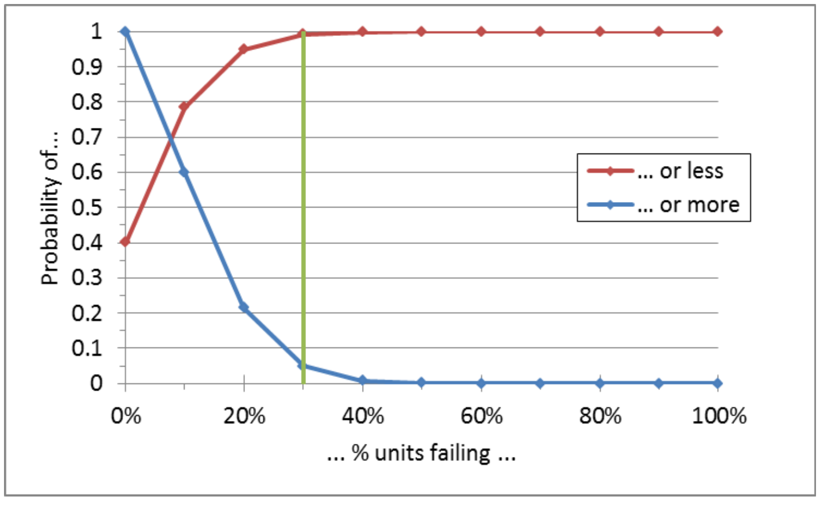
Binomial Confidence Limits (Solution 6.2)

UCL: Prob of 30% units failing *or less* is < 0.05



UCL = 60.7%

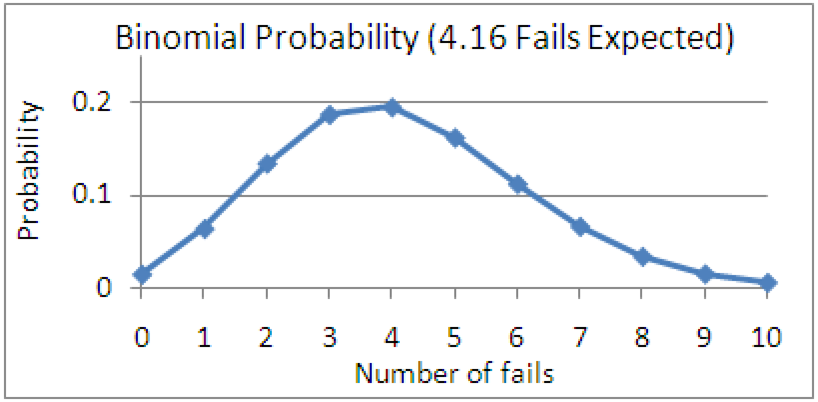
LCL: Prob of 30% units failing *or more* is < 0.05



LCL = 8.7%

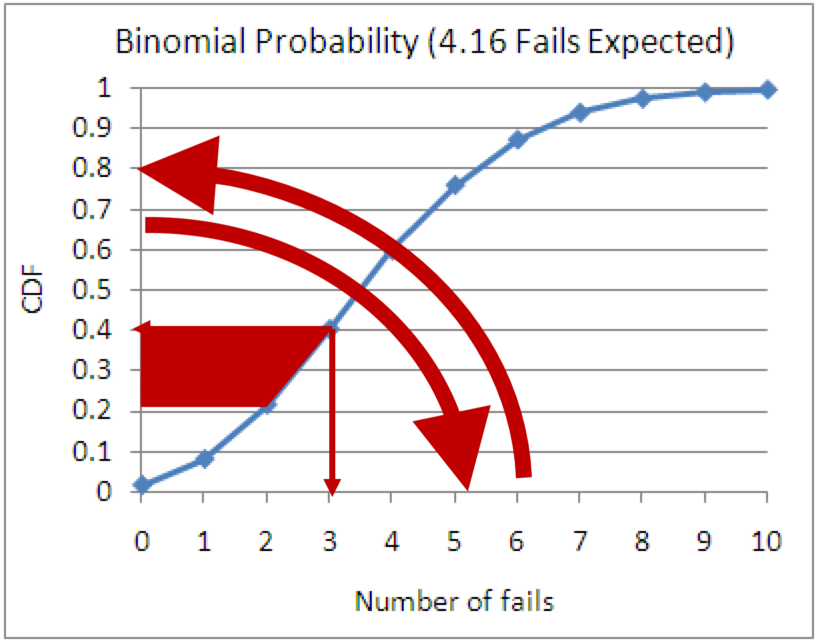
Synthesizing Binomial Data

1000 units
0.416% prob of fail
4.16 units expected

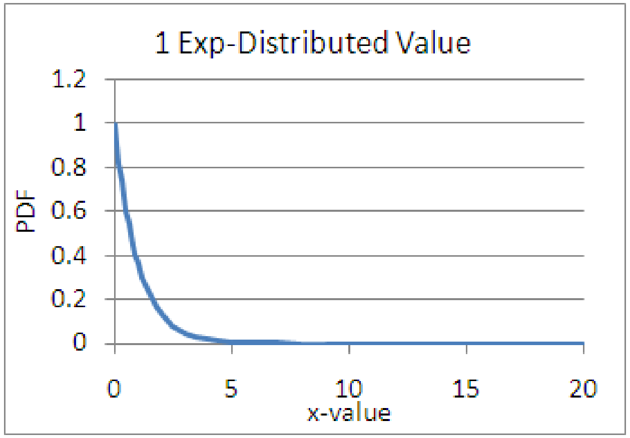


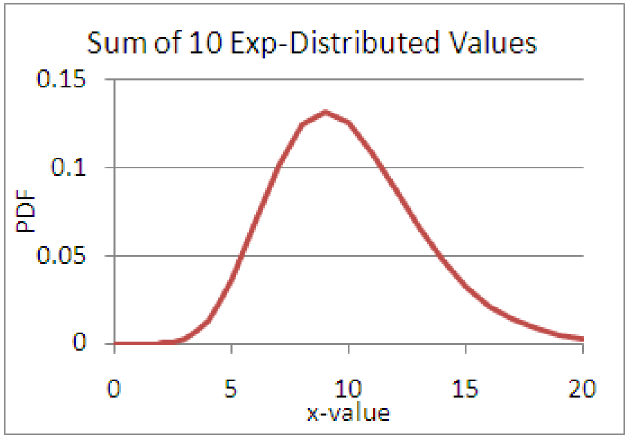
BINOMDIST(fails, samples, prob, TRUE**)**
CRITBINOM(samples, prob, CDF**)**

RAND()



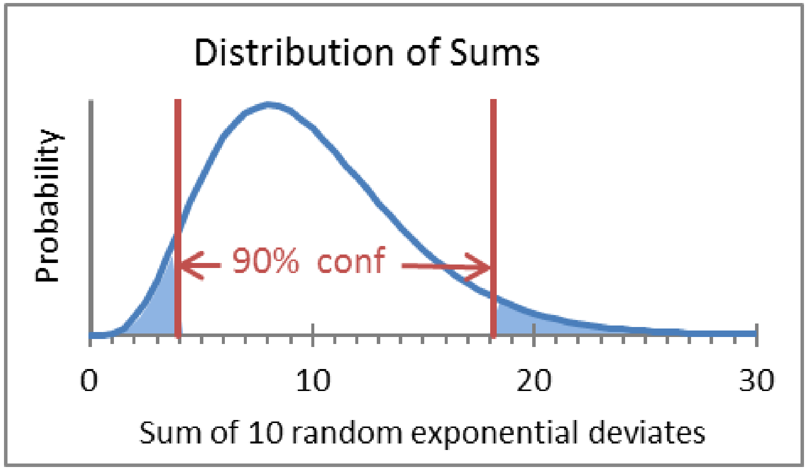
Why Chi-Square for Exponential CL?



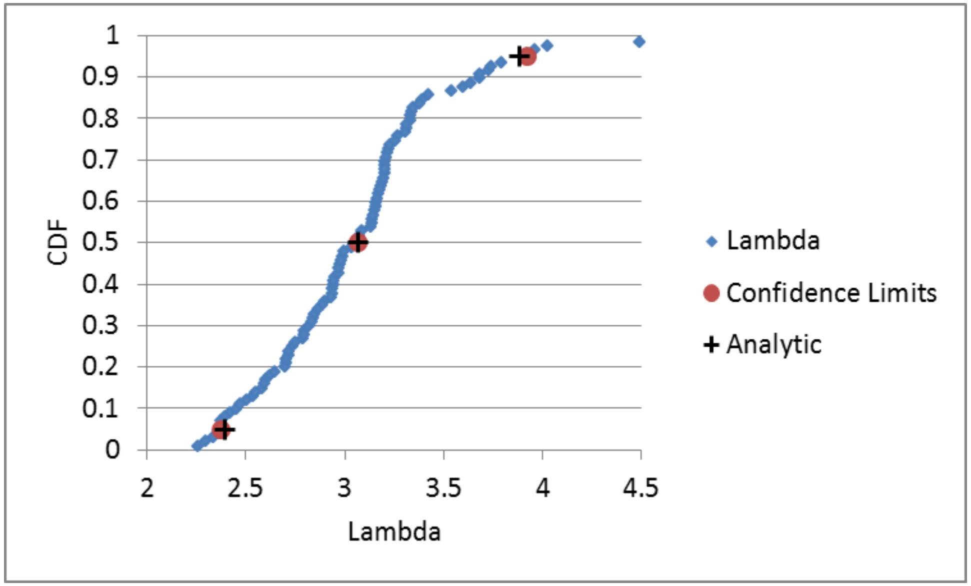


- For $f(t) = \lambda\, e^{-\lambda t}$, best estimate for $1/\lambda$ is $\frac{1}{\mathrm{N}}\sum t_i$ where t_i are the data
- So, what is the *distribution* of $\sum t_i$ where t_i are distributed exponentially?

- Answer: a gamma or a chi-square distribution
- Confidence intervals taken from that



Solution 6.3



Confidence Limits	CL values	MC	Analytic
Upper CL	0.95	3.920005	3.884124
Best estimate	0.5	3.068655	3.068655
Lower CL	0.05	2.376862	2.391386

=LARGE($C$24:$C$123, 100*(1-C12))

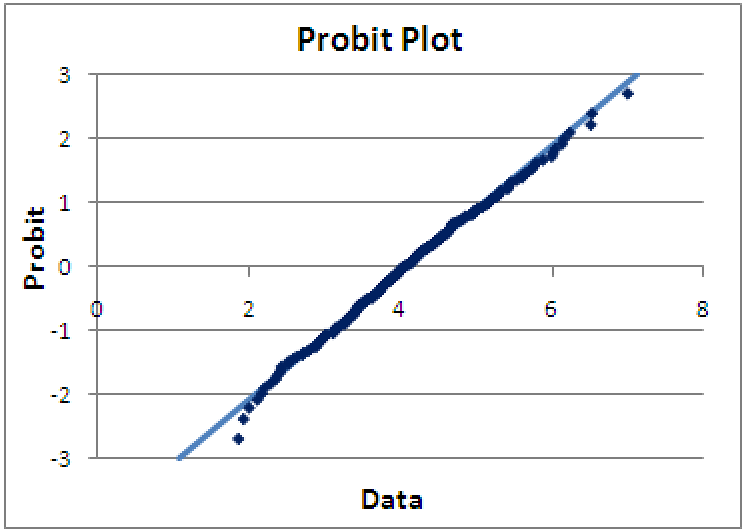
=E11 * CHIINV(95%, 2*($C$5))/(2*$C$5)

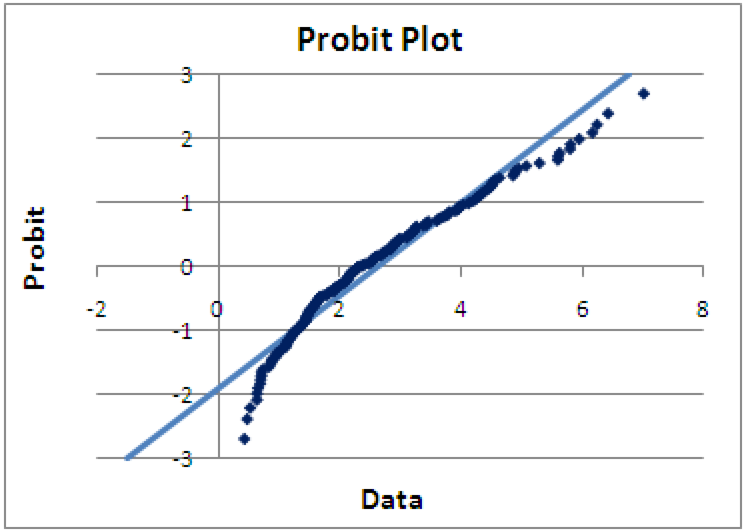
Confidence Limits Summary

- Confidence limits (UCL and LCL) are values between which (2-sided) or above or below which (1-sided) the true population value falls with <confidence level> probability
  - Units are whatever units your data uses
- Confidence level is the probability that the true value lies between (or above or below) your confidence limit(s).
- “CL” can mean either confidence limit or confidence level
  - Use context to decide
- Confidence limits can be calculated
  - Analytically (best if available)
  - Monte Carlo (will work for any distribution)
  - Likelihood methods (coming soon)
- Monte Carlo confidence limits work regardless of how you calculate the best estimate

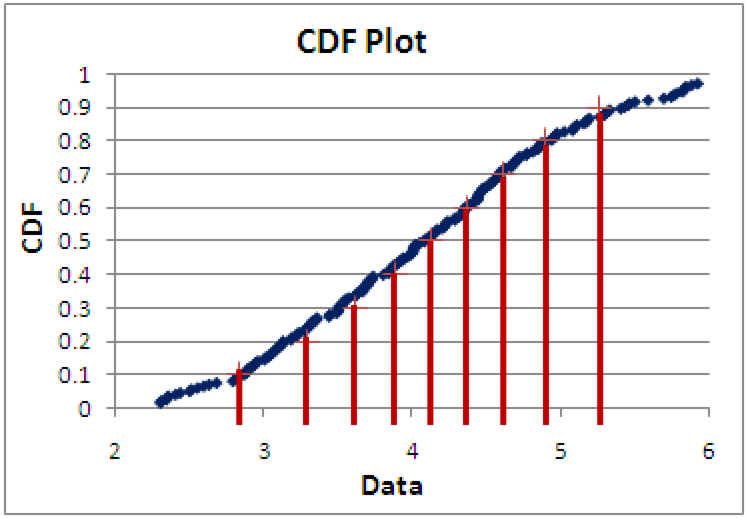
Goodness of Fit Tests

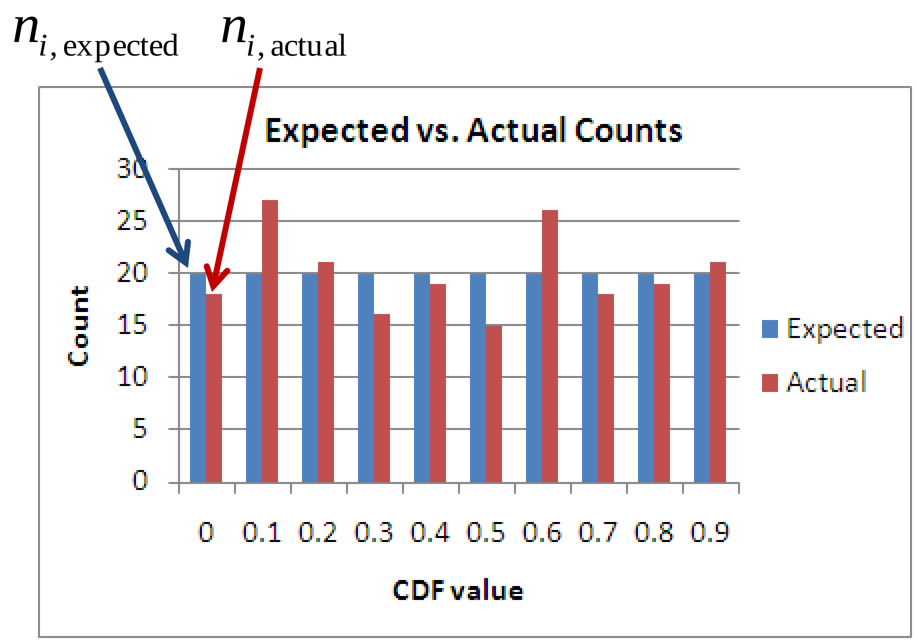
How Good Is Good Enough?

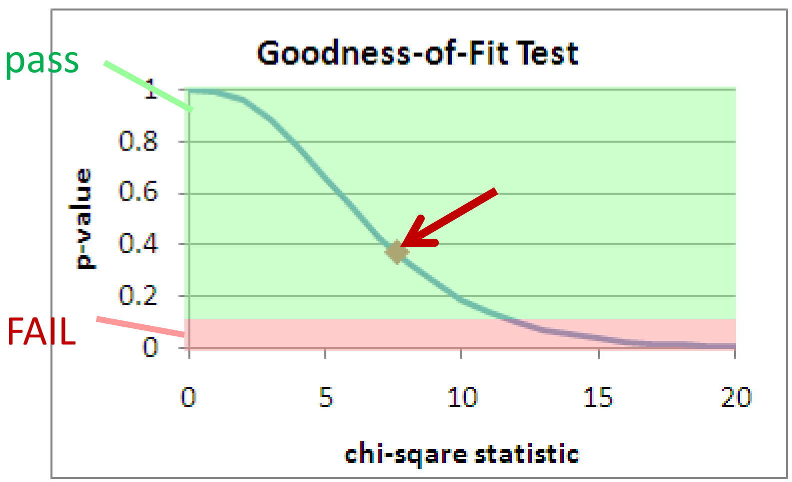




Pearson's Chi-Squared Test







$$\chi_i^2 = \frac{\left(n_{i,\text{actual}} - n_{i,\text{expected}}\right)^2}{n_{i,\text{expected}}}$$

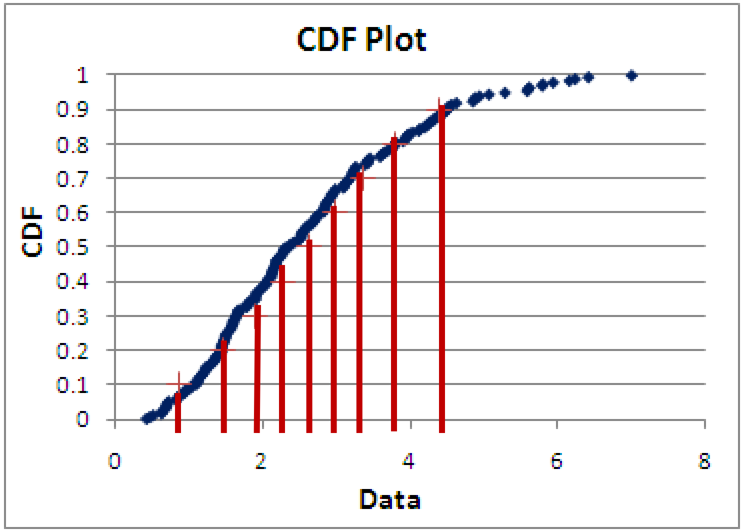
Chi-Sq:

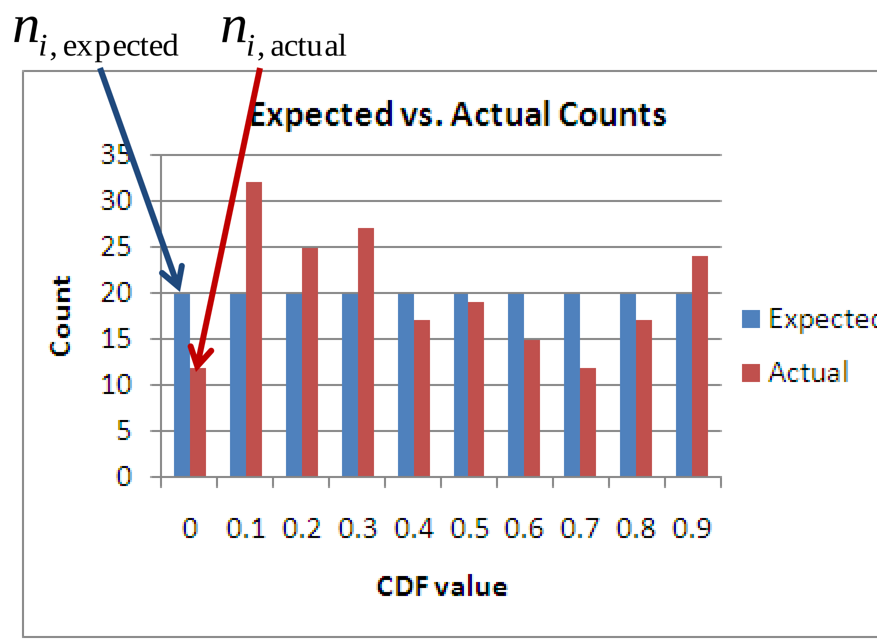
$$\chi^2 = \sum_i \chi_i^2$$

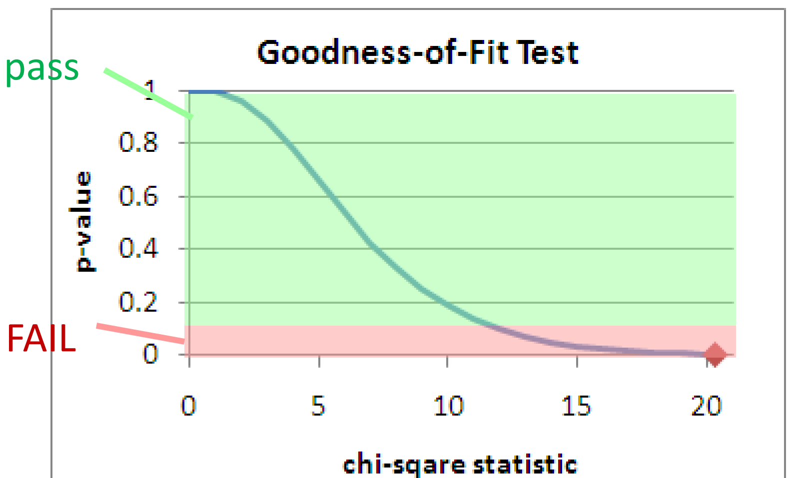
DoF = bins – parameters – 1 = 7

p-value = CHIINV(Chi-Sq, DoF)
=0.378

Pearson's Chi-Squared Test







$$\chi_i^2 = \frac{\left(n_{i,\text{actual}} - n_{i,\text{expected}}\right)^2}{n_{i,\text{expected}}}$$

Chi-Sq:

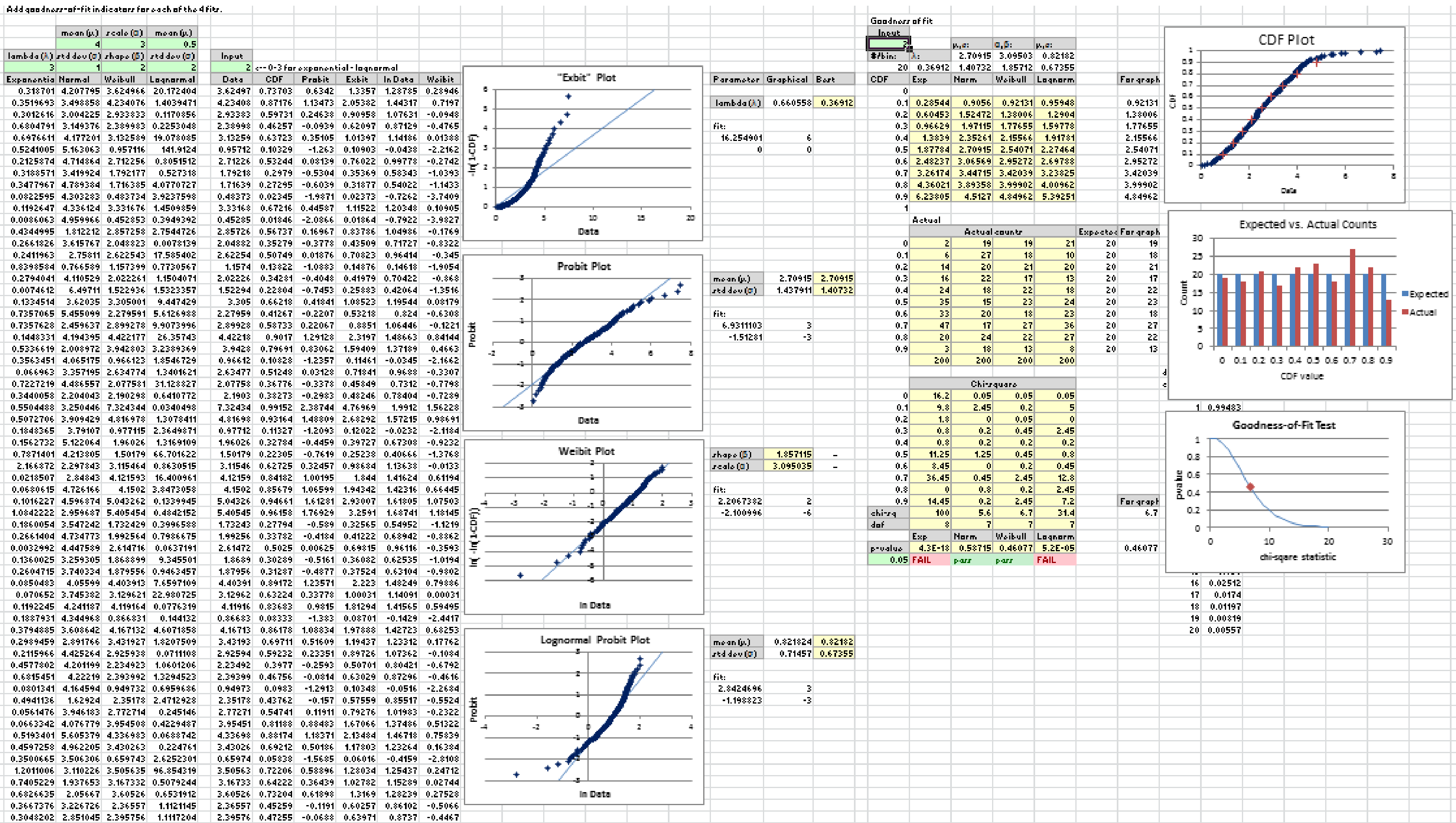
$$\chi^2 = \sum_i \chi_i^2$$

DoF = bins – parameters – 1 = 7

p-value = CHIINV(Chi-Sq, DoF)
=0.005

Exercise 7.1

- Add a chi-square goodness-of-fit test for each of the 4 fits in our synthesized data sheet



Other Goodness-of-Fit Tests

Exponential

Goodness-of-Fit Test

Kolmogorov's D

D		Prob>D
0.097489	>	0.1500

Note: Ho = The data is from the Exponential distribution. Small p-values reject Ho.

Weibull

Goodness-of-Fit Test

Cramer-von Mises W Test

W-Square		Prob>W^2
0.029696	>	0.2500

Note: Ho = The data is from the Weibull distribution. Small p-values reject Ho.

Normal

Goodness-of-Fit Test

Shapiro-Wilk W Test

W	Prob<W
0.947533	0.0270*

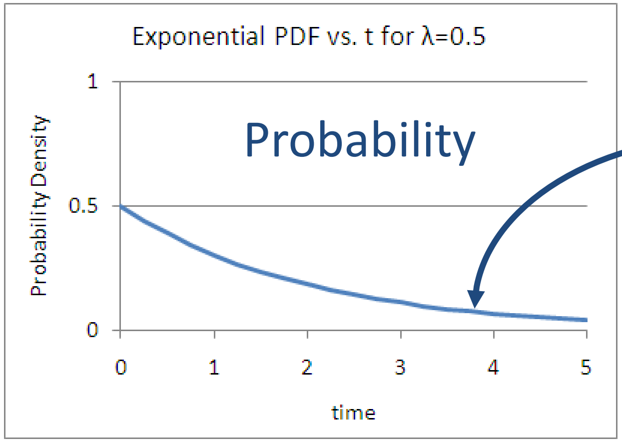
Note: Ho = The data is from the Normal distribution. Small p-values reject Ho.

Maximum Likelihood Method and the Exponential Distribution

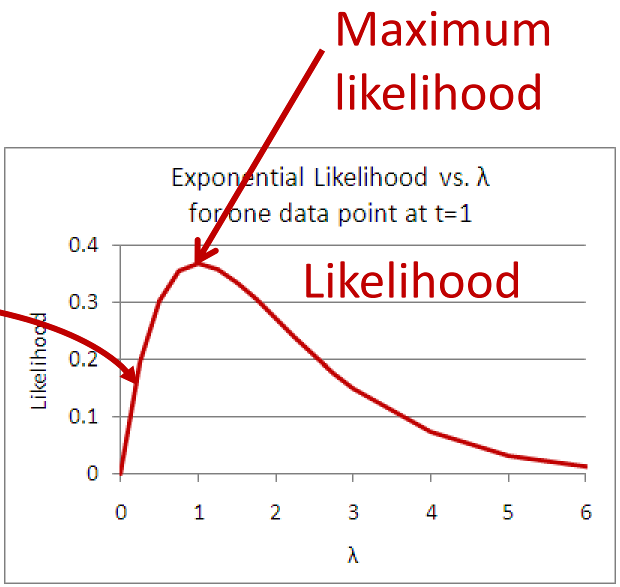
MLE

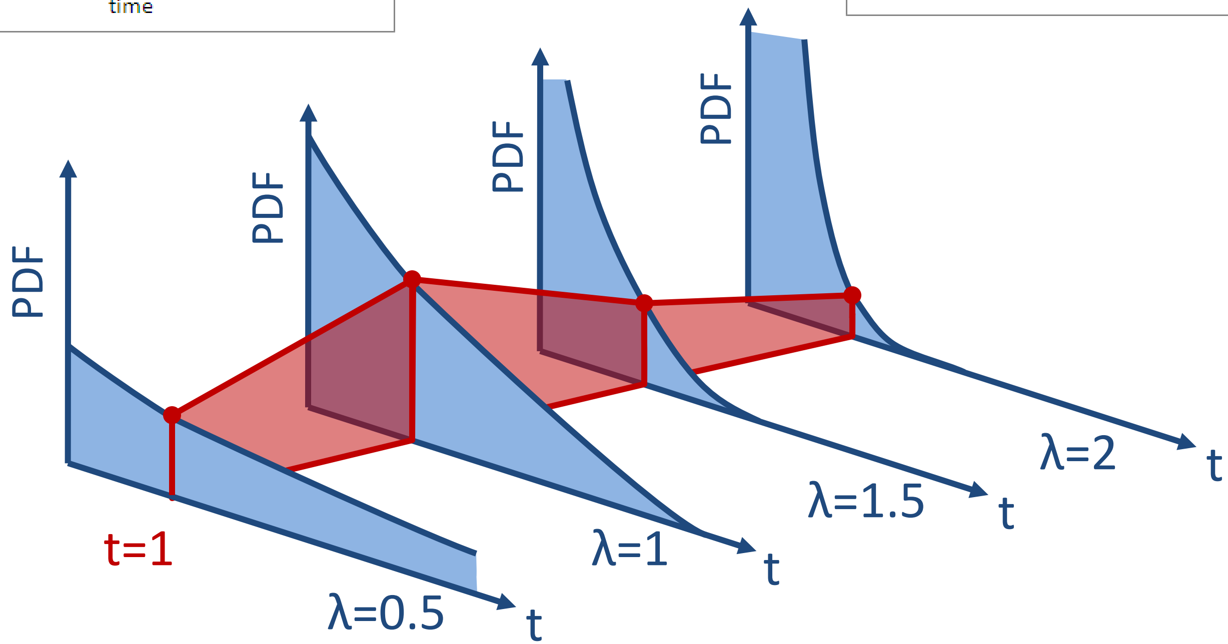
- Maximum Likelihood Estimation (MLE) is a fitting technique that is good for any model
- Principle
  - We can't ask: What is the most likely model?
    - Because we don't have some well-defined space of possible models
  - We can ask: Given this model, how likely is this data set?
  - (This is a fairly Bayesian approach. We are usually frequentists.)

Probability vs. Likelihood



$\lambda e^{-\lambda t}$





MLE

- Likelihood for each point
  - For exact values (exact times to fail), use the PDF
  - For ranges (failed between two readout times), use CDF delta
  - Multiply all together (or add logs)
- Use
  - Choose a model functional form with adjustable parameters
  - Adjust the parameters to maximize the likelihood

MLE for Exponential Data

- For a complete set of times to fail, likelihood is the PDF:

$$PDF_i = \lambda\, e^{-\lambda t_i}$$

- Take log of PDF:

$$\ln PDF_i = \ln \lambda - \lambda t_i$$

- Add up likelihood for each data point:

$$\text{Device hours} = \sum_i t_i$$

$$L = \sum_i \ln PDF_i = \sum_i \left(\ln \lambda - \lambda t_i\right) = N \ln \lambda - \lambda \sum_i t_i$$

$$\text{Sample Size} = N$$

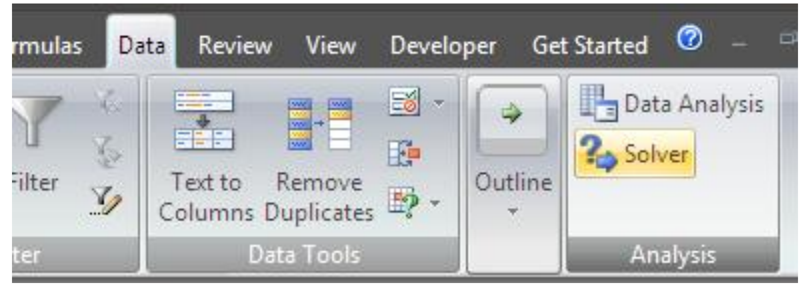
- Then choose λ to maximize L

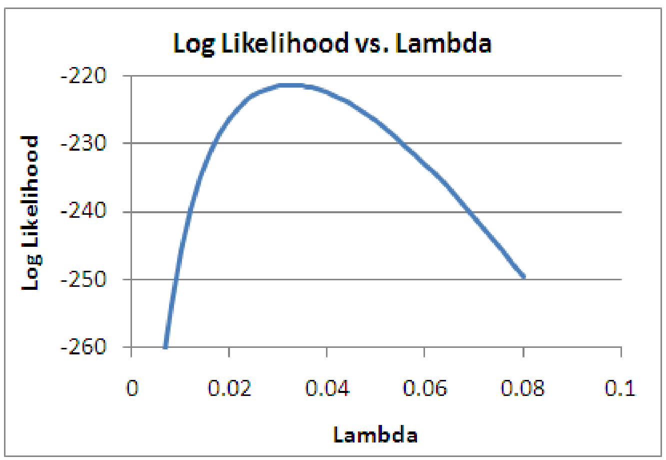
Ex 7.2a – MLE for Exponential

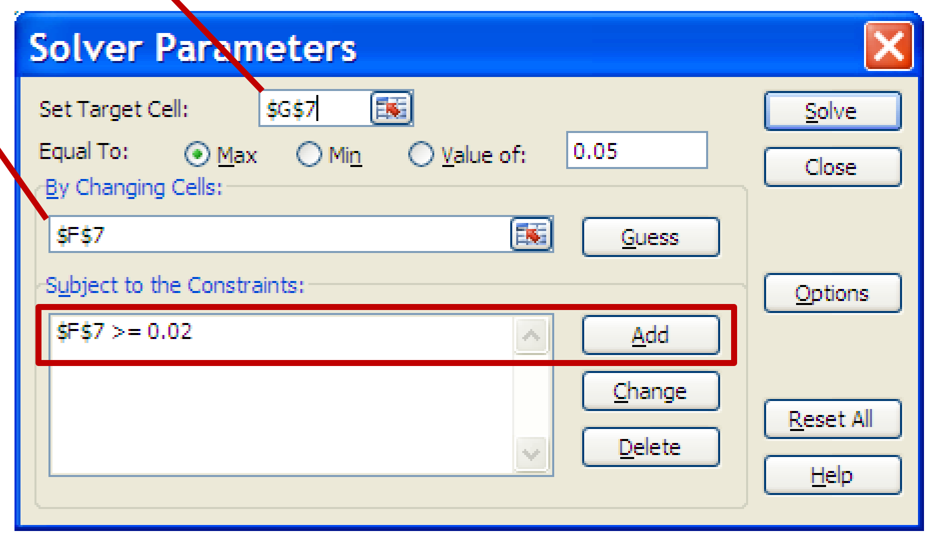
Maximum likelihood:			
	lambda	likelihood	Log LR
UCL			
estimate	0.03248	-221.357	
LCL			

guess

=$C$3*LN(F8) - F8*$C$4







Solution 7.2a

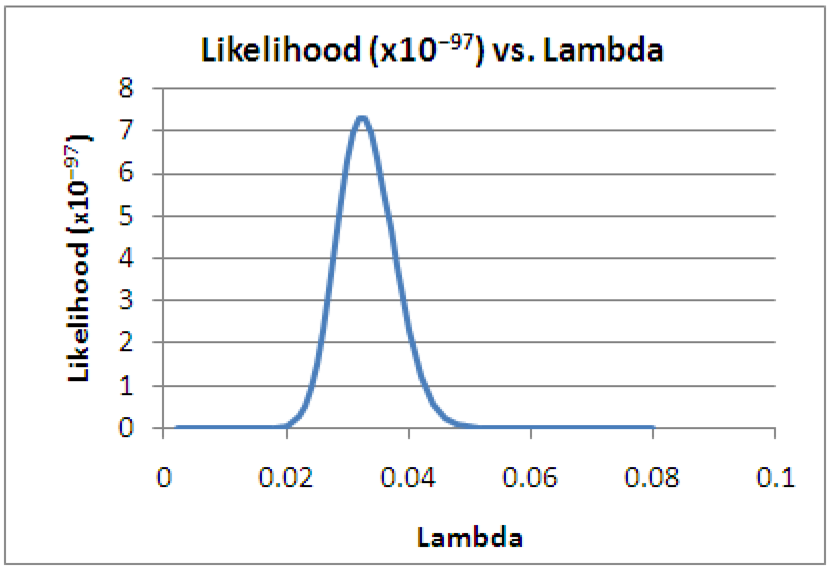
Maximum likelihood:			
	lambda	likelihood	Log LR
UCL			
estimate	0.03248	-221.357	
LCL			

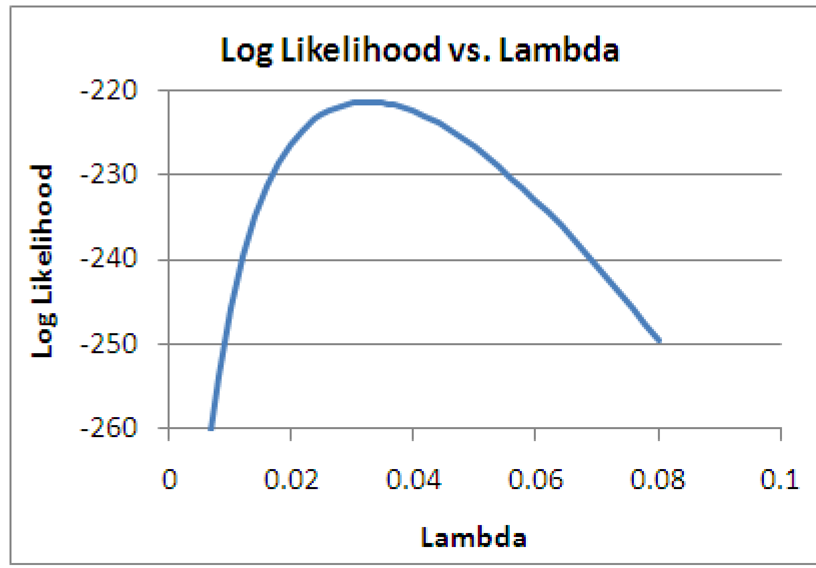
$\lambda = 0.032$ per hour = 3.2% per hour

MTTF = $1/\lambda$ = 30.8 hours

Graphs of Likelihood vs. Lambda

Maximum likelihood:			
	lambda	likelihood	Log LR
UCL			
estimate	0.03248	-221.357	
LCL			





Analytic λ

- For exponential, can maximize analytically:

$$L = N \ln \lambda - \lambda \sum_i t_i$$

$$\frac{dL}{d\lambda} = \frac{N}{\lambda} - \sum_i t_i = 0$$

$$\lambda = \frac{N}{\sum_i t_i} = \frac{\text{Number of fails}}{\text{Total device hours}}$$

Even works for type I censored data

Exercise 7.2b

- Calculate λ for the Ex12 data set using the analytic expression and compare it to what you got from the MLE technique

Solution 7.2b

- Same as MLE technique

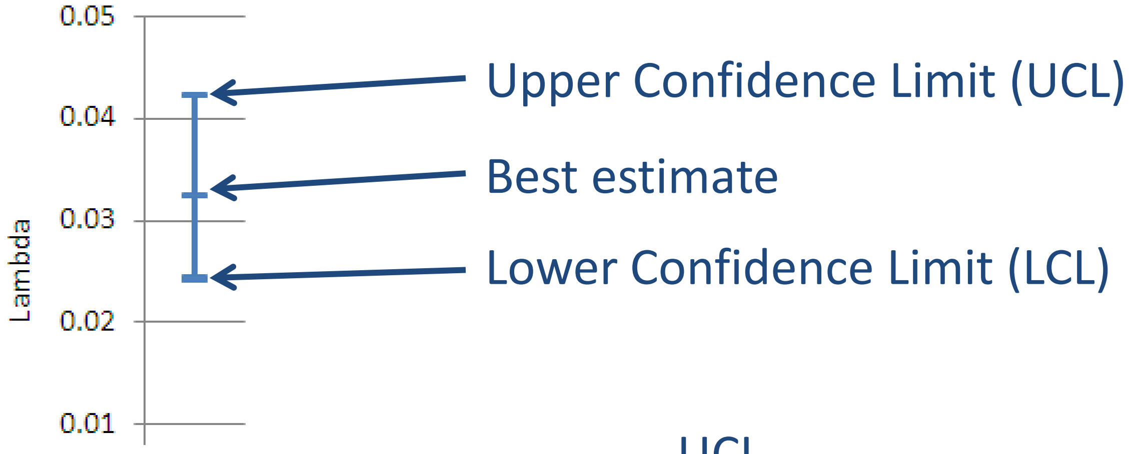
Analytic

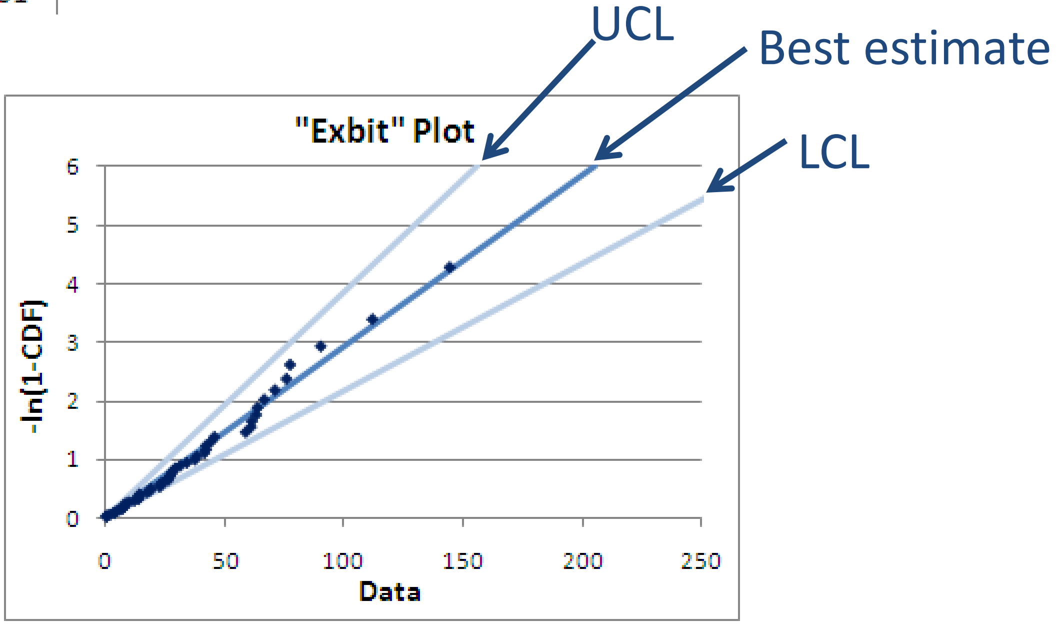
fail count	50
device hours	1539.413
lambda (fails / dev hrs)	0.03248

MLE

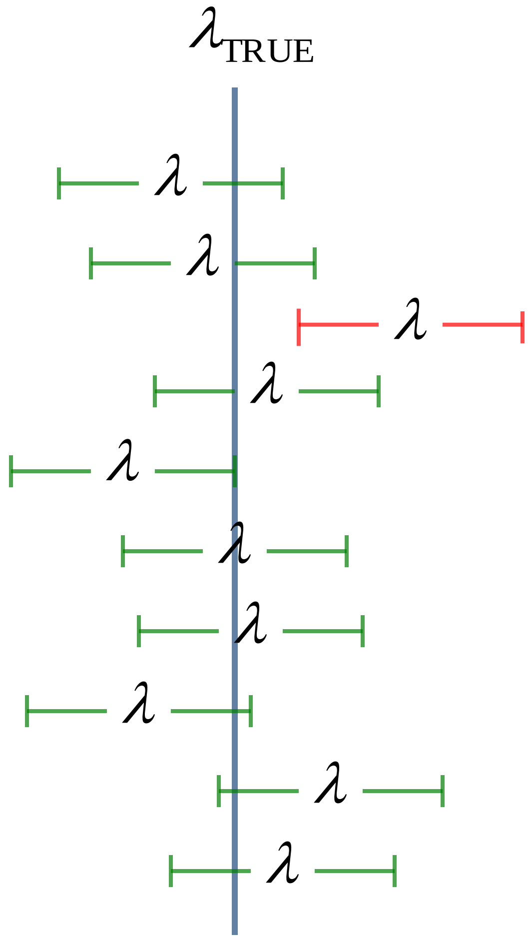
Maximum likelihood:			
	lambda	likelihood	Log LR
UCL			
estimate	0.03248	-221.357	
LCL			

Uncertainty Range of Lambda



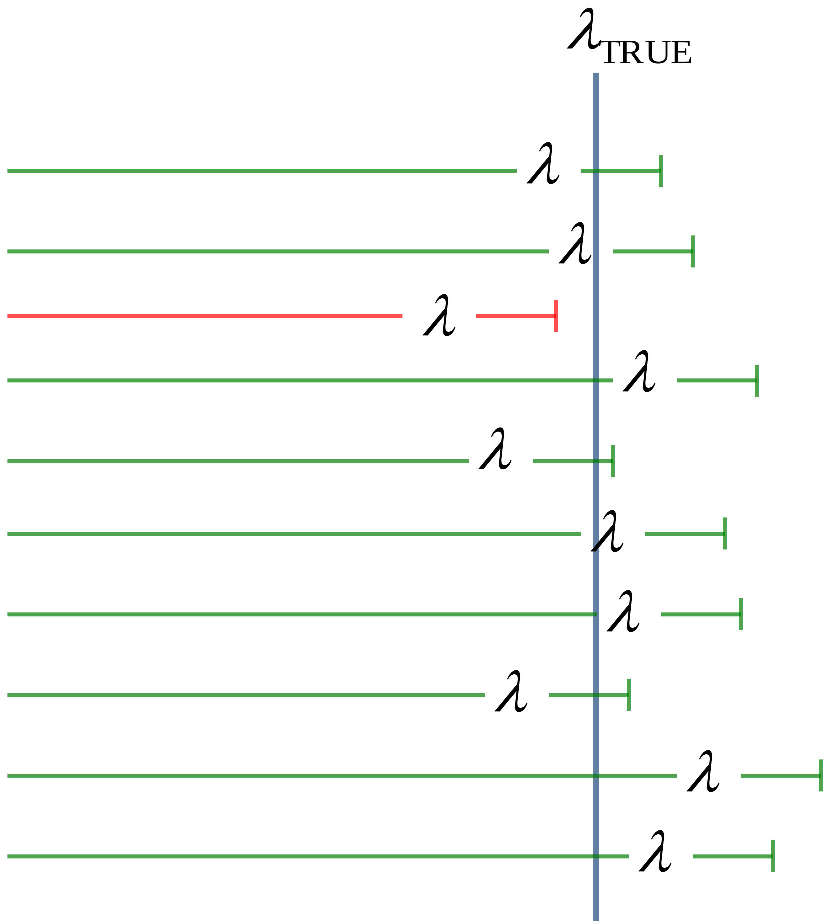


Confidence Interval (2-Sided)



- 90% of random sample λ's with this confidence interval include the true population λ

Confidence Interval (1-Sided)

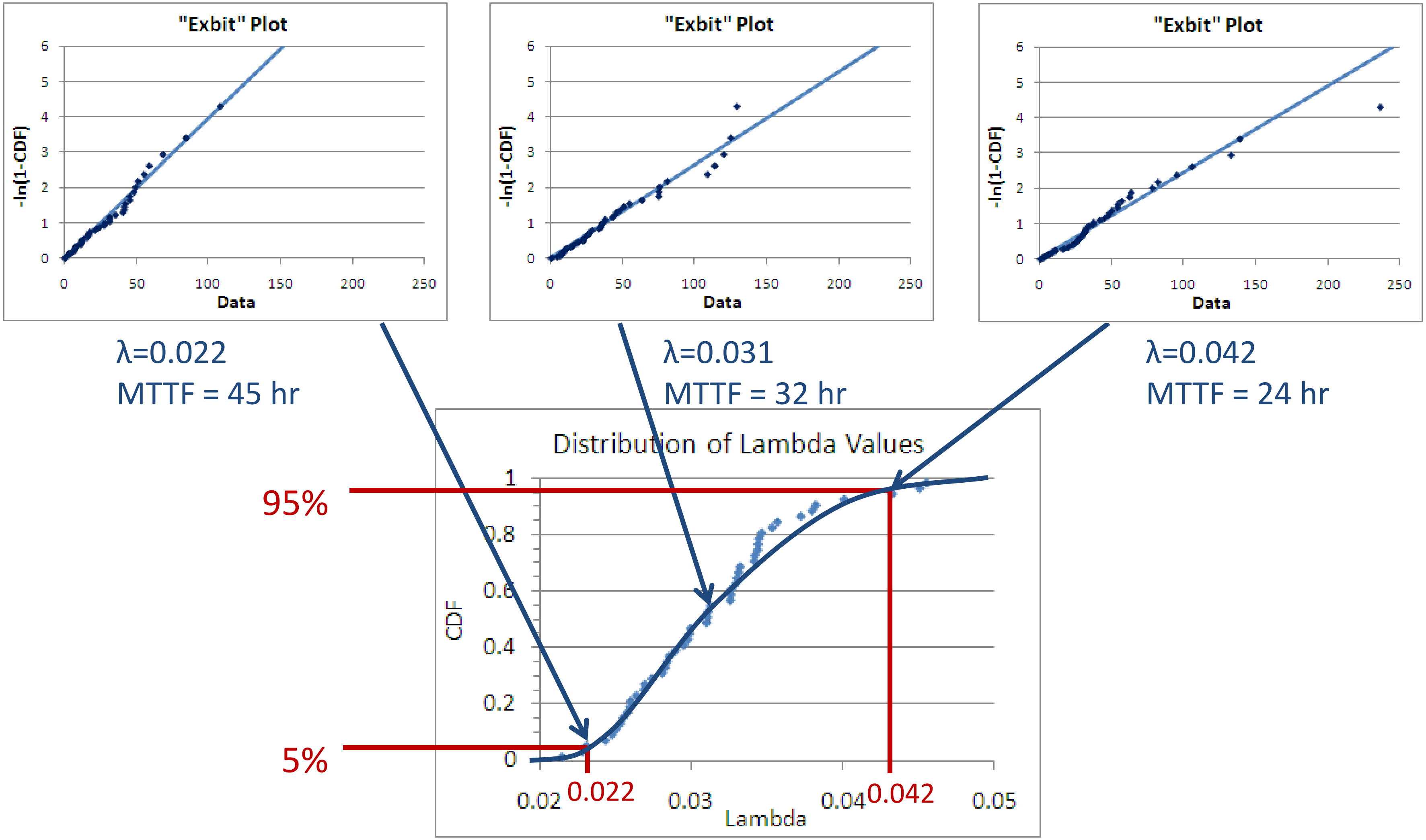


Uncertainties on Parameters

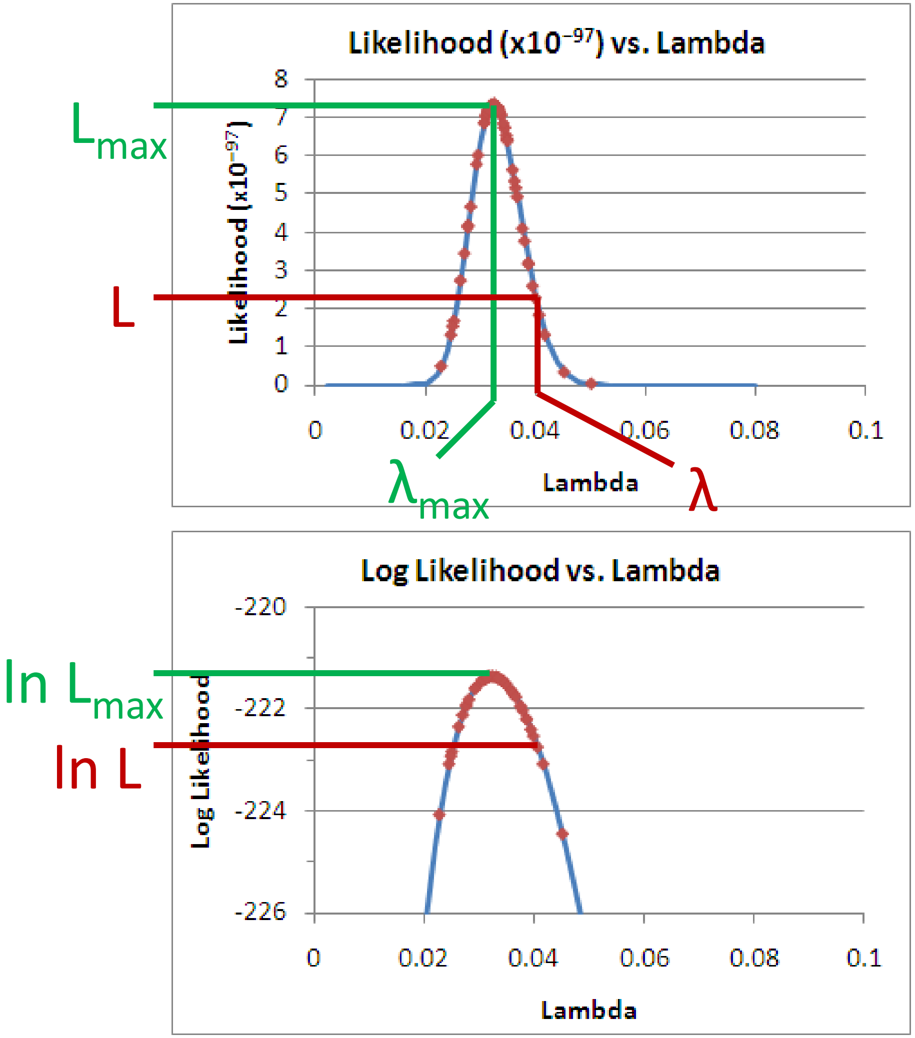
To calculate:

- Monte Carlo
- Likelihood ratio
- Analytic

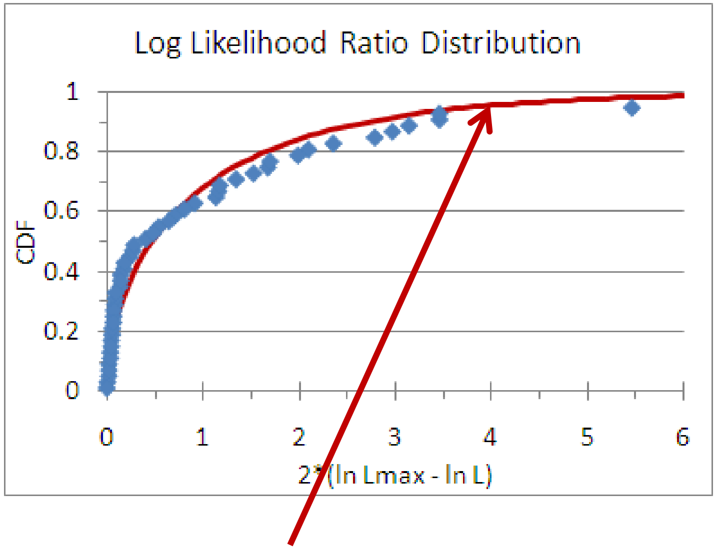
Recall Monte Carlo Lambda Uncertainty



Likelihood Ratio Lambda Uncertainty



$$\ln\left(\frac{L_{max}}{L}\right)^2 = 2\times\left(\ln L_{max} - \ln L\right)$$



1–CHIDIST(Log LR, 1)

Number of parameters in model (=1 for exponential)

Exercise 7.2c

- Calculate UCL and LCL for lambda:
  - Calculate Log LR for each (below)
  - Choose lambda for each to set Log LR = 0.1
    - Do by hand first, then use Solver to fine-tune

Likelihood of best estimate

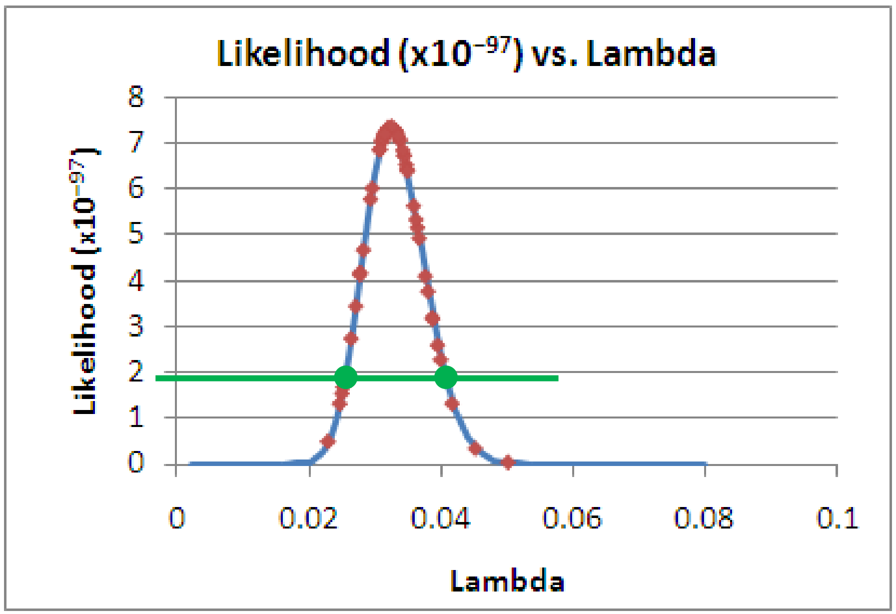
Likelihood of UCL

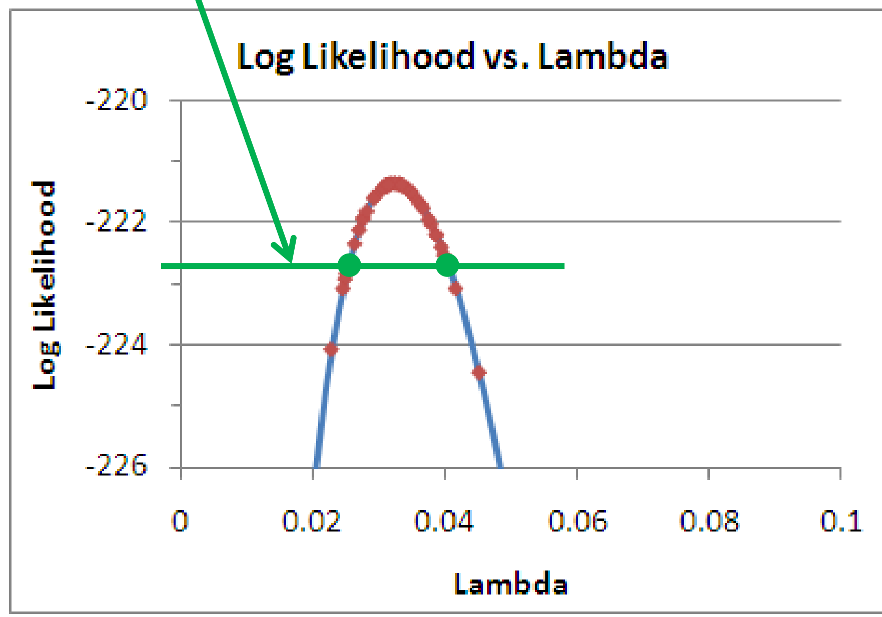
```
=CHIDIST(2*($G$8-G7), 1)
```

Maximum likelihood:			
	lambda	likelihood	Log LR
UCL	0.040632	-222.709	0.100001
estimate	0.03248	-221.357	1
LCL	0.025499	-222.709	0.100001

Solution 7.2c

Maximum likelihood:			
	lambda	likelihood	Log LR
UCL	0.040632	-222.709	0.100001
estimate	0.03248	-221.357	1
LCL	0.025499	-222.709	0.100001





Analytic Lambda Uncertainty

for 90% CL

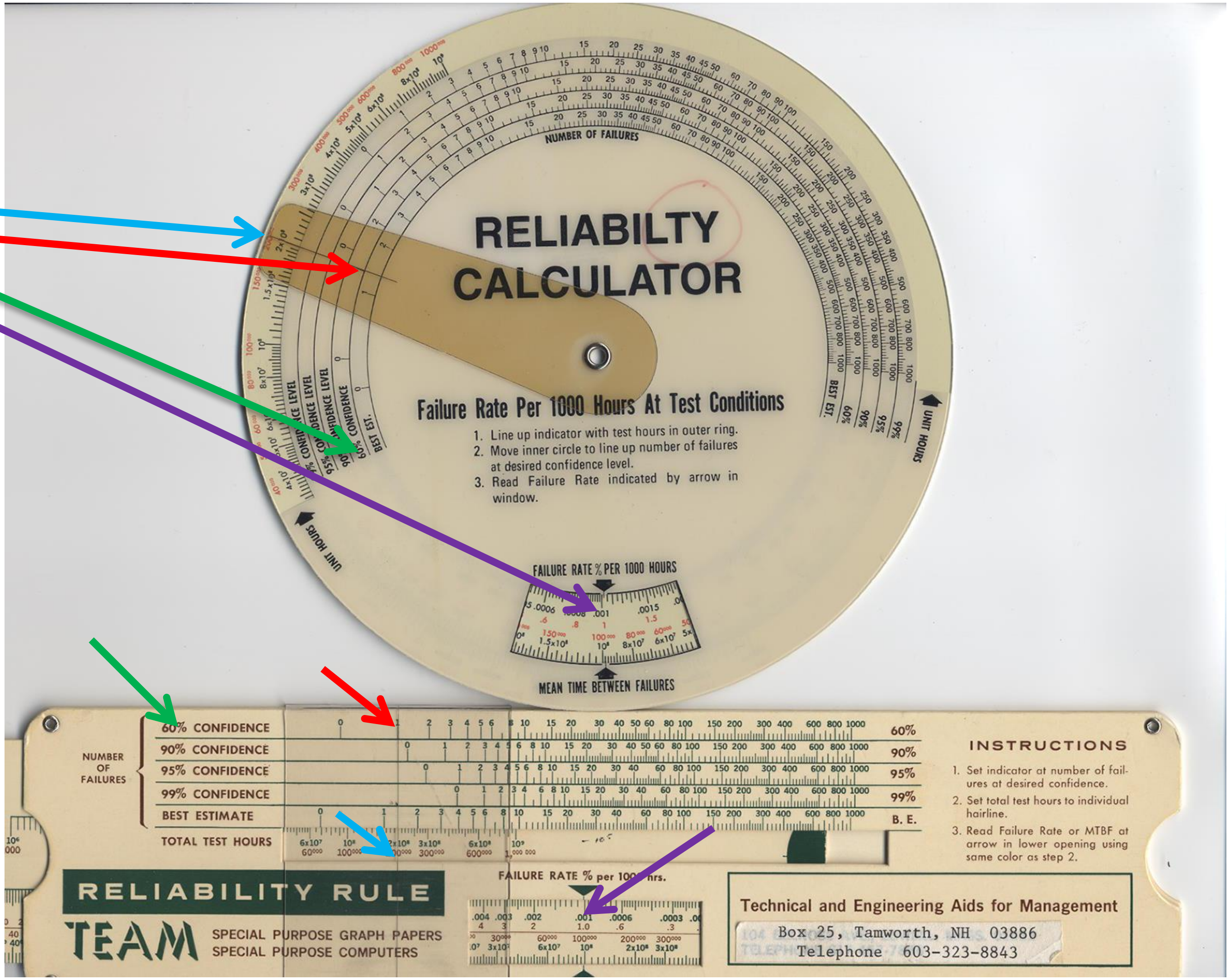
Note N+1 vs. N

$$\lambda_{UCL} = \lambda_{BE} \frac{\mathrm{CHIINV}\left(5\%, 2(N+1)\right)}{2N}$$

$$\lambda_{LCL} = \lambda_{BE} \frac{\mathrm{CHIINV}\left(95\%, 2N\right)}{2N}$$

Venerable Calculation

Reliability Calculator	
Device hours	200
Fails	1
Confidence	60%
Fail rate (%/hr)	1.01%



Exercise 7.2d

- Calculate lambda UCL and LCL analyticall

Solution 7.2d

Maximum likelihood:	
	lambda
UCL	0.040632
estimate	0.03248
LCL	0.025499

Analytic:	
UCL	0.041111
estimate	0.03248
LCL	0.025311

Exercise 7.3

- This is Tobias & Trindade exercise 3.1
- How many units do we need to verify a 500,000 hr MTTF with 80% confidence, given that we can run a test for 2500 hours and 2 fails are allowed?
- Hint: you can do this by trial and error. Calculate the UCL on λ as a function of sample size SS and then adjust SS until the UCL equals the target λ.

Solution 7.3

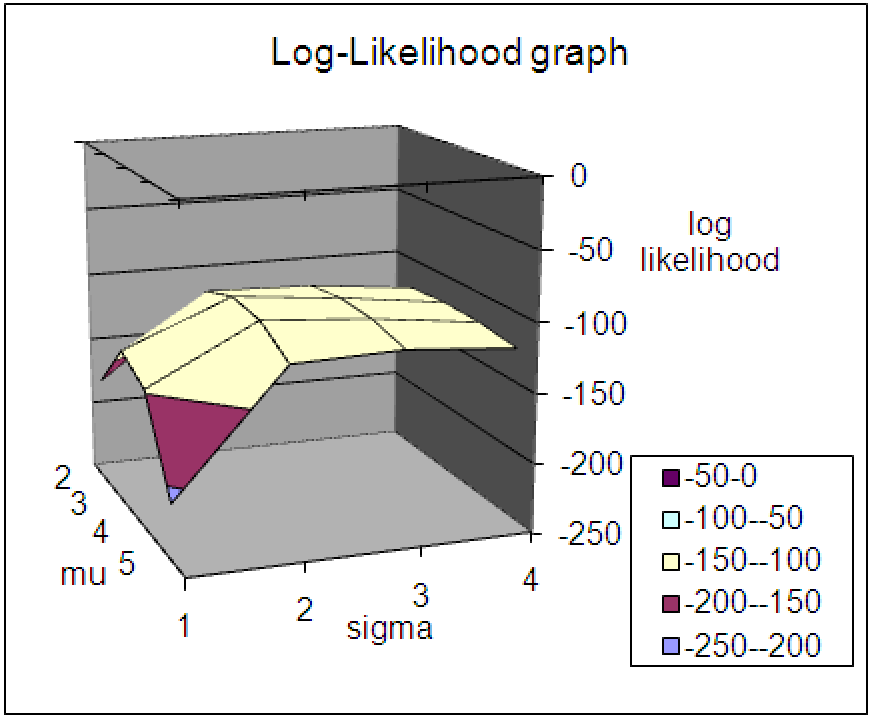
	A	B	C	D
1		**Exercise 14 – Find sample size to meet a MTTF target**		
2		How many units do we need to verify a 500,000 hr MTTF with 80% confidence, given that we can run a te		
3				
4		1. Calculate the target lambda as 1/MTTF.		
5		(2. Note that all lambda values below are multiplied by 1,000,000 to make them easier to evaluate.)		
6		3. Guess at a sample size SS (>1) and list all other givens.		
7		4. Calculate the point (best) estimate lambda_BE as fails / (hours*SS)		
8		5. Calculate the upper confidence value lambda_UCL as CHIINV(1-alpha, 2*(fails+1))/(2*hours*SS)		
9		6. By trial and error, adjust SS until lambda_UCL is as close as you can get to the target		
10				
11		MTTF	500000	
12		alpha	0.8	
13		hr	2500	
14		fails	2	
15		SS	855	
16		lambda_target	2	/ 1,000,000
17		lambda_BE	0.935673	/ 1,000,000
18		lambda_UCL	2.001885	/ 1,000,000

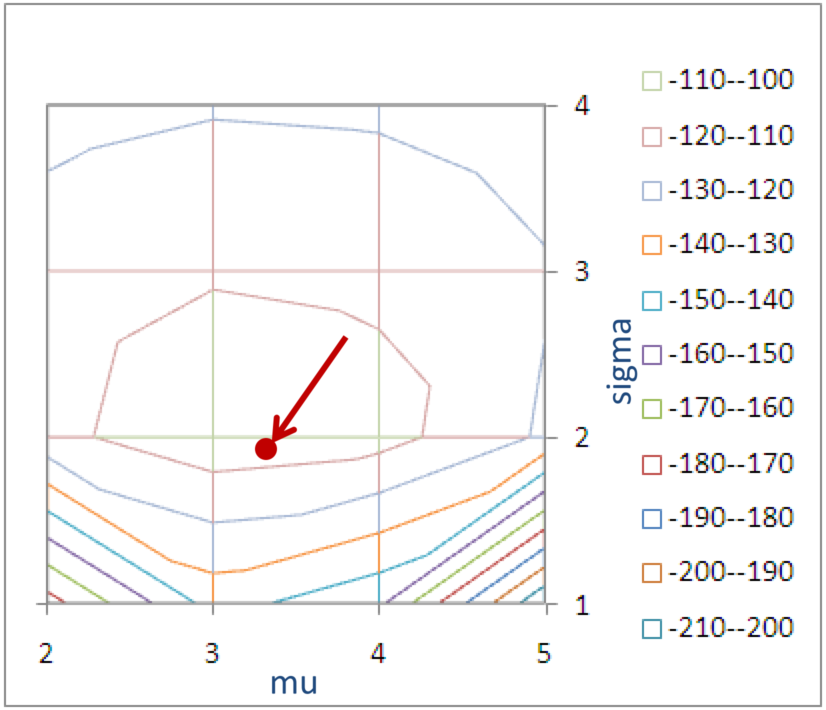
=CHIINV(1-C12, 2*(C14+1))/(2*C13*C15) *1000000

Normal Distribution MLE and Analytic

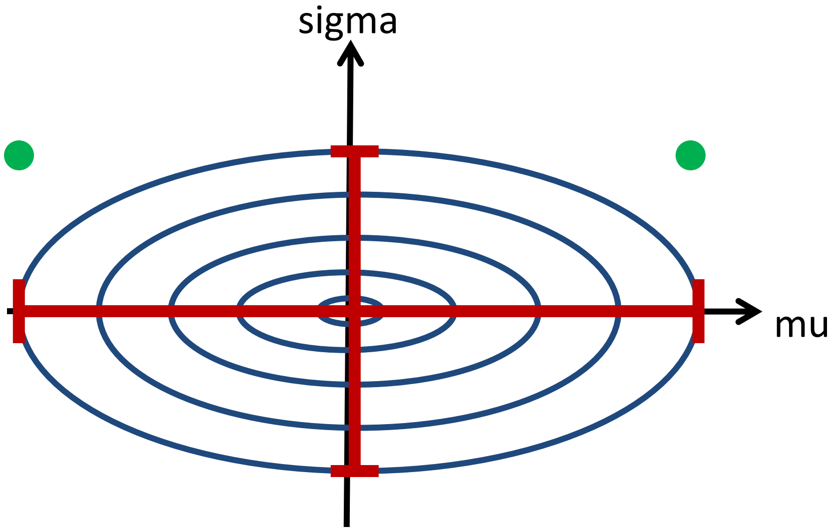
MLE for the Normal

$$L_i = \ln\left(\text{NORMDIST}\left(\text{data}_i, \mu, \sigma, \text{false}\right)\right)$$

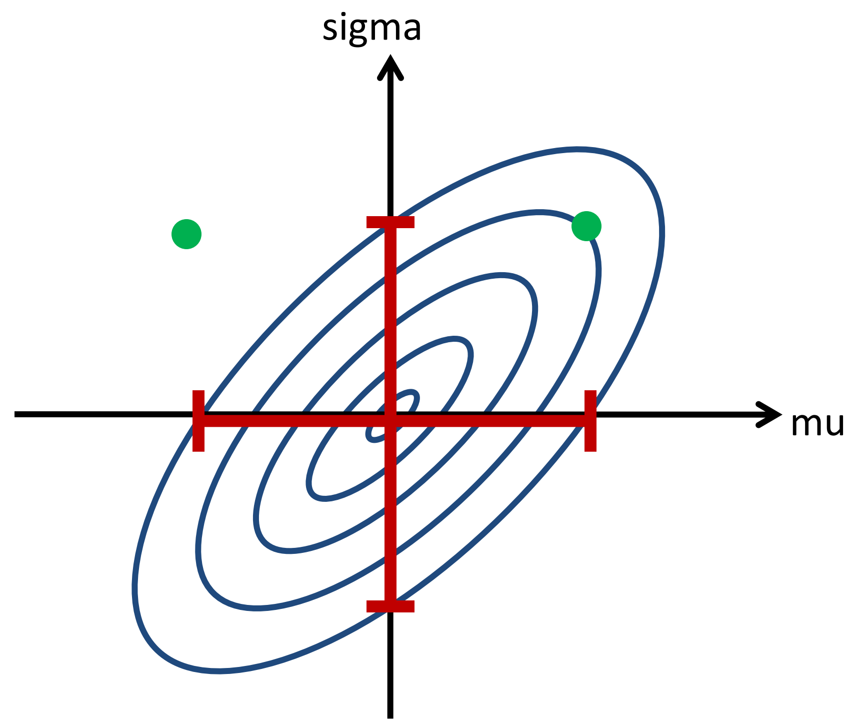




2D MLE



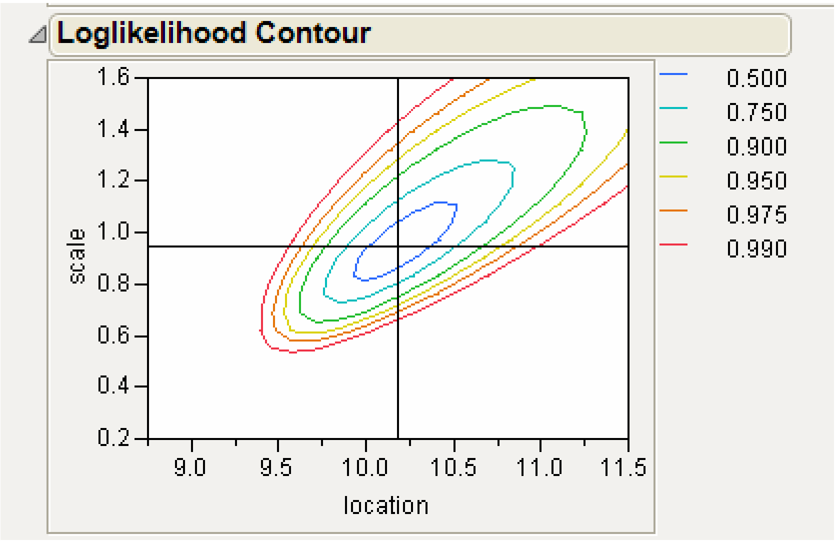
Uncorrelated



Correlated

Described by covariance matrix

JMP MLE Correlations

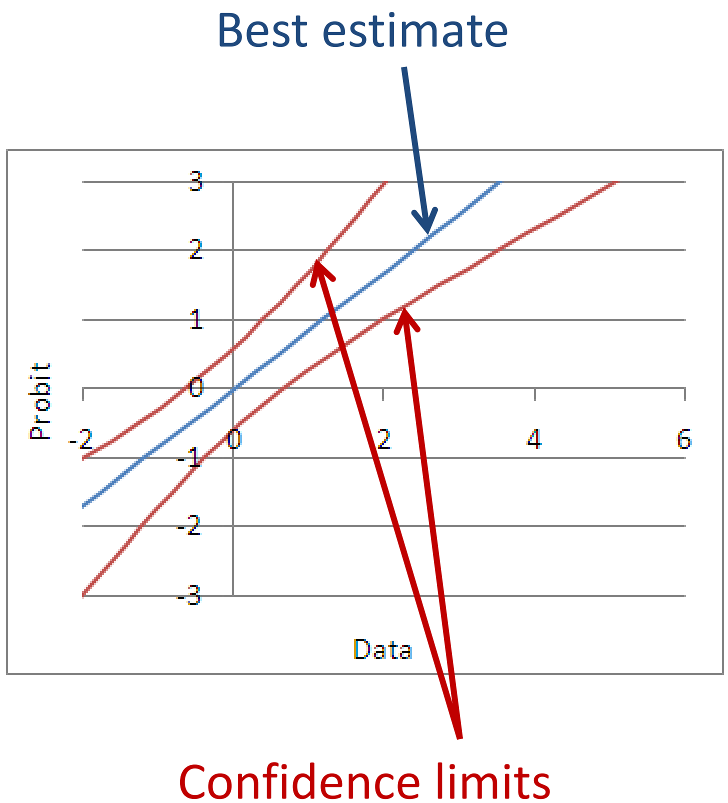


Analytic Normal Parameters

	Best estimate	Uncertainty
Mean μ	μ = AVERAGE(data)	m = NORMSINV(CL) * σ / SQRT(N)
Stdev σ	σ = STDEV(data)	s = σ * (SQRT(CHIINV(1-CL, N-1) / (N-1)) - 1)
Percentile	μ + z*σ	UCL = μ + z*σ + SQRT(m^2 + z^2*s^2) LCL = μ + z*σ – SQRT(m^2 + z^2*s^2)

CL = confidence level (e.g., 95%)
z = probit value at which to evaluate distribution (e.g., -2)
N = number of samples in data set

Normal Distribution Uncertainties



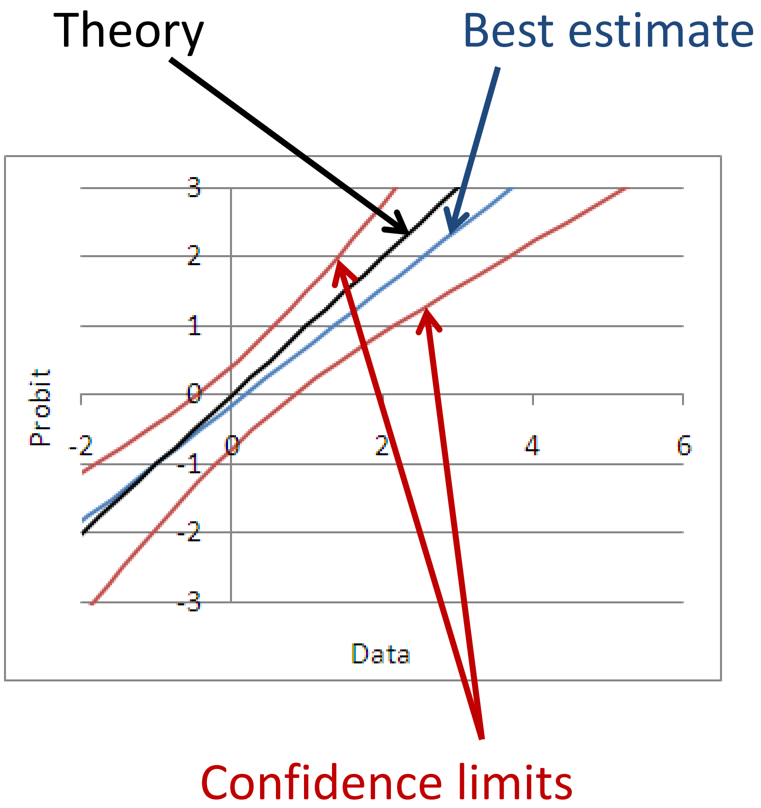
Exercise 7.4a

- For the 9 data points given, extract mu, sigma, and their uncertainties, and calculate the 95% confidence interval for the 2-sigma point of their parent distribution.

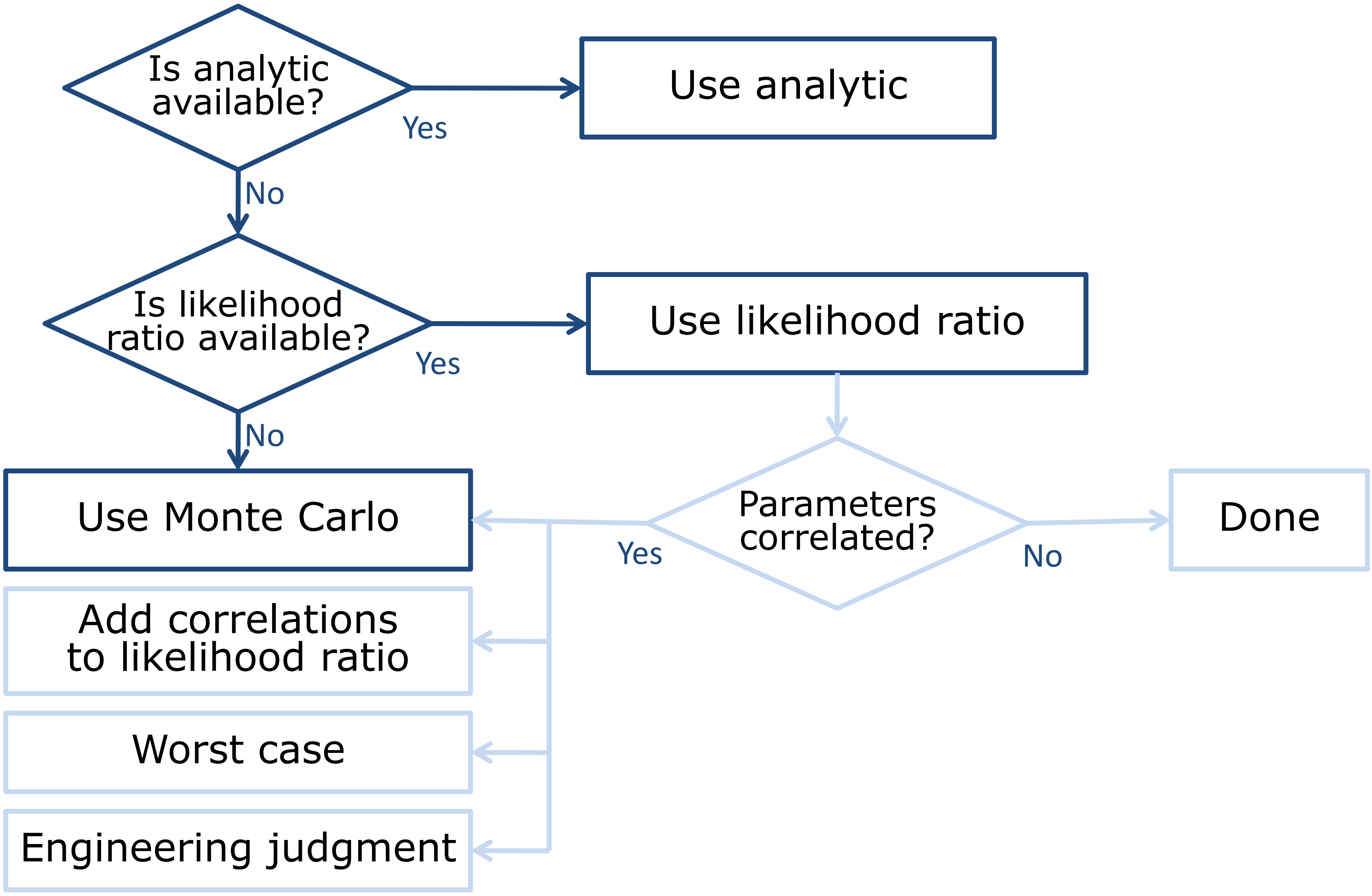
Solution 7.4a

CL	95%			fit	delta
count	9		mean	0.05451	0.522683
			stdev	0.953307	0.373953
Data					
-1.05884			probit	2	
-0.70025					
0.17781			UCL	2.873571	
-0.17661			best est	1.961123	
1.49588			LCL	1.048675	
0.923093					
-1.30856					
0.274838					
0.86323					

Exercise 7.4b

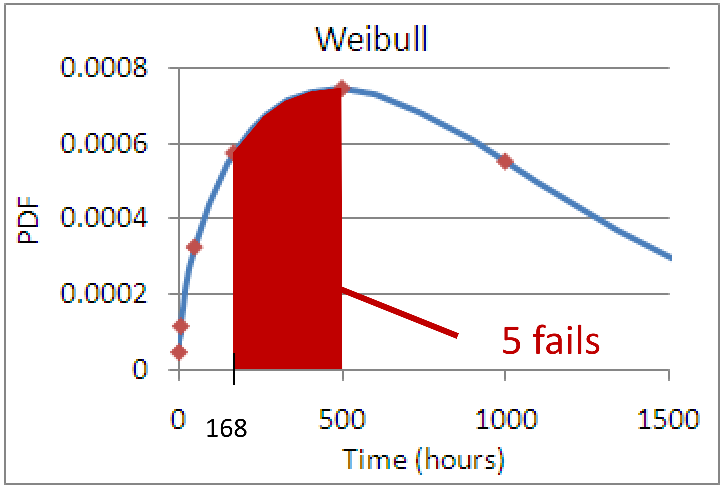


Calculation Method Flowchart



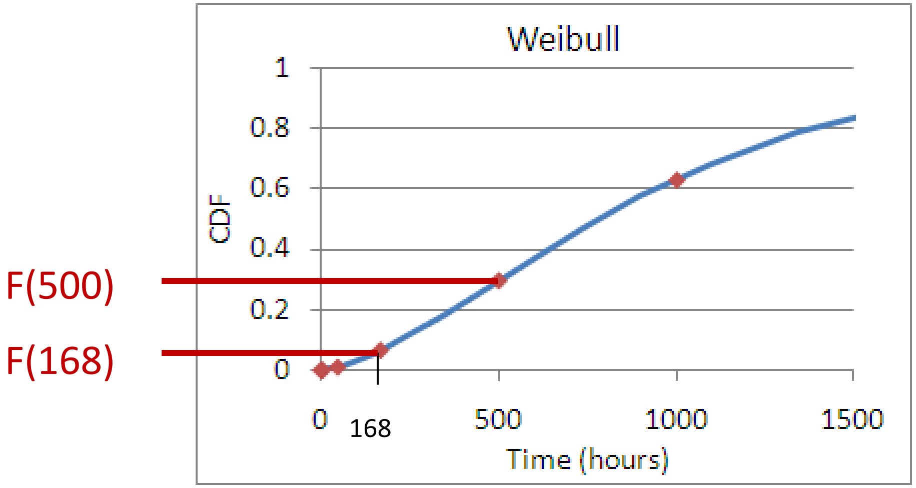
Weibull MLE with Readout Data

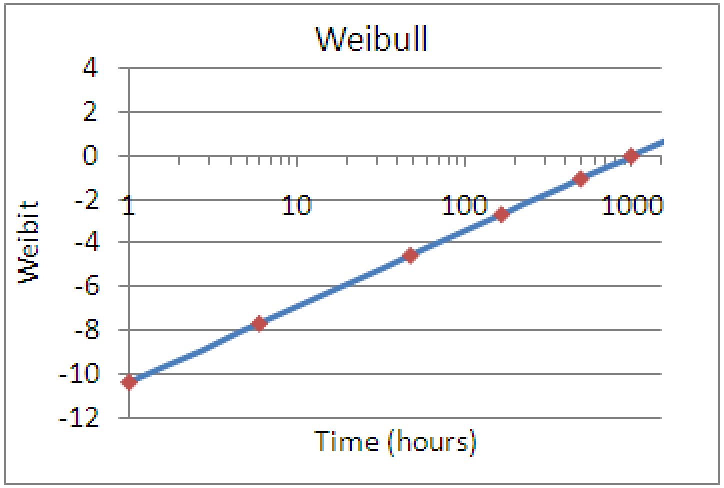
Weibull Readout Data



$$LIK = [F(500) - F(168)]^5$$

$$L = 5 \cdot \ln[F(500) - F(168)]$$

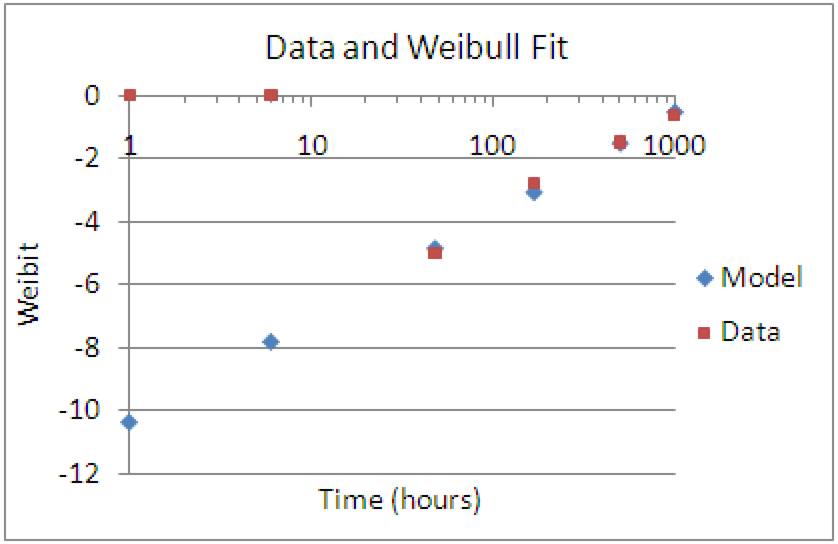




MLE for Weibull

Vary these to maximize this

shape	1.166114		
lifetime	3333.955		
SS	300		
		model	-224.747586
time	fails	F	log-likelihood
0		0	
1	0	7.79E-05	0
6	0	0.00063	0
48	1	0.007093	-5.04168174
168	7	0.03021	-26.3702422
500	27	0.103654	-70.5029942
1000	30	0.217739	-65.1245434
survivors	235	0.782261	-57.7081249



$$L = \sum_{r=1}^{R} \left[n_r \cdot \ln\{F(t_r) - F(t_{r-1})\} + d_r \cdot \ln S(t_r) \right] + s_R \ln S(t_R)$$

$$F(t) = 1 - \exp\left\{ -\left(\frac{t}{\alpha}\right)^{\beta} \right\}$$

$$S(t) = \exp\left\{ -\left(\frac{t}{\alpha}\right)^{\beta} \right\}$$

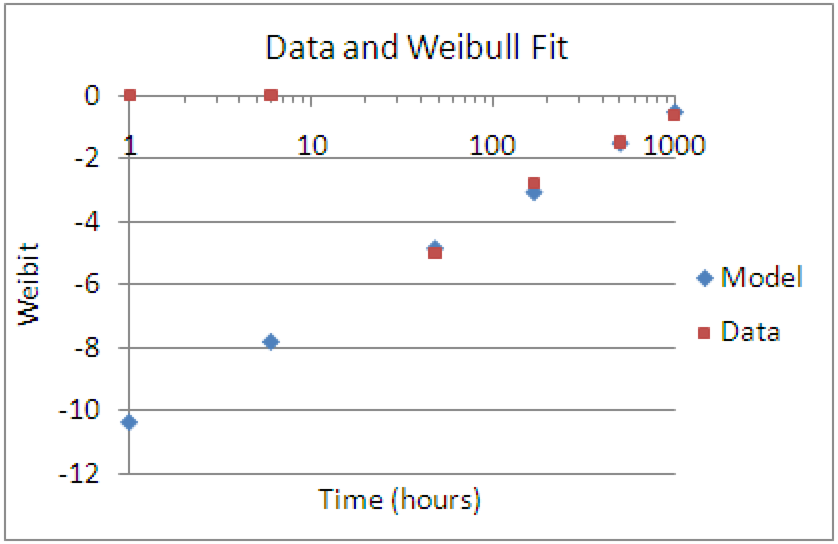
Exercise

- Use MLE to determine Weibull fit parameters for the readout data given below and on the Ex16 tab.
- Also, find (separate) 90% confidence intervals for each parameter using the likelihood ratio technique. (That is the confidence where the LR=0.1.)

SS	300
time	fails
0	
1	0
6	0
48	2
168	16
500	43
1000	63

Solution

		LCL	Best	UCL
shape	1.260344	1.117712	1.260344	1.413664
lifetime	1642.709	1464.712	1642.709	1852.951
SS	300			



					Weibits	
time	fails	model F	L	data F	Data	Model
0		0				
1	0	3.16E-05	0	0	#NUM!	-10.3631
6	0	0.000404	0	0	#NUM!	-7.81346
48	2	0.007764	-9.82356	0.006667	-5.00729	-4.85442
168	16	0.045283	-52.5263	0.06	-2.78263	-3.07174
500	43	0.196489	-81.2318	0.203333	-1.4814	-1.51976
1000	63	0.443784	-88.0219	0.413333	-0.62867	-0.53341
survivors	176	0.556216	-103.241	0.586667		

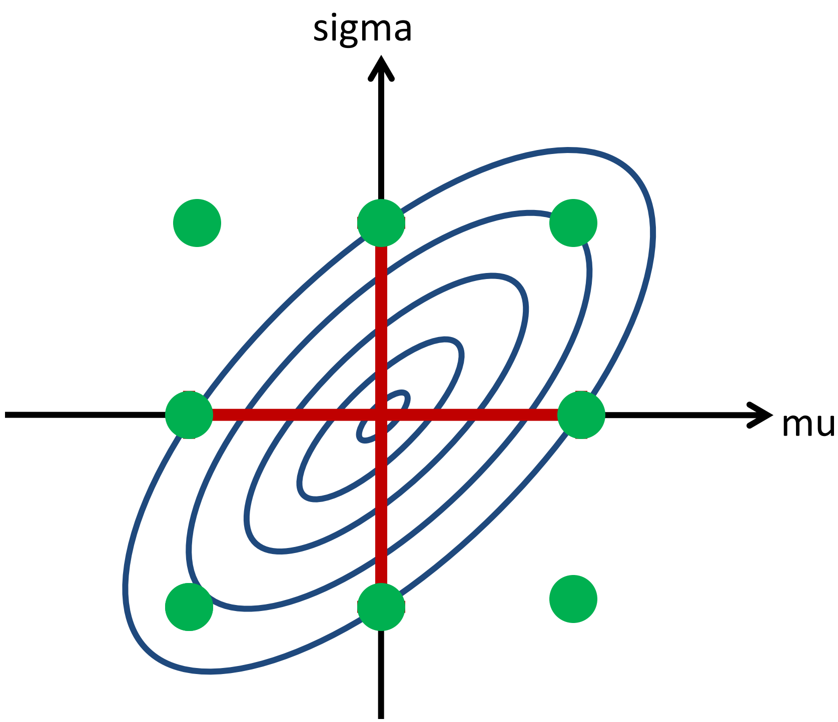
Exercise

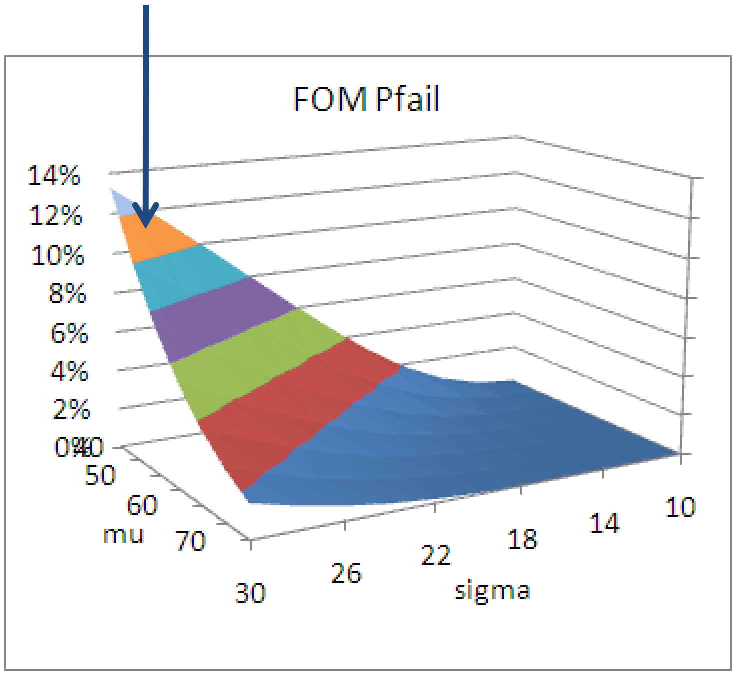
- Add a chi-square goodness-of-fit test to your fit from part (a). Recall that the bins should have more than about 5 fails each, so you will need to combine the first few readouts into 1 bin. So we do things the same way, combine the first 3 readouts into 1 bin, even though they only have 2 total fails.

Solution

					Weibits		Goodness of fit test		
time	fails	model F	L	data F	Data	Model	pred fails	chi-sq stat	
0		0							
1	0	8.86E-05	0	0	#NUM!	-9.33171			
6	0	0.000847	0	0	#NUM!	-7.07348			
48	2	0.01158	-9.06888	0.006667	-5.00729	-4.45267	3.473957	0.625382	
168	16	0.054921	-50.2186	0.06	-2.78263	-2.87376	13.0022	0.691176	
500	43	0.200137	-82.9697	0.203333	-1.4814	-1.49917	43.56502	0.007328	
1000	63	0.414306	-97.0824	0.413333	-0.62867	-0.62557	64.25062	0.024343	
survivors	176	0.585694	-94.1526	0.586667			175.7082	0.000485	
		Ltotal	-333.492				chi-sq	1.348713	
		best L	-333.492				dof	2	
		LR p-value	1				p-value	0.509484	pass

Confidence and Figures of Merit





Exercise

- Use the pfail at 2000 hours as the FOM for the Weibull model of Ex16. Evaluate this pfail as a function of the shape and lifetime parameters.
- Try various corners of the "space" of shape and lifetime values and find the worst case corner. Report these values as your worst case shape and lifetime values.

Solution

		LCL	Best	UCL		Worst case		FOM time	2000
shape	1.413664	1.117712	1.260344	1.413664		1.4136642		FOM pfail	0.788438
lifetime	1464.712	1464.712	1642.709	1852.951		1464.7121			0.722

Ltotal	-334.69
best L	-333.492
LR p-value	0.121738

The End