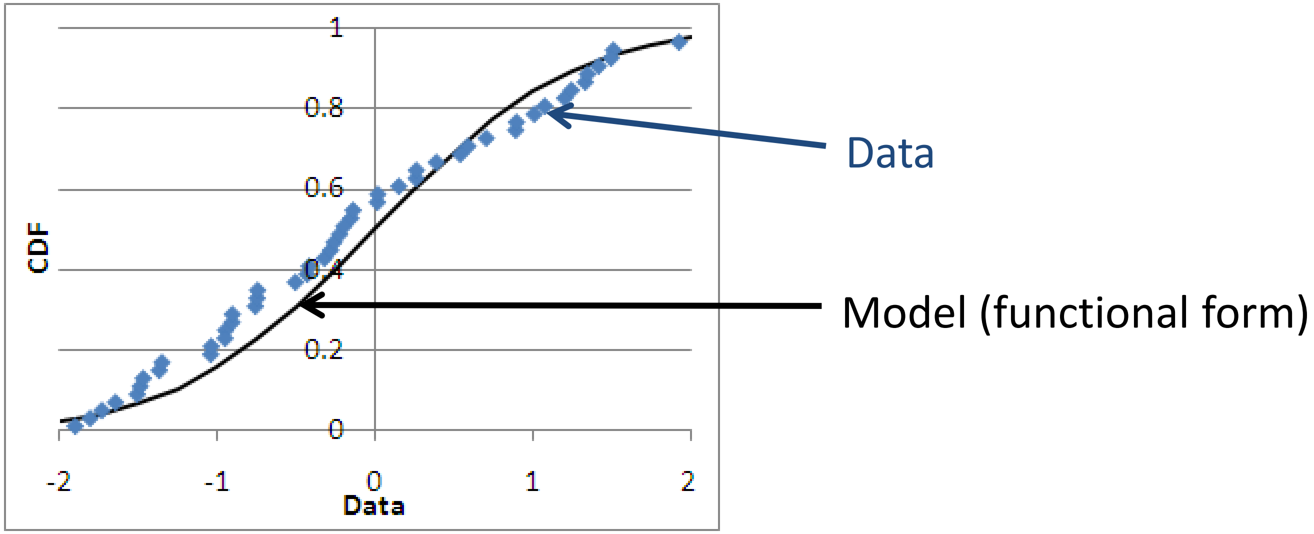
ECE 510 Lecture 4
Reliability Plotting
T&T 6.1-6

Scott Johnson
Glenn Shirley

Functional Forms

Reliability Functional Forms



- Choose functional form for model to fit data

A Function Bestiary

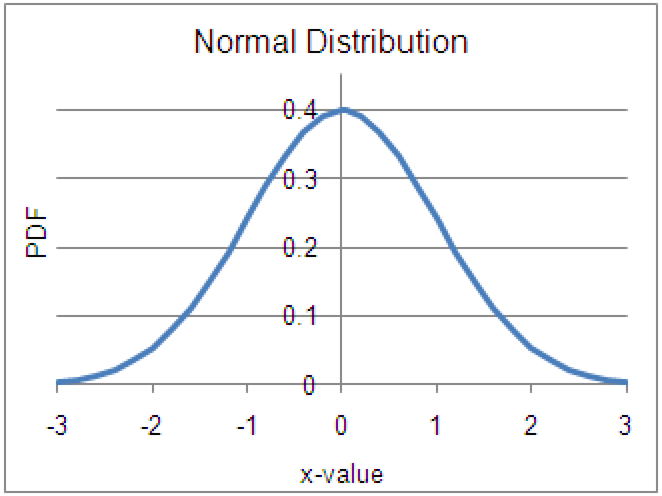
- *Bestiary: A medieval collection of stories providing physical and allegorical descriptions of real or imaginary animals*

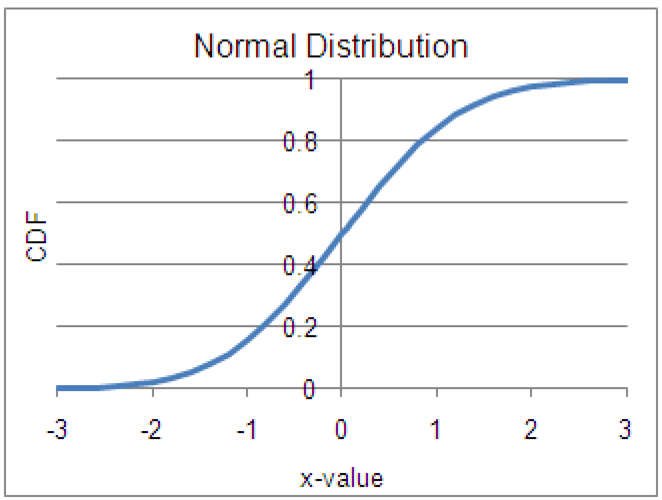
- Continuous distributions
  - Normal
  - Exponential
  - Lognormal
  - Weibull
  - Gamma
  - Beta

Most common for reliability (Normal, Exponential, Lognormal, Weibull)

- Discrete distributions
  - Hypergeometric
  - Binomial
  - Poisson
- Statistical distributions
  - Chi-square
  - Student's t
  - F

Normal Distribution





μ = mean
σ = standard deviation

σ^2 = variance

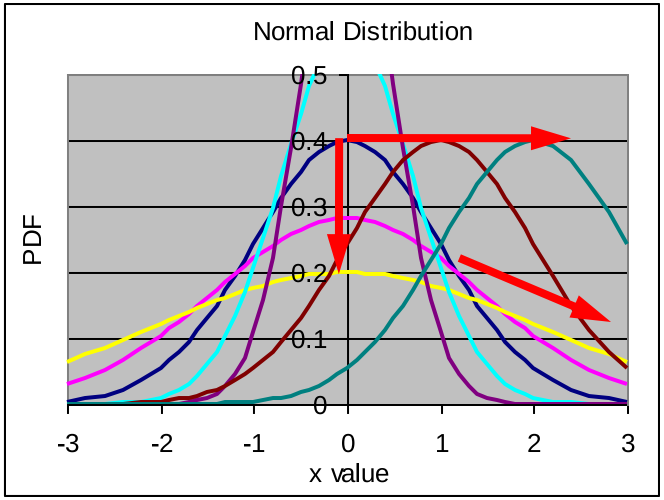
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} \leftarrow e^{-x^2}$$

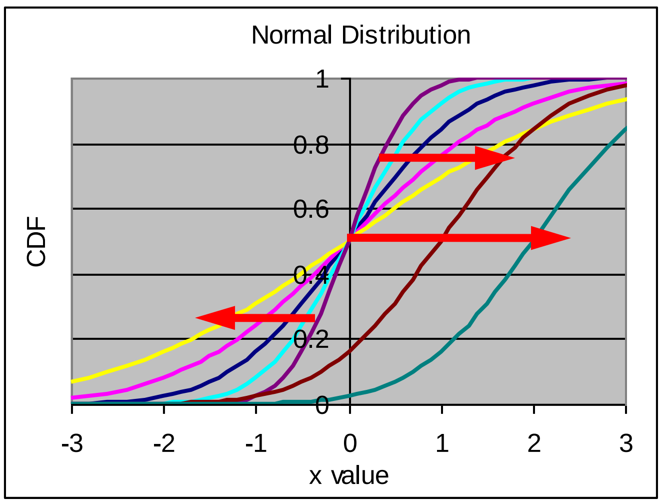
$$F(x) = \int_{-\infty}^{x} dx' \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x'-\mu}{\sigma}\right]^2}$$

$$\text{rand normal} = NORMSINV(CDF)$$
$$\text{where CDF is rand uniform}$$

- Using Excel:
  - PDF = NORMDIST(x,μ,σ,FALSE)
  - CDF = NORMDIST(x,μ,σ,TRUE)
- Plot using:
  - y-axis = probit = NORMSINV(CDF)
  - x-axis = x
  - σ = 1/slope
  - μ = x-intercept = – (y-intercept) / slope

Normal Distribution





mean	0	0	0	0	0	1	2
std	1	1.41	2	0.71	0.5	1	1

μ = mean
σ = standard deviation

σ^2 = variance

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}$$

$$F(x) = \int_{-\infty}^{x} dx' \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x'-\mu}{\sigma}\right]^2}$$

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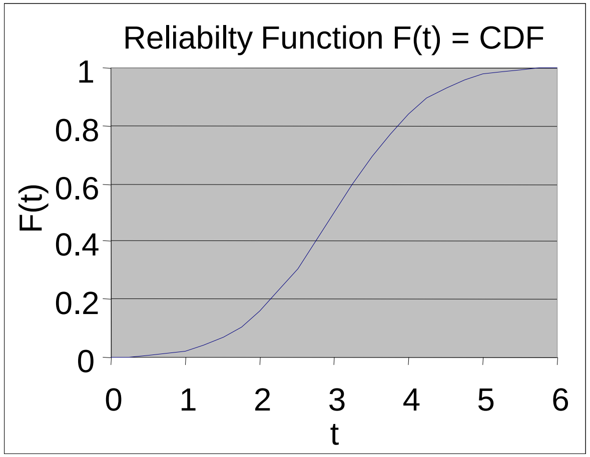
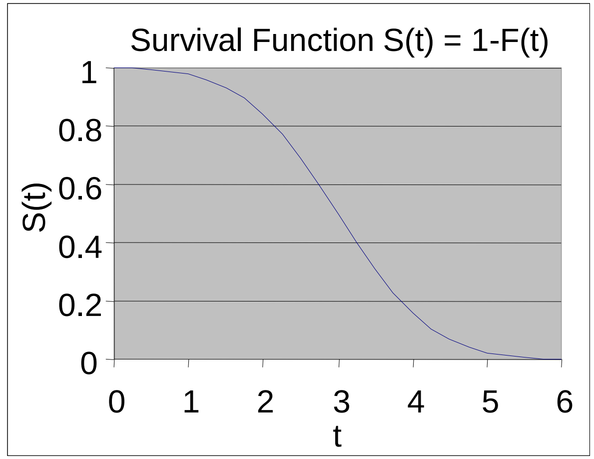
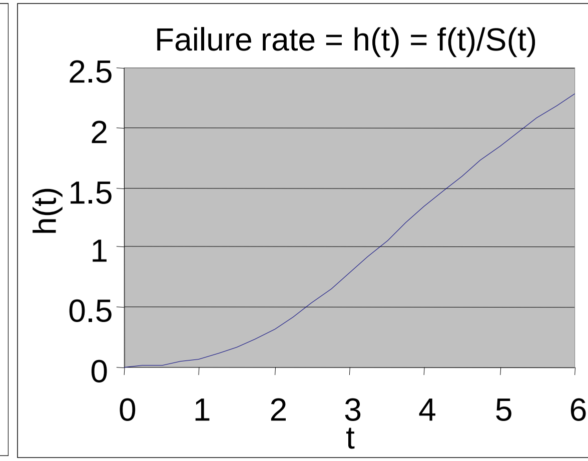
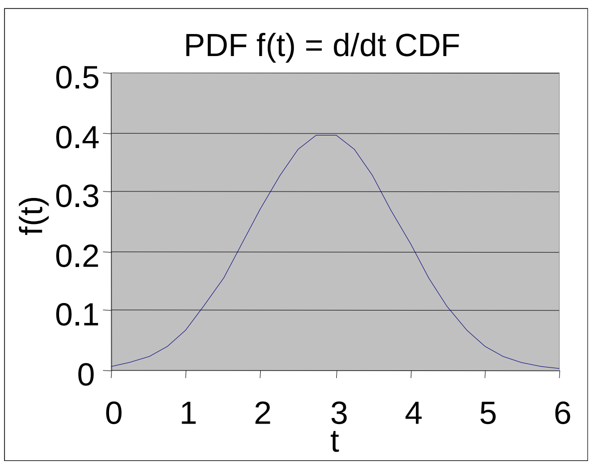
  - x-axis = x
  - σ = 1/slope
  - μ = x-intercept

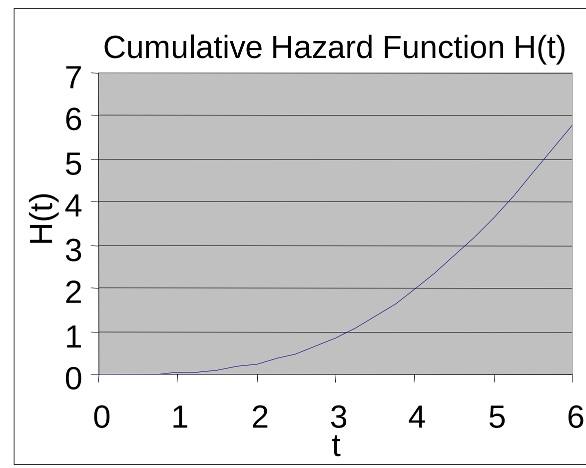
Normal Distribution Reliability Plots



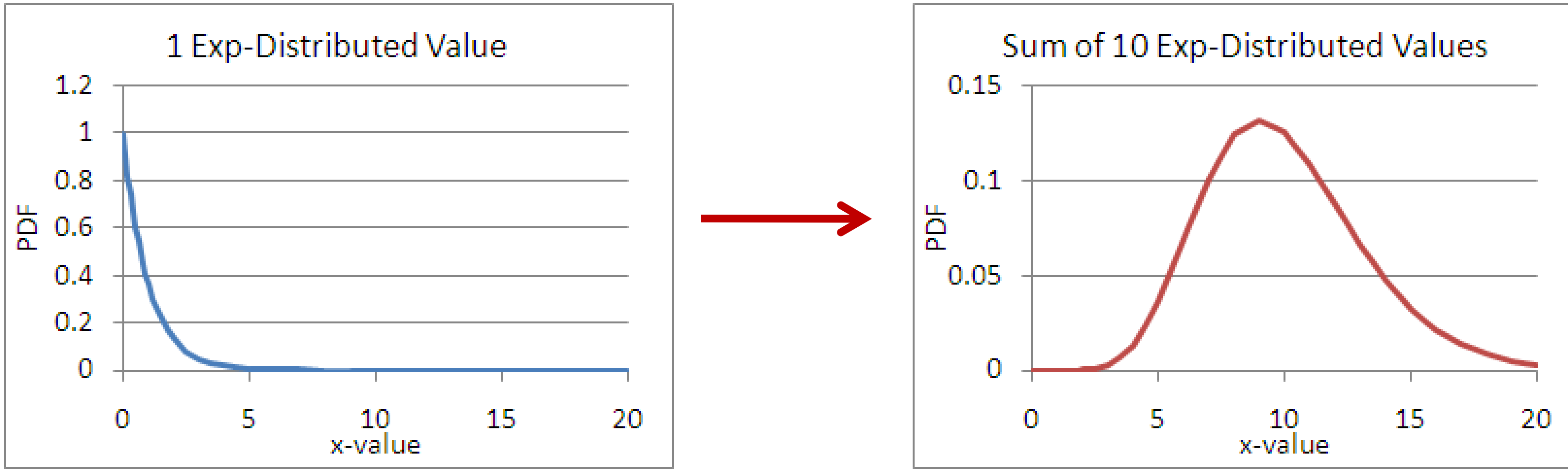






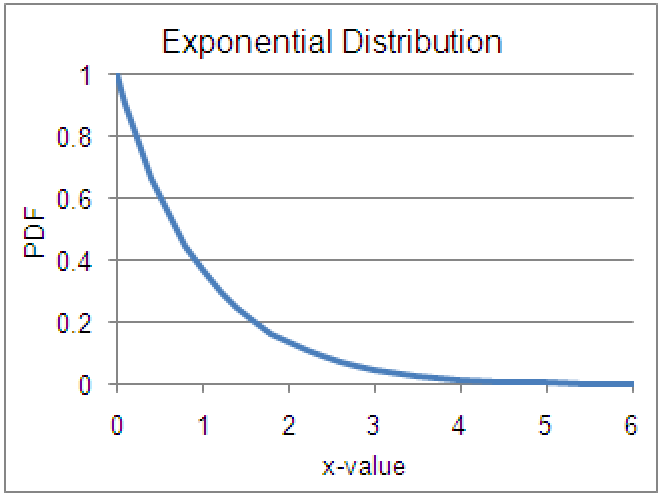


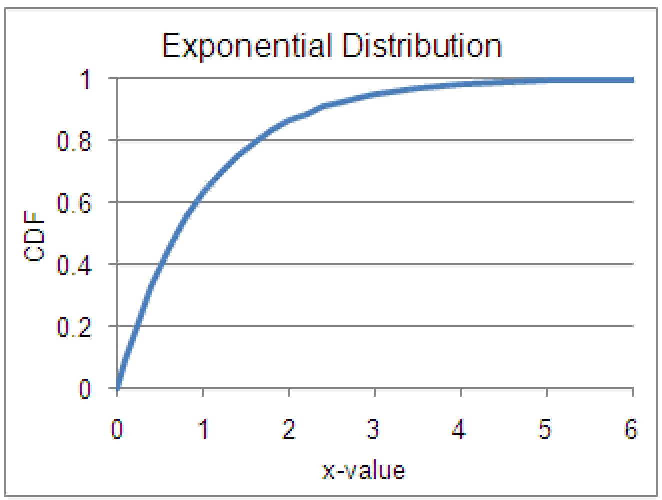
Use of Normal Distributions



- Most measurement error
- Sum of random things is normal

Exponential Distribution





λ = scale factor

$$f(x) = \lambda e^{-\lambda x}$$

$$e^{-t/\tau}$$

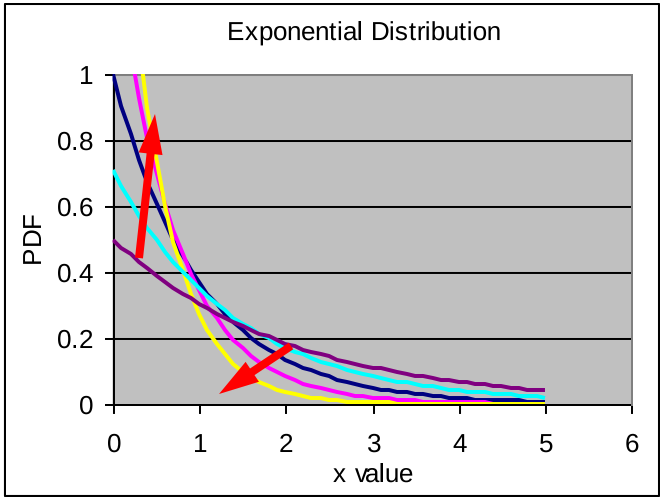
$$F(x) = 1 - e^{-\lambda x}$$

$$\text{rand exponential} = -\frac{\ln(1 - CDF)}{\lambda}$$

where CDF is rand uniform

- Using Excel:
  - PDF = λ*EXP(-λx)
  - CDF = 1-EXP(-λx)
- Plot using:
  - y-axis = “exbit” = -LN(1-CDF)
  - x-axis = x
  - λ = slope

Exponential Distribution



lambda	1	1.41	2	0.71	0.5

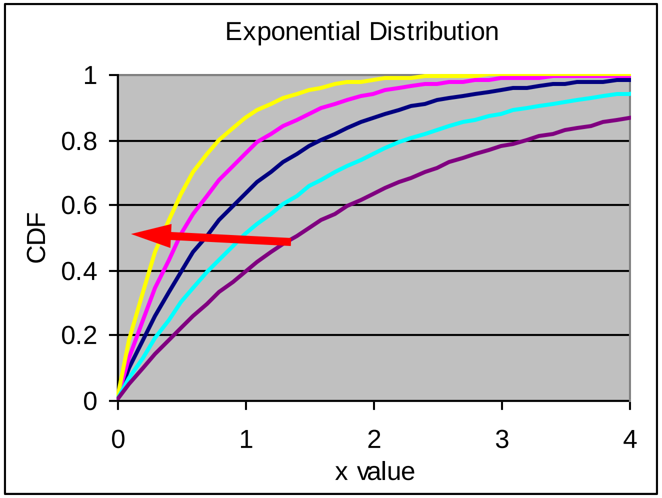
λ = scale factor

$$f(x) = \lambda e^{-\lambda x}$$

$$F(x) = 1 - e^{-\lambda x}$$

$$\text{rand exponential} = -\frac{\ln(1 - CDF)}{\lambda}$$

where CDF is rand uniform



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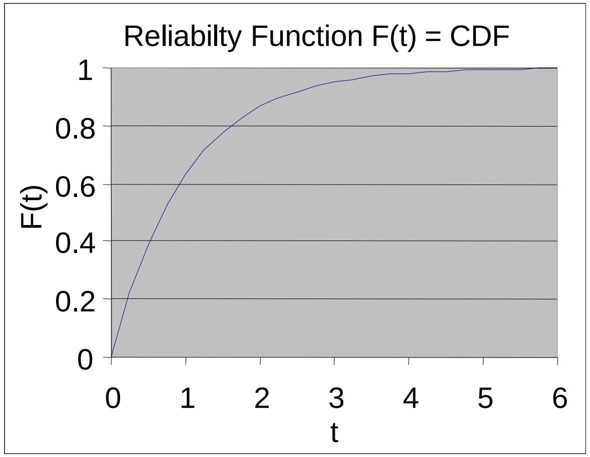
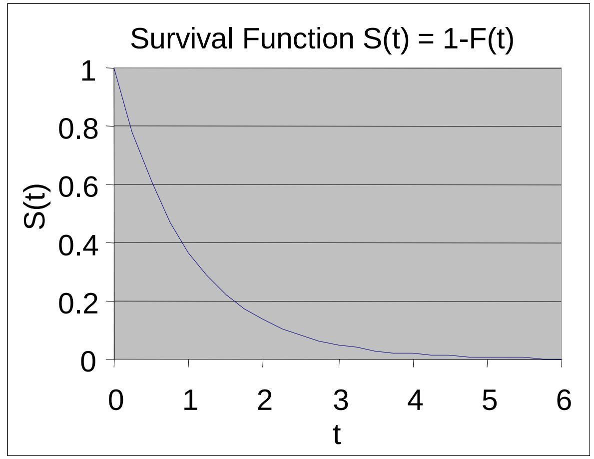
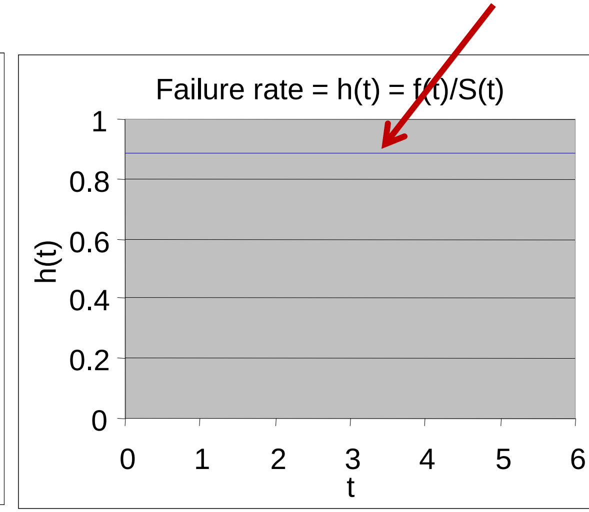
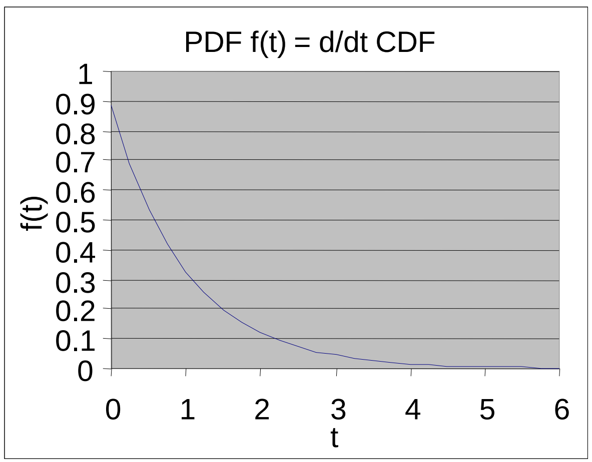
  - x-axis = x
  - λ = slope

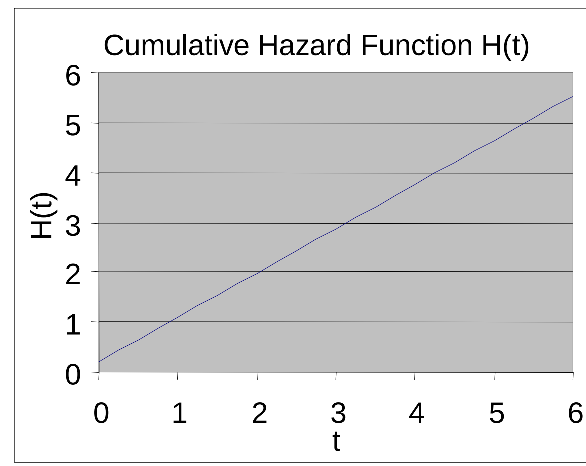
Exponential Reliability Plots











Use of Exponential Distributions

- Constant fail rate
  - No "memory" of the past; no age
  - Radioactive decay
  - Soft errors, external environment
- Easy to calculate
  - MTTF = 1/λ
  - Median time to fail from $F(t_{50}) = 1 - e^{-\lambda t_{50}} = 0.5$ so $t_{50} = \frac{\ln 2}{\lambda}$

Exercise 4.1

- Given an exponential fail distribution with

$$\lambda = \frac{0.04\%}{\text{khr}}$$

what is the probability of failure within 15,000 hours of use?
What is the MTTF?

Solution 4.1

- Convert to "pure" units

$$\lambda = \frac{0.04\%}{\text{khr}} = 0.000\,000\,4\,\frac{\text{fails}}{\text{hour}}$$

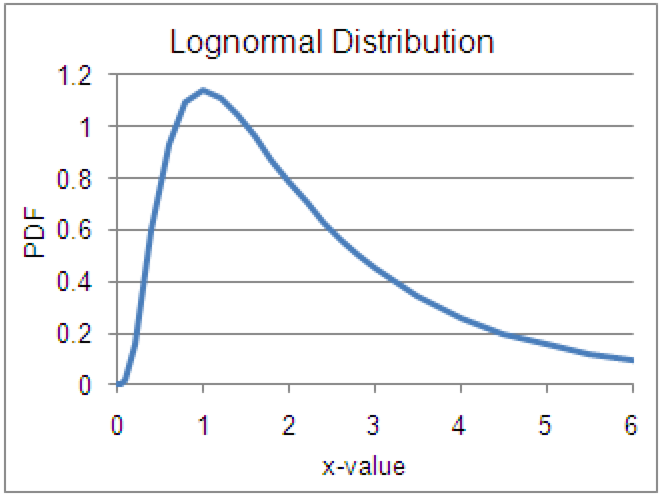
then evaluate the fail function at 15,000 hours

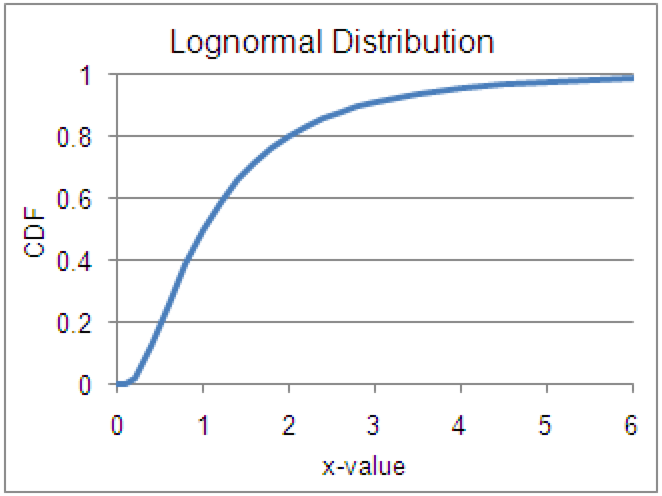
$$F(t) = 1 - e^{-\lambda t} = 1 - e^{-0.000\,000\,4 \times 15{,}000} = 0.006 = 0.6\%$$

The MTTF is even easier

$$MTTF = \frac{1}{\lambda} = 2{,}500{,}000\ \text{hours}$$

LogNormal Distribution





t50 = median time to fail
σ = standard deviation

$$f(t) = \frac{1}{\sigma\, t\sqrt{2\pi}}\, e^{-\frac{1}{2}\left[\frac{\ln(t)-\ln(t50)}{\sigma}\right]^2}$$

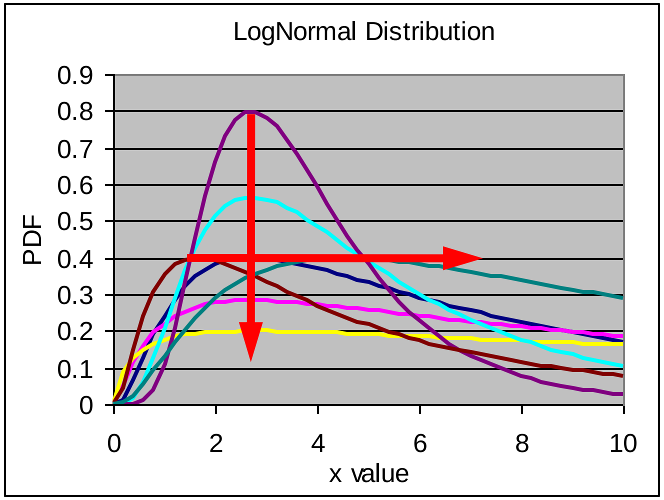
$$F(t) = \int_{-\infty}^{t} dt' \frac{1}{\sigma\, t'\sqrt{2\pi}}\, e^{-\frac{1}{2}\left[\frac{\ln(t')-\ln(t50)}{\sigma}\right]^2}$$

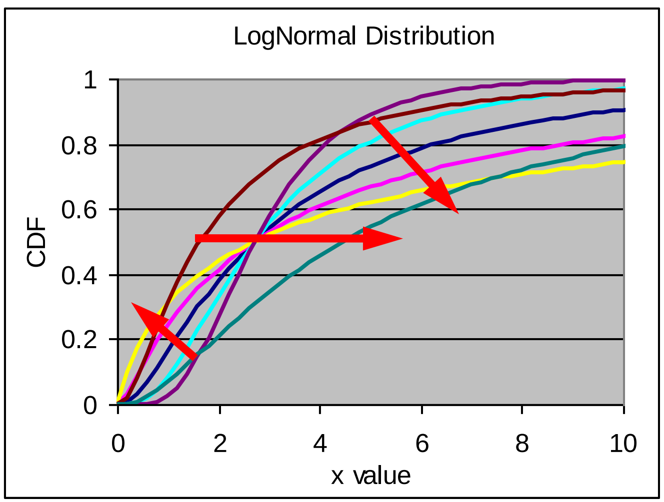
$$\text{rand normal} = \exp(NORMSINV(CDF))$$

where CDF is rand uniform

- Using Excel:
  - PDF = NORMDIST(ln(t),ln(t50),σ,FALSE)/t
  - CDF = NORMDIST(ln(t),ln(t50),σ,TRUE)
- Plot using:
  - y-axis = probit = NORMSINV(CDF)
  - x-axis = ln(t)
  - σ = 1/slope
  - ln(t50) = x-intercept

LogNormal Distribution





t50	1	1	1	1	1	0.5	1.5
std	1	1.41	2	0.71	0.5	1	1

t50 = median time to fail

σ = standard deviation

$$f(t) = \frac{1}{\sigma\, t\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{\ln(t)-\ln(t50)}{\sigma}\right]^2}$$

$$F(t) = \int_{-\infty}^{t} dt' \frac{1}{\sigma\, t'\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{\ln(t')-\ln(t50)}{\sigma}\right]^2}$$

$$\text{rand normal} = \exp\left(NORMSINV\left(CDF\right)\right)$$

where CDF is rand uniform

- Using Excel:
  - PDF = NORMDIST(ln(t),ln(t50),σ,FALSE)/t
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- Plot using:
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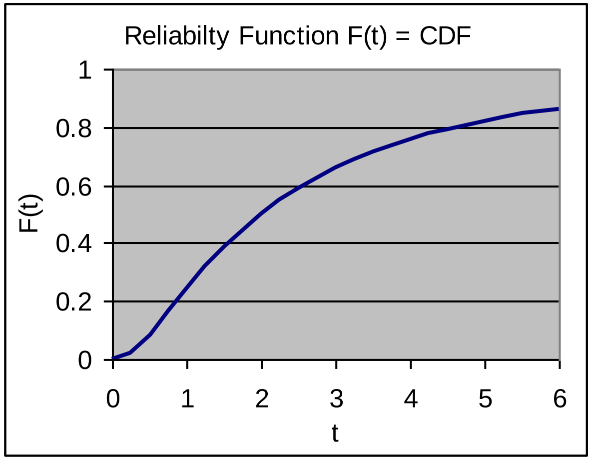
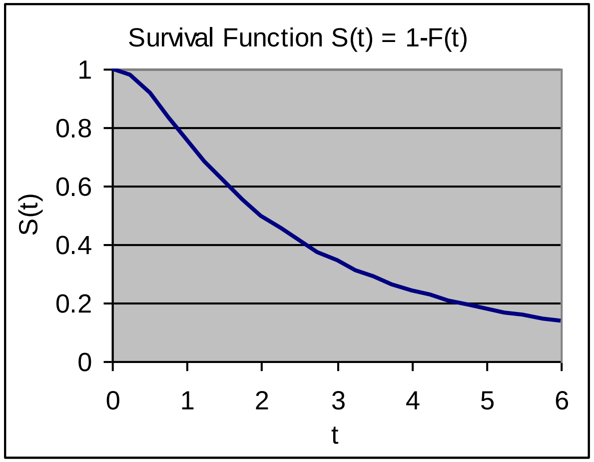
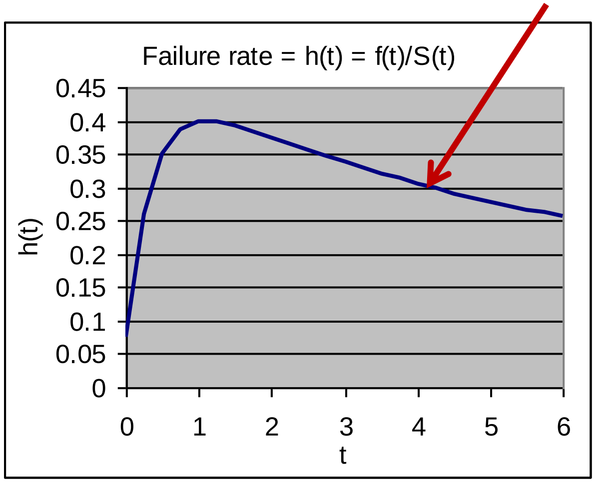
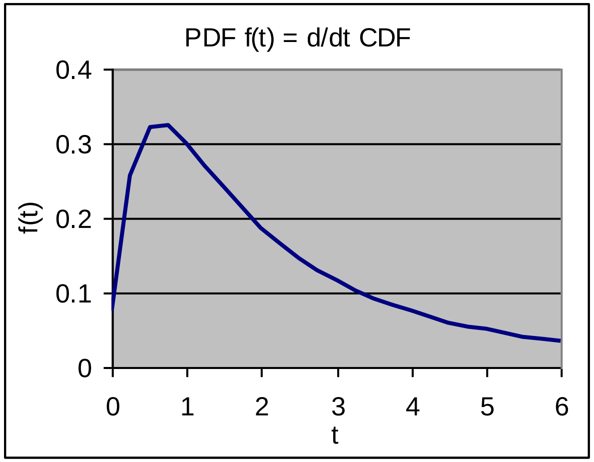
  - x-axis = ln(t)
  - σ = 1/slope
  - ln(t50) = x-intercept

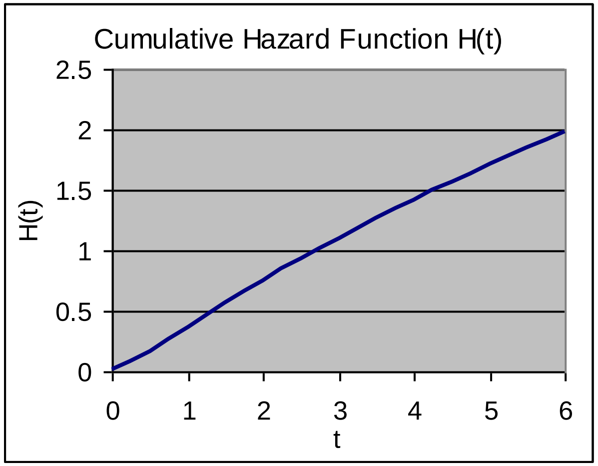
Lognormal Reliability Plots





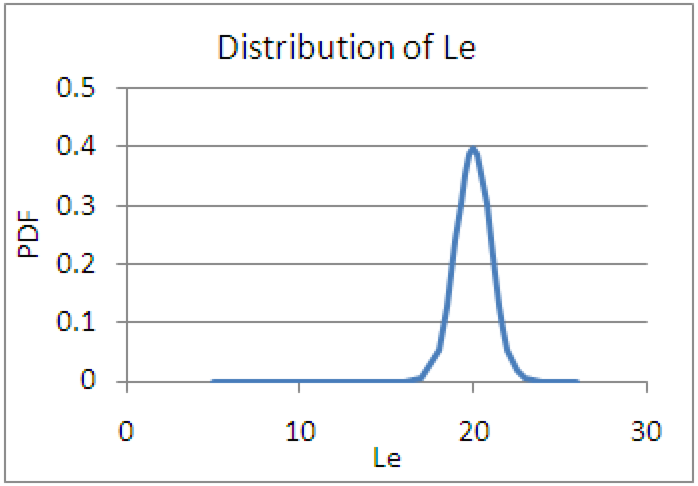




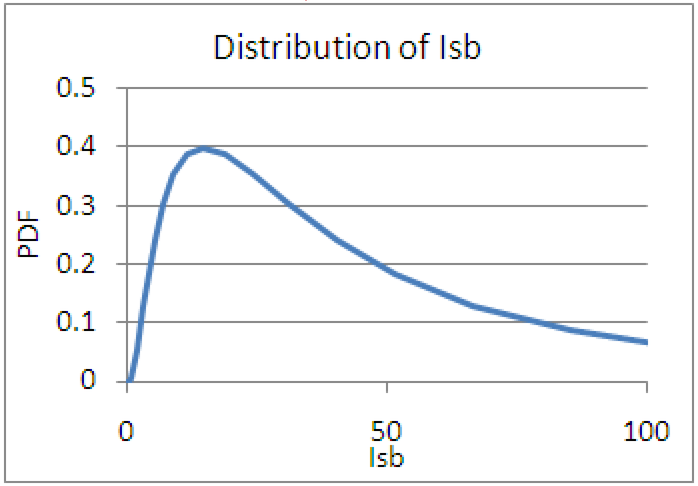


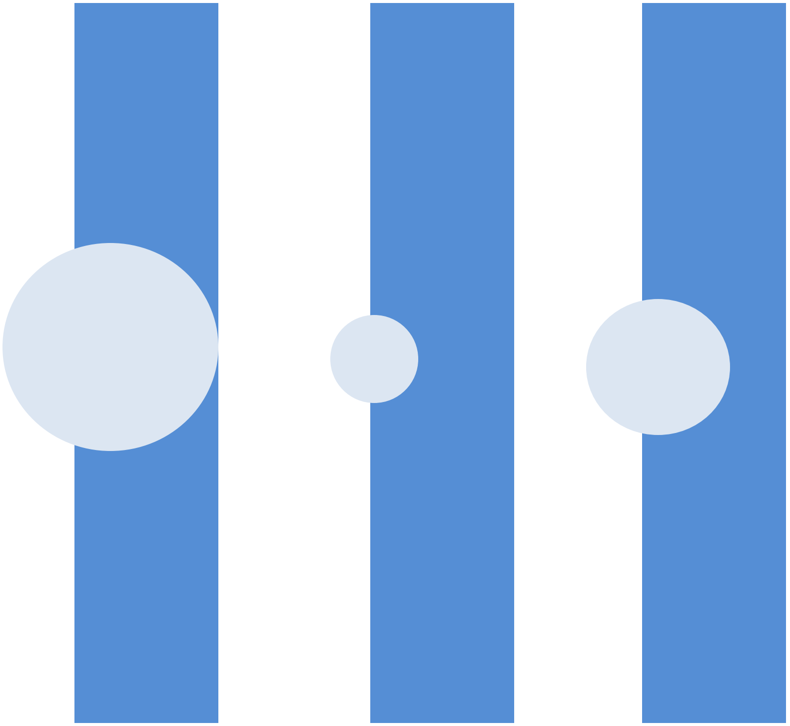
Mostly decreasing failure rate:
IM-type mechanism

Use of Lognormal Distributions



$$I_{SB} \sim e^{Le}$$

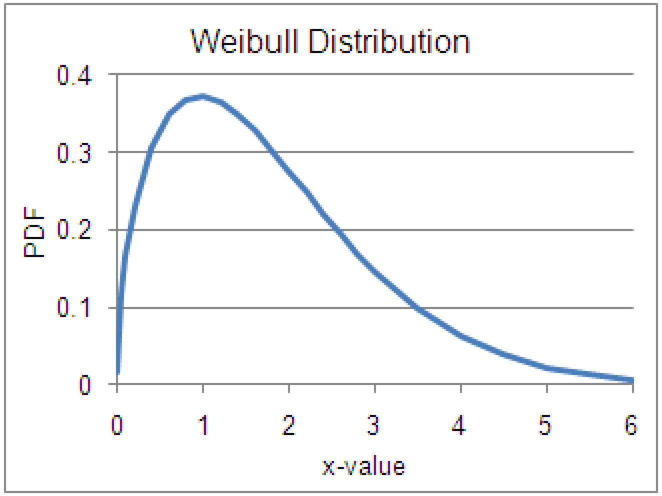


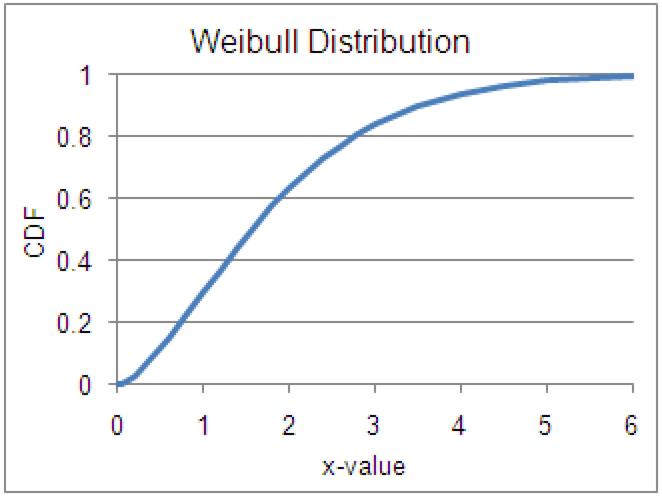


$$R_t = (1+\delta) \times R_{t-1}$$

Weibull Distribution

$$e^{-\left(\frac{t}{\alpha}\right)^{\beta}}$$





$$f(x) = \frac{\beta}{\alpha}\left(\frac{x-\gamma}{\alpha}\right)^{\beta-1} \exp\left[-\left(\frac{x-\gamma}{\alpha}\right)^{\beta}\right]$$

β = shape parameter
α = scale parameter
γ = location parameter

$$F(x) = 1 - \exp\left[-\left(\frac{x-\gamma}{\alpha}\right)^{\beta}\right]$$

Note: α and β are often swapped in meaning!

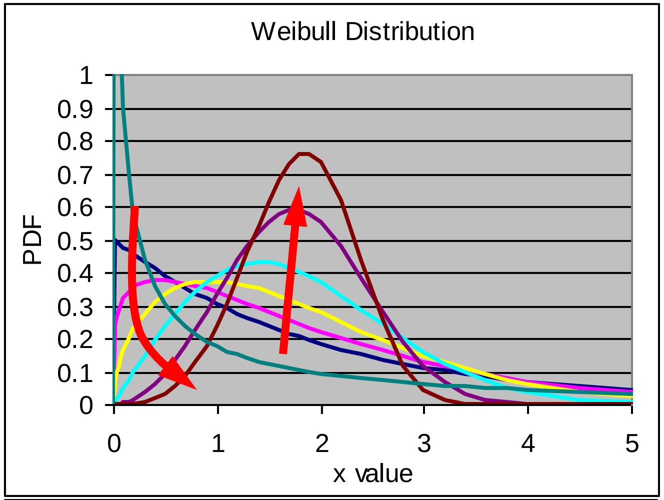
Excel swaps them (below).
T&T use β→m and α→c.

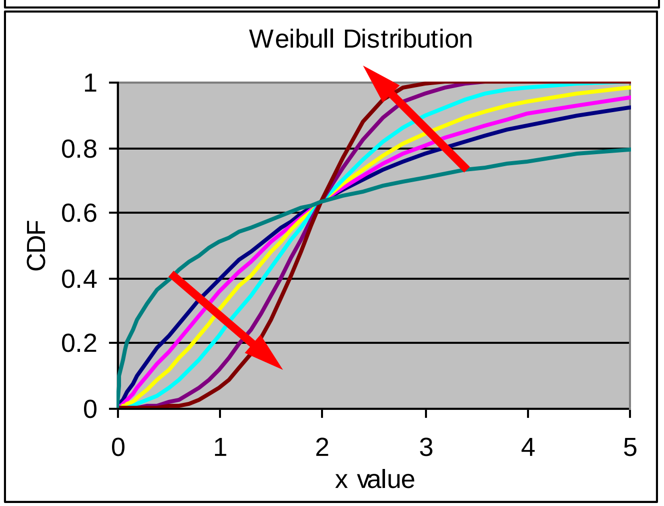
$$\text{rand Weibull} = \alpha\left[-\ln(1-CDF)\right]^{1/\beta}$$

where CDF is rand uniform

- Using Excel:
  - PDF = WEIBULL(x,β,α,FALSE)
  - CDF = WEIBULL(x,β,α,TRUE) =1-EXP(-((x/α)^β))
  - Note γ=0 in Excel
- Plot using:
  - y-axis = weibit = ln(-ln(1-CDF))
  - x-axis = ln(x)
  - β = slope
  - α = exp(-intercept/slope)

Weibull Distribution





beta	1	1.2	1.5	2	3	4	0.5
alpha	2	2	2	2	2	2	2

$$f(x)=\frac{\beta}{\alpha}\left(\frac{x-\gamma}{\alpha}\right)^{\beta-1}\exp\left[-\left(\frac{x-\gamma}{\alpha}\right)^{\beta}\right]$$

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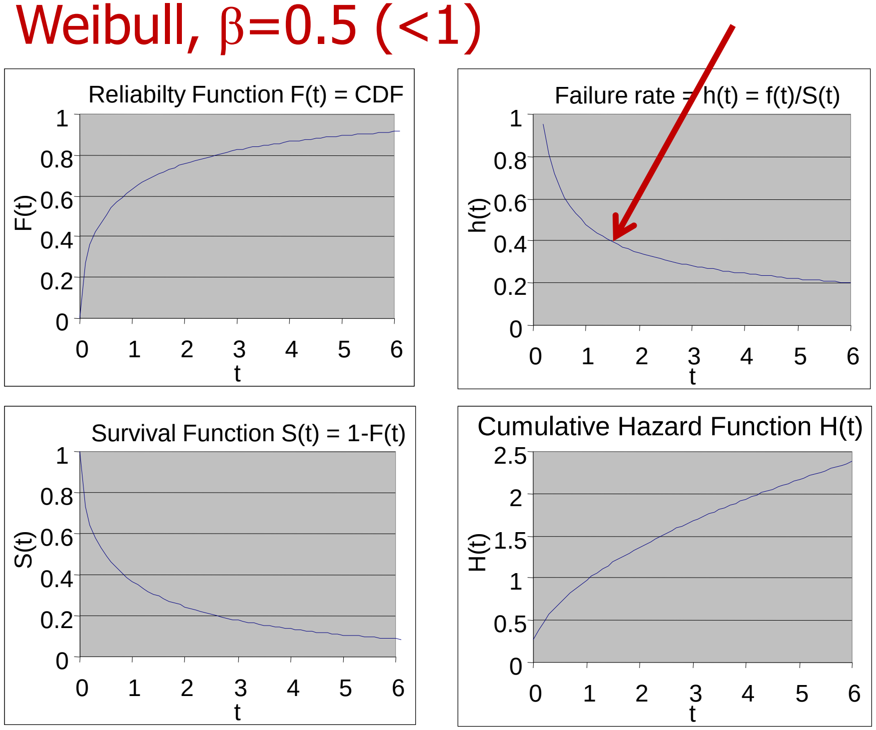
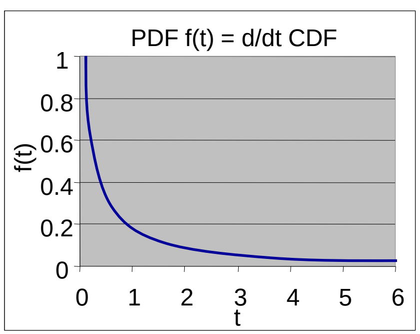
Weibull Reliability Plots

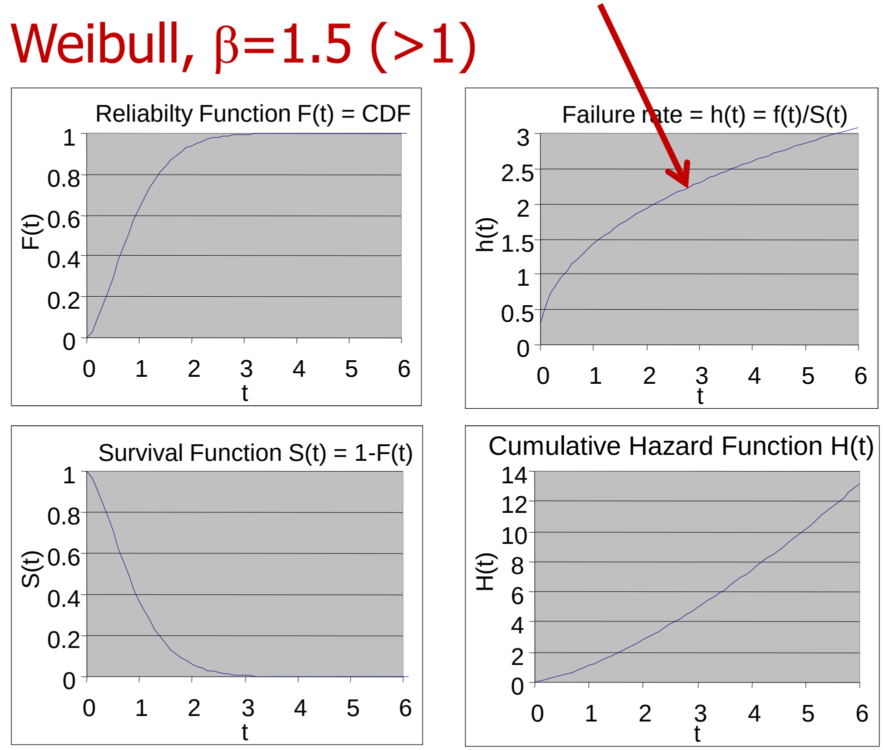
Weibull, β=0.5 (<1)

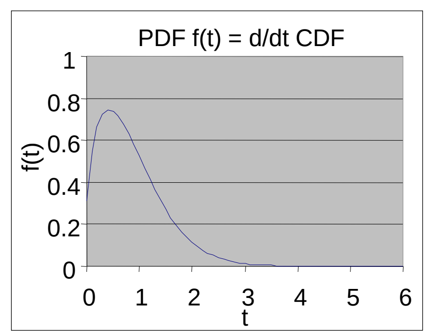




Decreasing failure rate:
Infant Mortality (IM) type mechanism

Weibull, β=1.5 (>1)

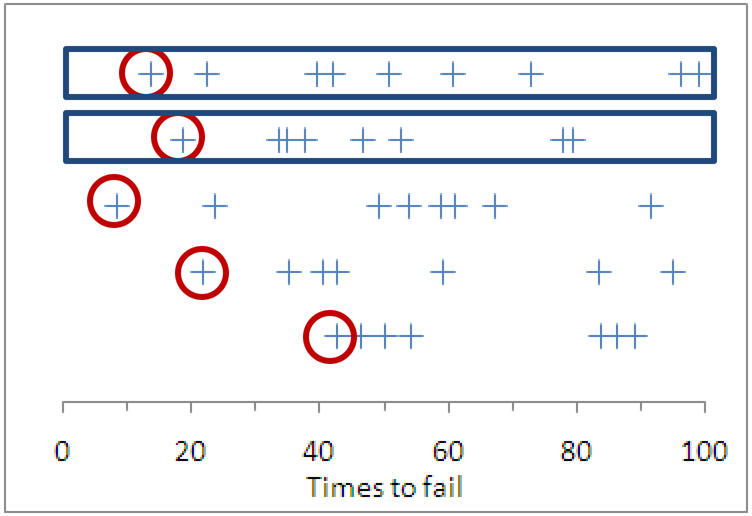


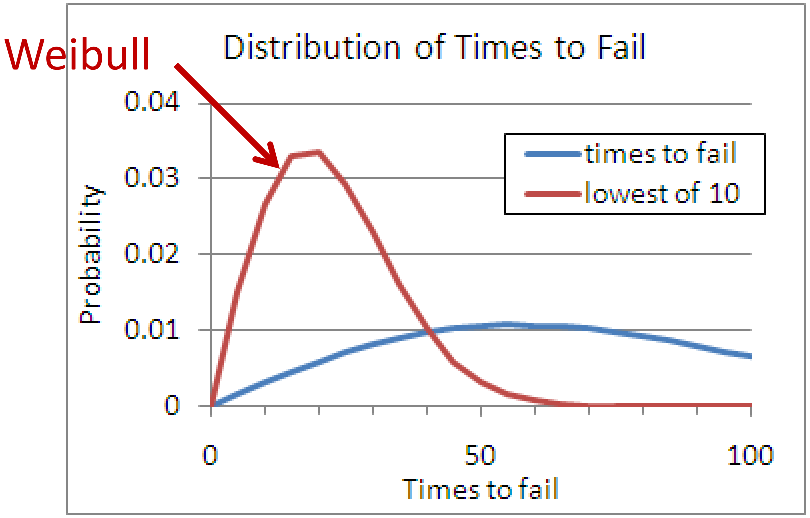


Increasing failure rate:
Wearout (WO) type mechanism

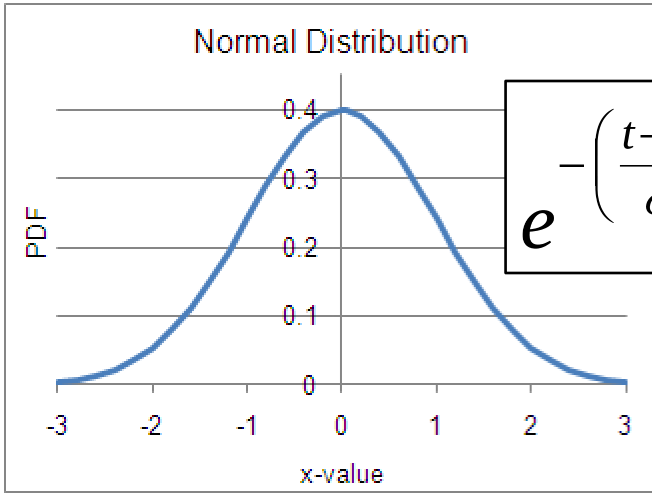
Use of Weibull Distributions

- When fail is caused by the worst of many items
- When it fits the data well

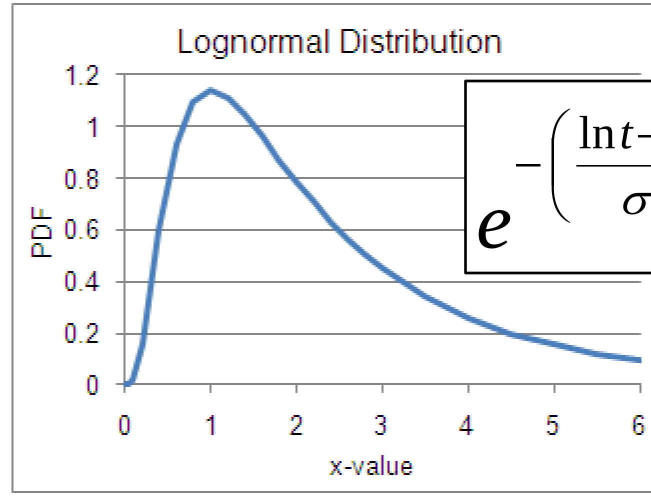




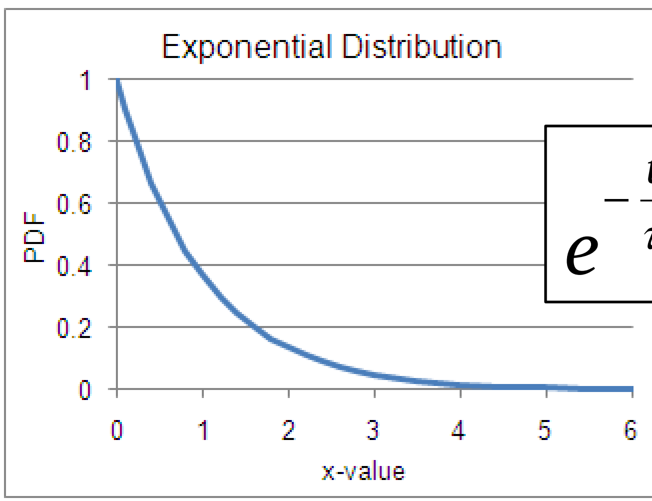
Main Reliability Functions



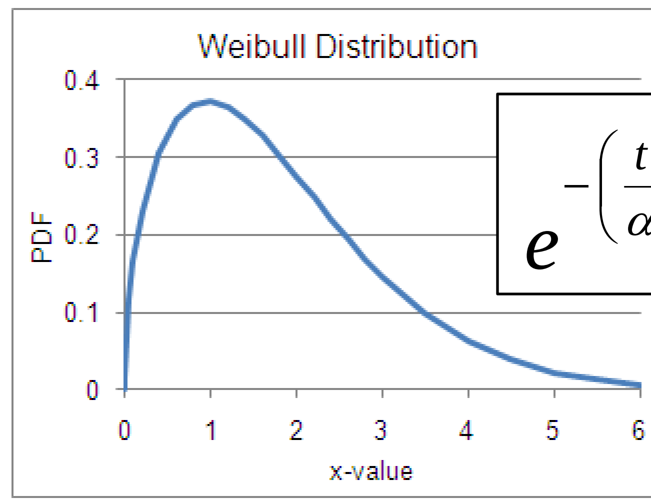
$$e^{-\left(\frac{t-\mu}{\sigma}\right)^2}$$



$$e^{-\left(\frac{\ln t-\mu}{\sigma}\right)^2}$$



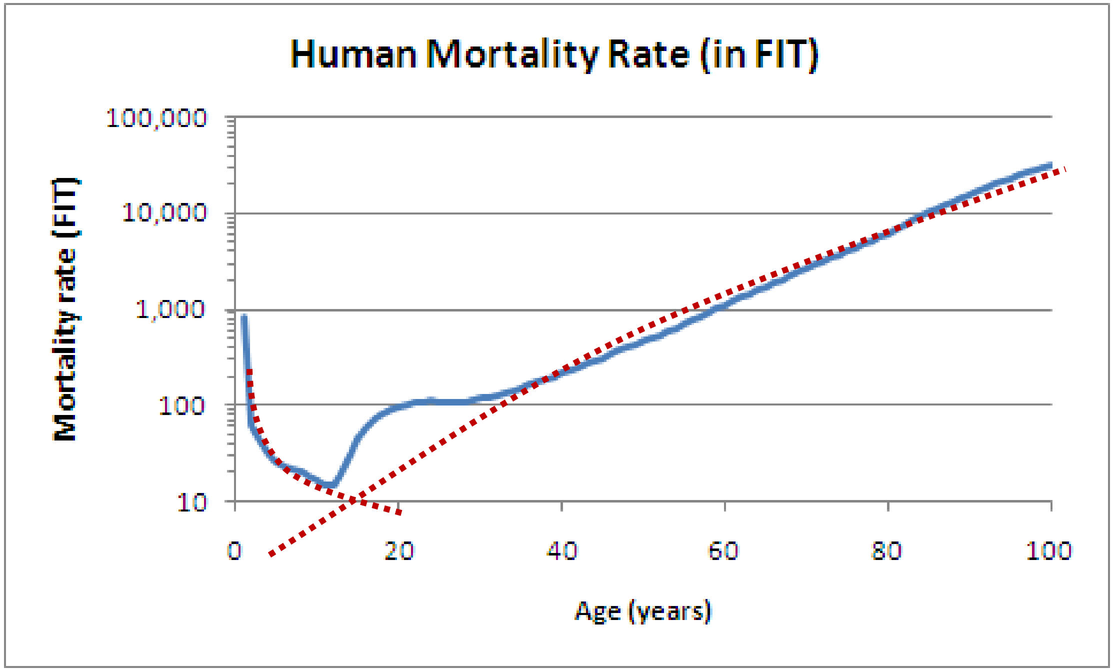
$$e^{-\frac{t}{\tau}}$$



$$e^{-\left(\frac{t}{\alpha}\right)^\beta}$$

Multiple Mechanisms

Multiple Mechanisms



Survivals multiply, hazard rates add:

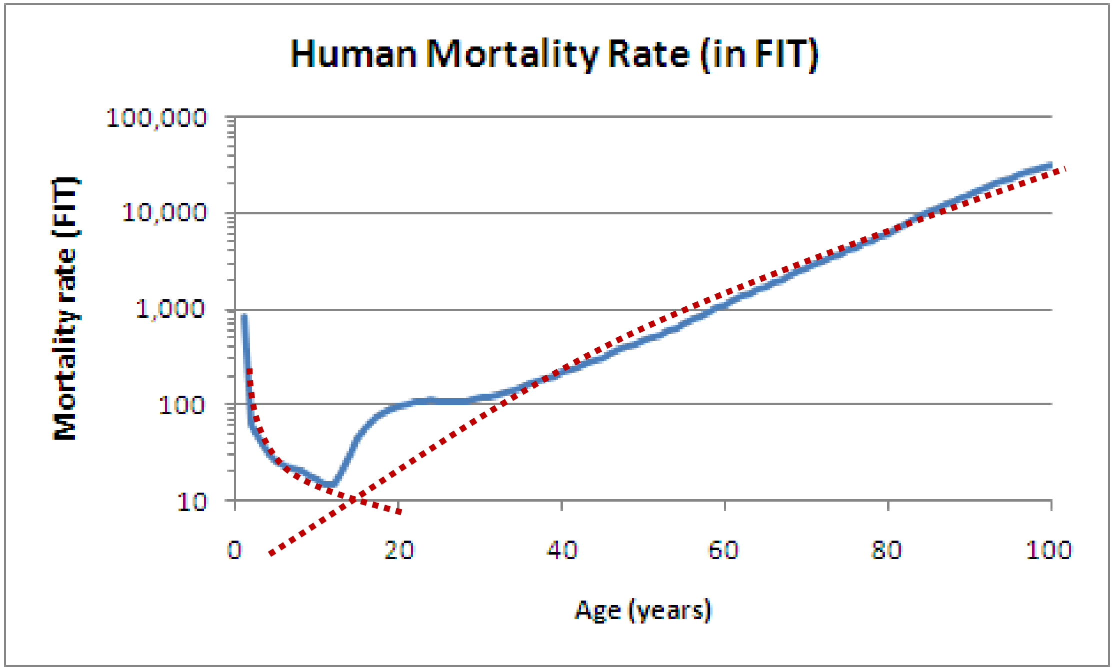
$$S_{tot}(t) = S_1(t)\, S_2(t)$$

$$F_{tot}(t) = 1 - S_1(t)\, S_2(t) \approx F_1(t) + F_2(t)$$

$$h_{tot}(t) \approx h_1(t) + h_2(t)$$

Exercise 4.2

Hand fit 2 Weibull distributions to the human mortality data like this:



Plot both the hazard rate h(t) (like above) and the fail function F(t).

Useful: for the Weibull, from T&T table 4.3:

$$h(t) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta}$$

Solution 4.2

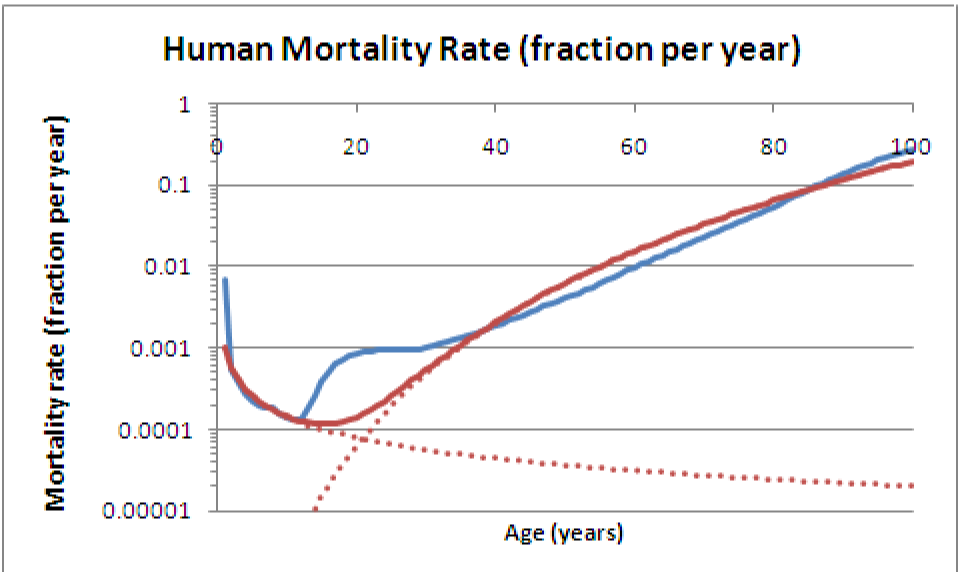
$$h(t)=\frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta}$$

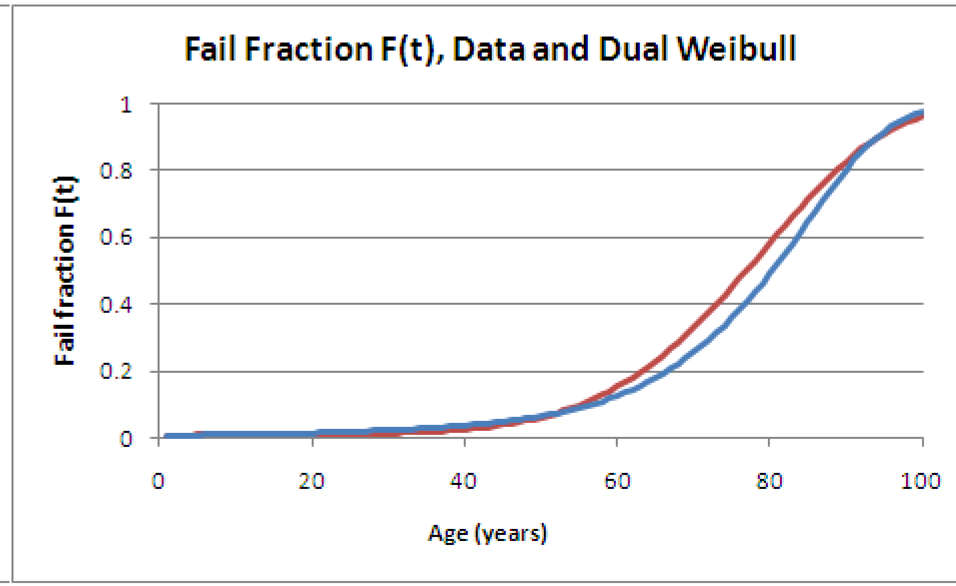
$$h_1(t)+h_2(t)$$

$$F(t)=1-e^{\left(\frac{t}{\alpha_1}\right)^{\beta_1}}e^{\left(\frac{t}{\alpha_2}\right)^{\beta_2}}$$

			alpha	3E+14	82		
			beta	0.15	6		
Age	data h(t)	data H(t)	data F(t)	Weib1 h(t)	Weib2 h(t)	Weib h(t)	Weib F(t)
1	0.00706	0.00706	0.007035	0.0010105	1.974E-11	0.00101	0.006714
2	0.00053	0.00759	0.007561	0.0005606	6.316E-10	0.000561	0.007447
3	0.00036	0.00795	0.007918	0.0003972	4.796E-09	0.000397	0.007912
4	0.00027	0.00822	0.008186	0.000311	2.021E-08	0.000311	0.008259

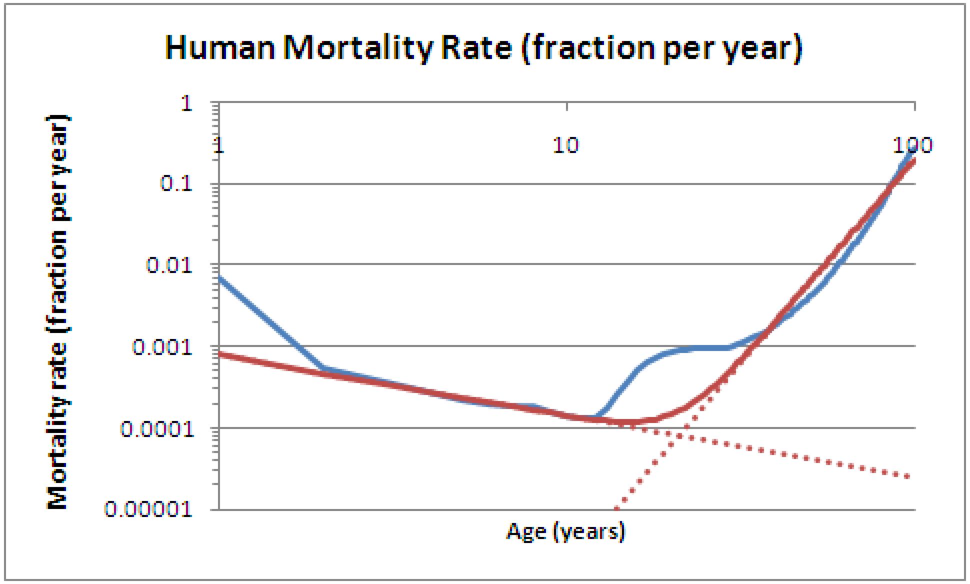
alpha	3E+14	82
beta	0.15	6





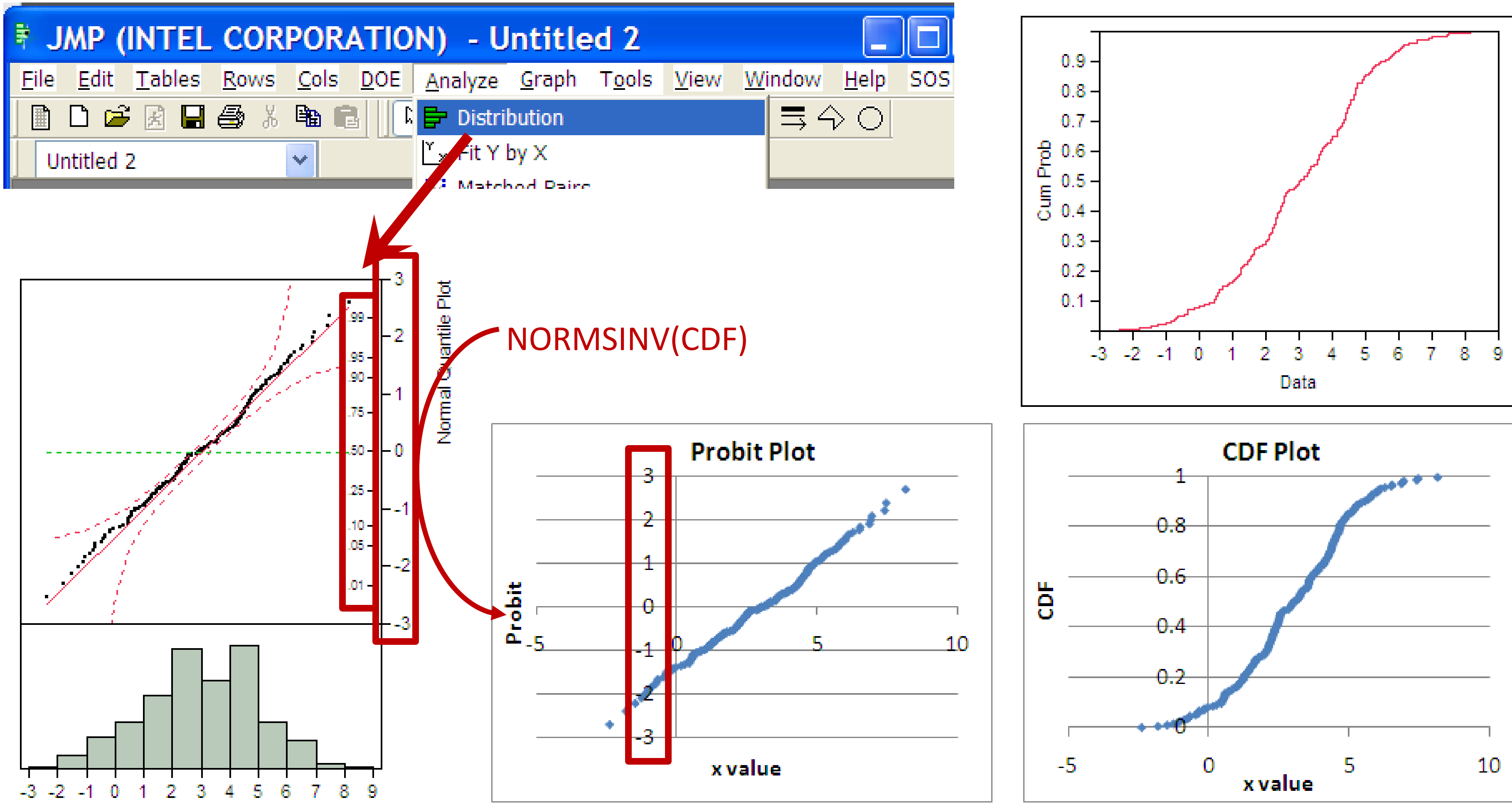
Reliability Plotting

Reliability Plotting



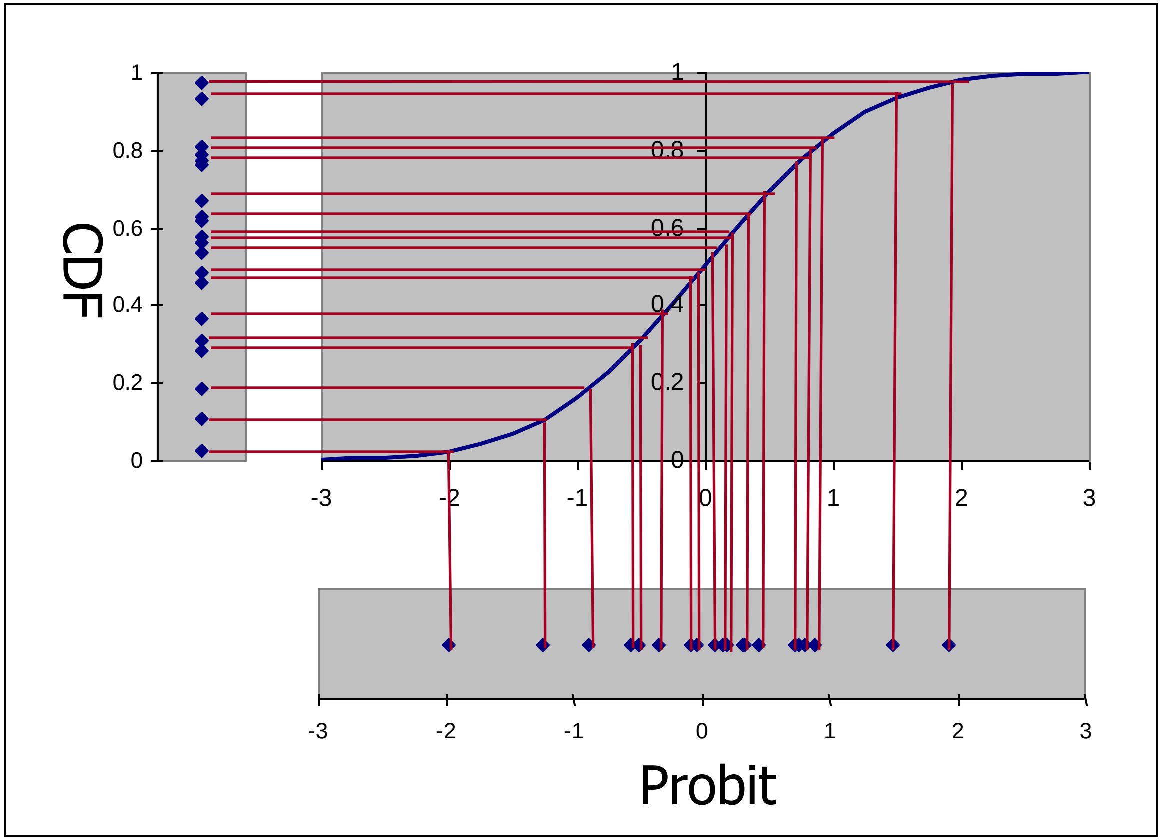
- Note straight lines (dotted, each Weibull)

Probit Plot



- Our eyes detect straight lines

Excel NORMxxx Functions

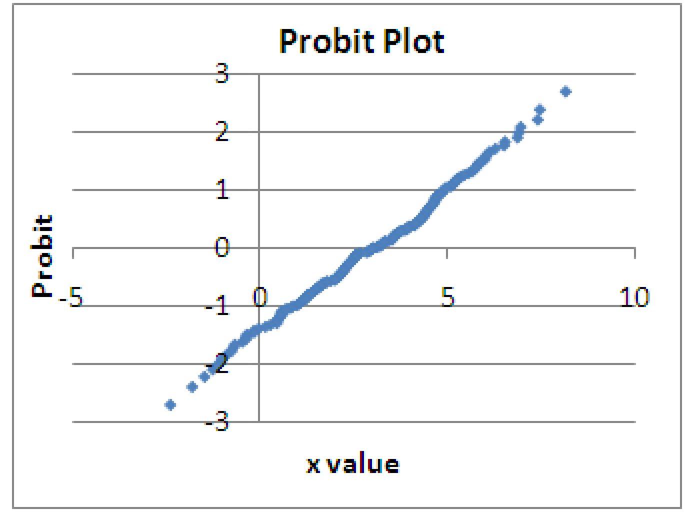


- Probit = NORMSINV(CDF)
- CDF = NORMSDIST(Probit)

Probit Plots in Excel

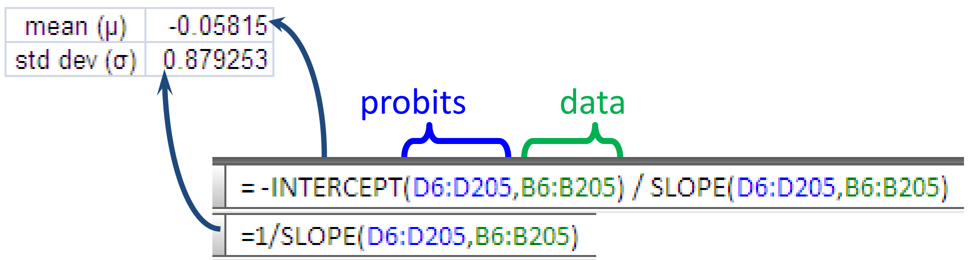
=NORMSINV(C6)

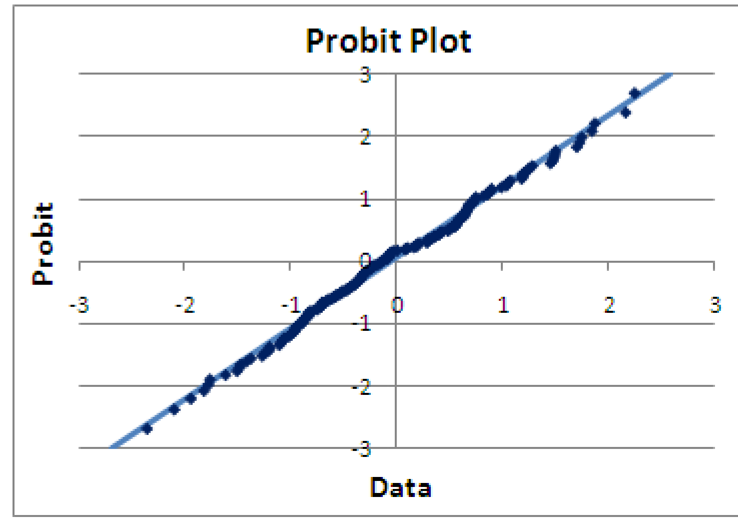
	200	
Data	**CDF**	**Probit**
2.92116	0.482535	-0.04379
4.69107	0.796906	0.830621
3.863768	0.622255	0.31141
0.556751	0.118263	-1.18371



- Plot using:
  - y-axis = probit = NORMSINV(CDF)
  - x-axis = x
  - $\sigma = 1/\text{slope}$
  - μ = x-intercept = – (y-intercept) / slope

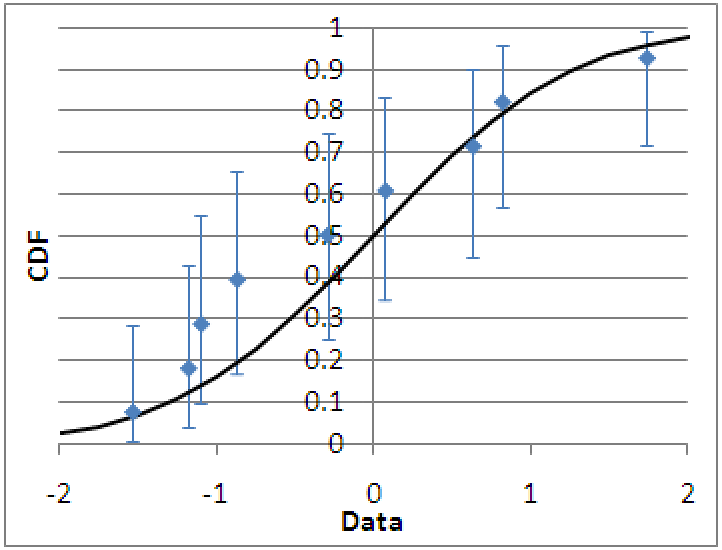
Probit Plots in Excel

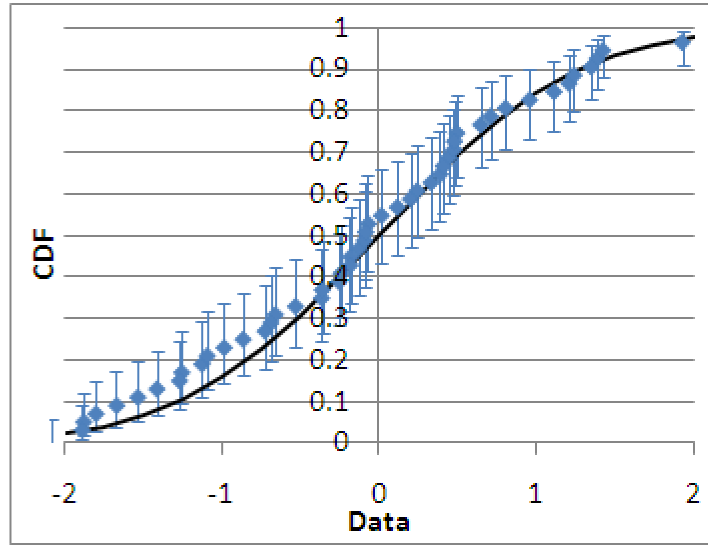


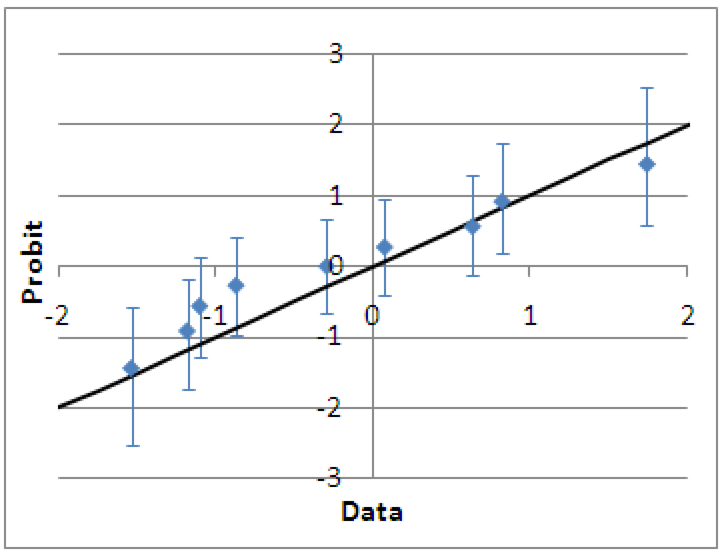


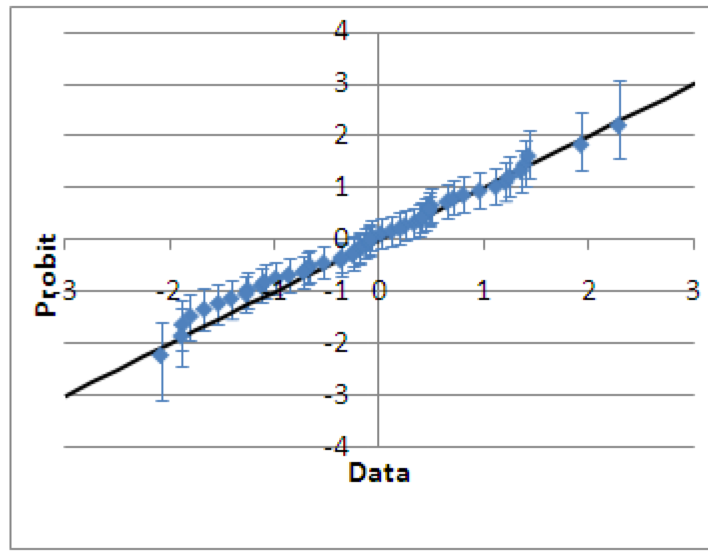
- Plot using:
  - y-axis = probit = NORMSINV(CDF)
  - x-axis = x
  - σ = 1/slope
  - μ = x-intercept = – (y-intercept) / slope

Uncertainties in Probit Plots





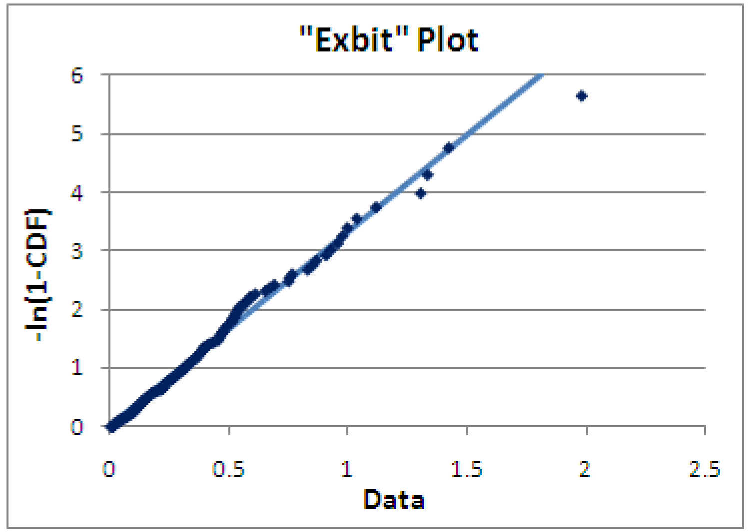




“Exbit” Plots

=-LN(1-H7)

Data	CDF	Probit	Exbit
0.257295	0.557385	0.144343	0.815055
0.04842	0.128244	-1.13473	0.137245
0.134112	0.347804	-0.39125	0.427411
0.032308	0.083333	-1.38299	0.087011



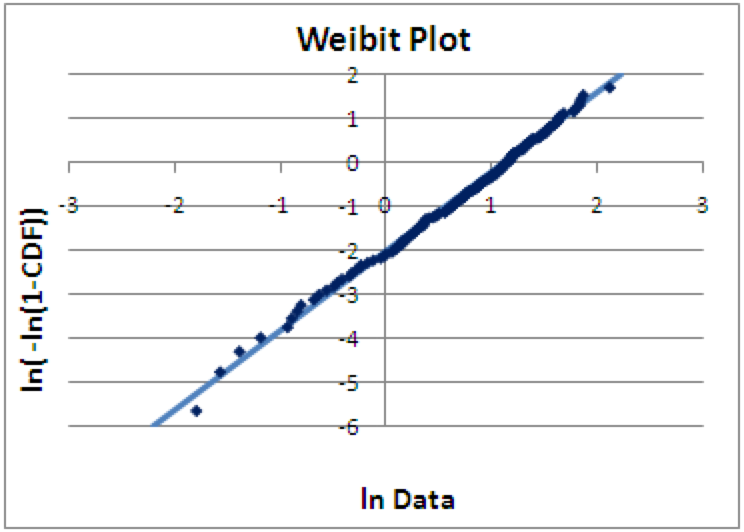
“exbits” data

=SLOPE(J7:J206,G7:G206)

lambda (λ)	3.29628329

- Plot using:
  - y-axis = “exbit” = -LN(1-CDF)
  - x-axis = x
  - λ = slope
- Note that “exbit” is not a standard name

Weibit Plots



=LN(-LN(1-H7))

Data	CDF	Probit	Exbit	ln Data	Weibit
3.857623	0.796906	0.830621	1.594087	1.350051	0.466301
3.044861	0.627246	0.324567	0.986835	1.113455	-0.01325
2.905862	0.582335	0.207871	0.873076	1.06673	-0.13573

Weibit ln data

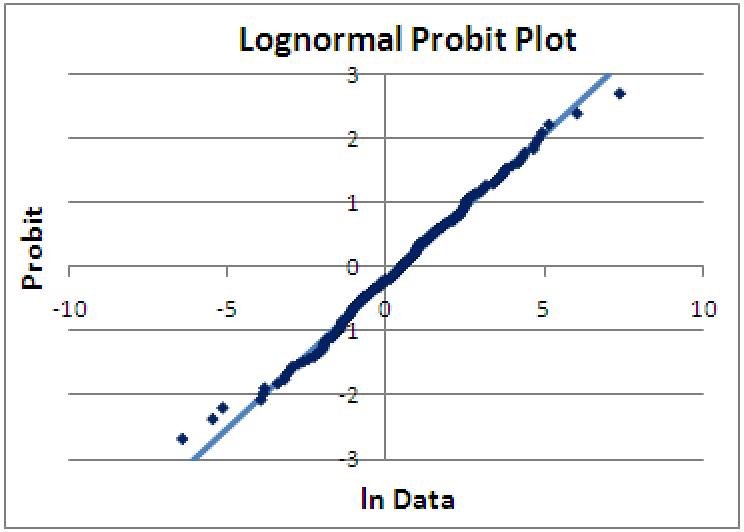
=SLOPE(L7:L206,K7:K206)

shape (β)	1.80701926
scale (α)	3.05820444

=EXP(-INTERCEPT(L7:L206,K7:K206)/T38)

- Plot using:
  - y-axis = Weibit = ln(-ln(1-CDF))
  - x-axis = ln(x)
  - β = slope
  - α = exp(-intercept/slope)
- Note that “Weibit” *is* a standard name

Lognormal Probit Plot



Data	CDF	Probit	Exbit	ln Data	Weibit
0.072804	0.068363	-1.48809	0.070812	-2.61998	-2.64772
5.155989	0.722056	0.58896	1.280335	1.640159	0.247122
171.1415	0.986527	2.212298	4.307064	5.142491	1.460256

probits ln data

=-INTERCEPT(I7:I206,K7:K206) / SLOPE(I7:I206,K7:K206)

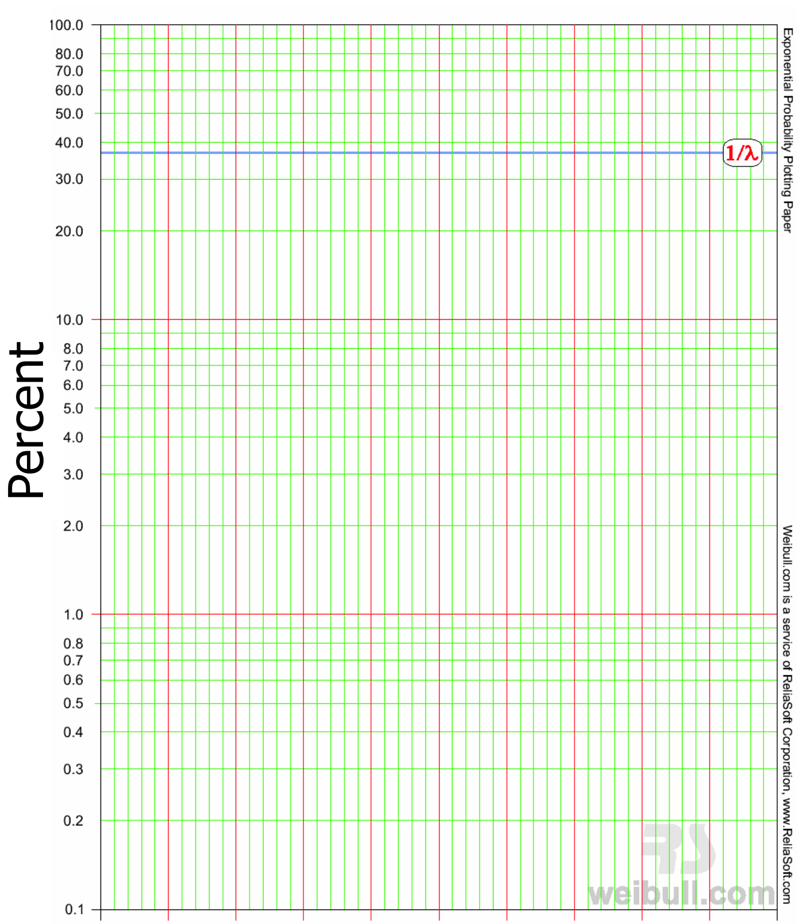
mean (μ)	0.46846058
std dev (σ)	2.21137733

=1/SLOPE(I7:I206,K7:K206)

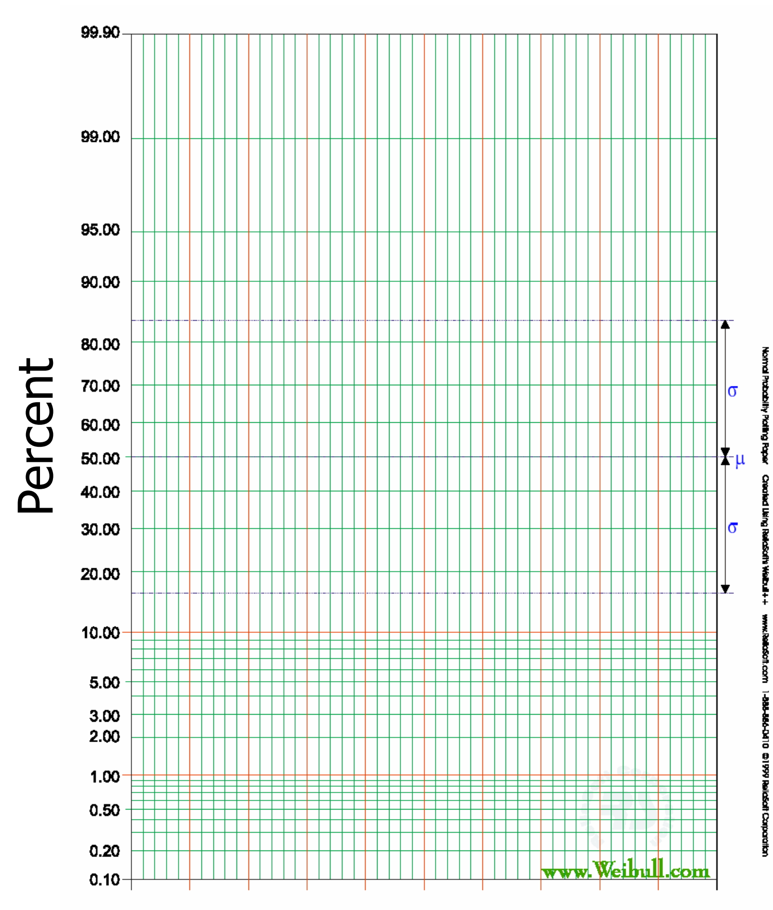
- Plot using:
  - y-axis = probit = NORMSINV(CDF)
  - x-axis = ln(t)
  - σ = 1/slope
  - ln(t50) = x-intercept

The Graph Paper Method

Exponential (semi-log)

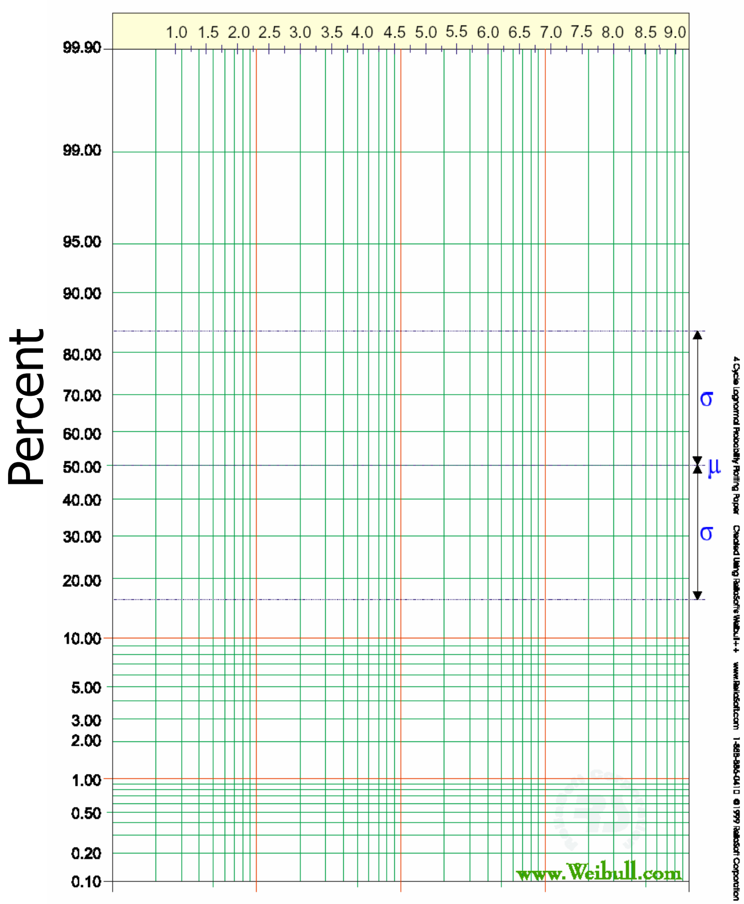


Normal

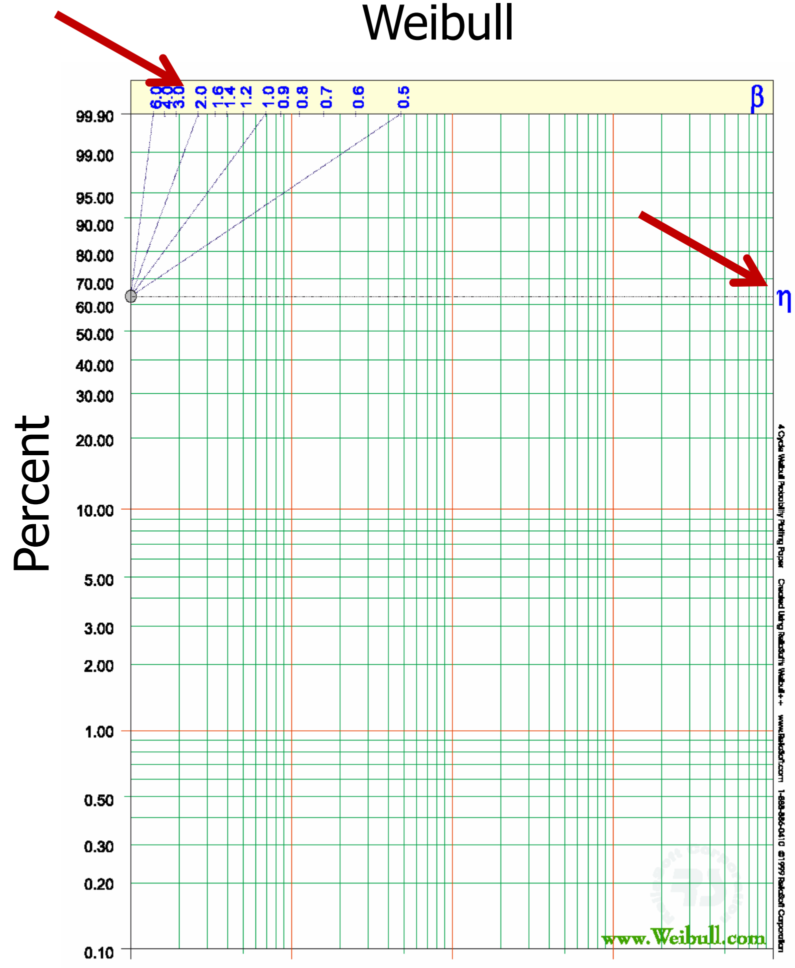


More Graph Paper

Lognormal



Weibull

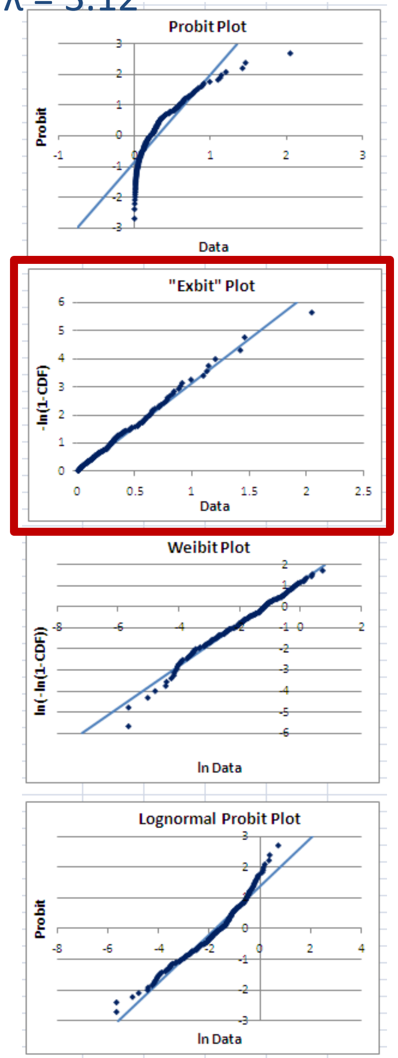


Exercise 4.3

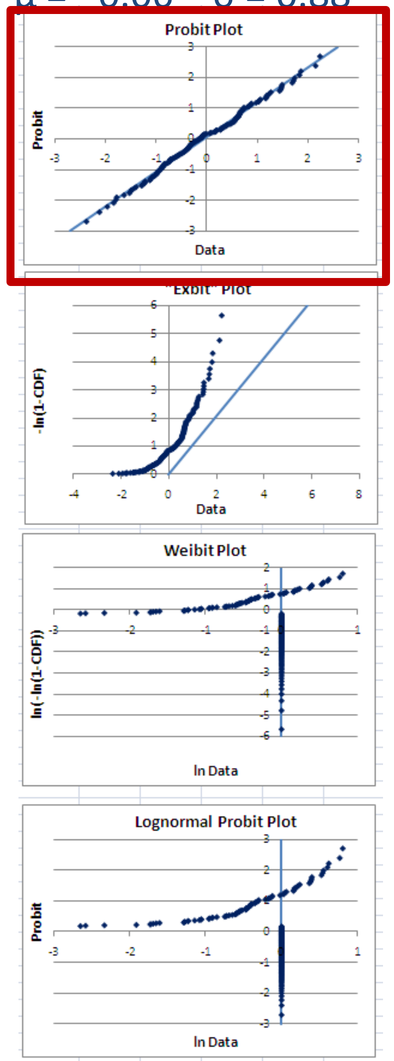
- Make probit, “exbit”, Weibit, and lognormal probit plots
- Determine parameters for each plot
- Look at all 4 data sets (0 – 3)
- Determine which type each distribution is
  - Give the parameters for each correct distribution

Solution 4.3

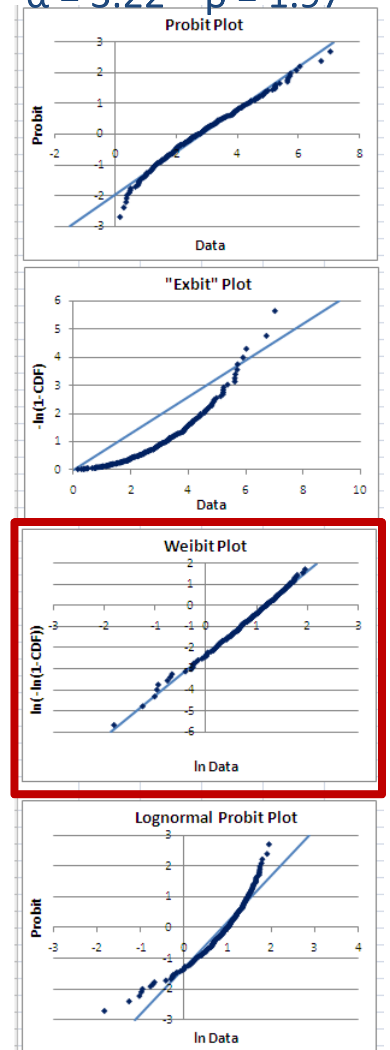
Data0 – exponential
λ = 3.12



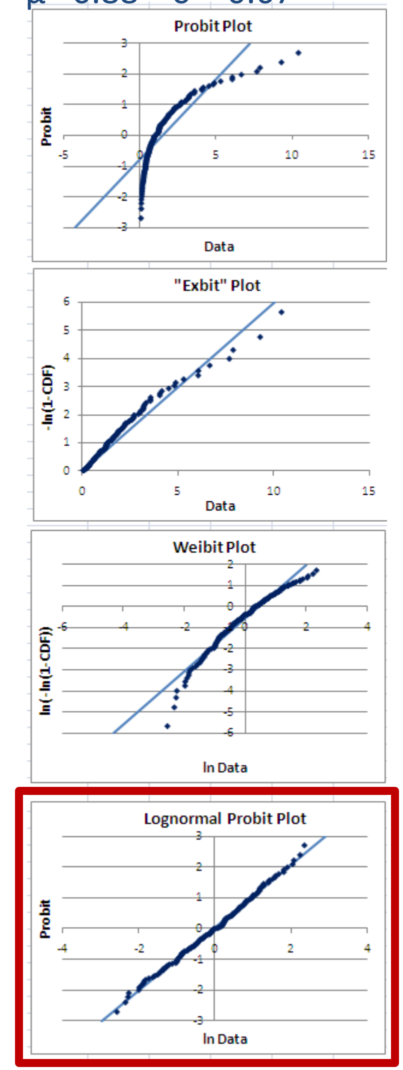
Data1 – normal
μ = –0.06 σ = 0.88



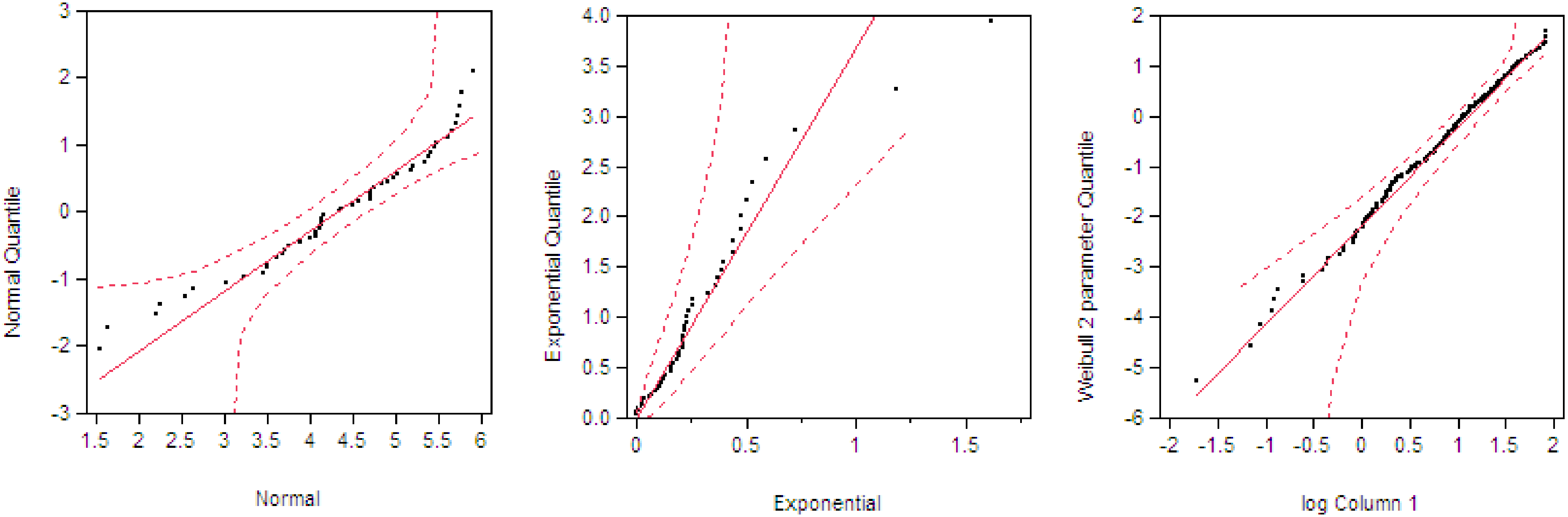
Data2 – Weibull
α = 3.22 β = 1.97



Data3 – lognormal
μ =0.88 σ = 0.67

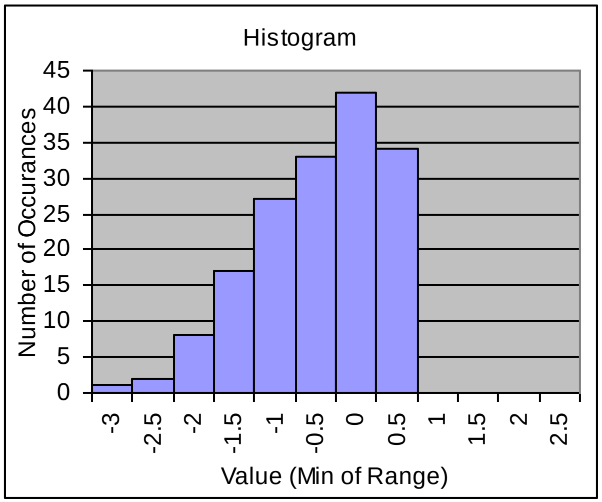


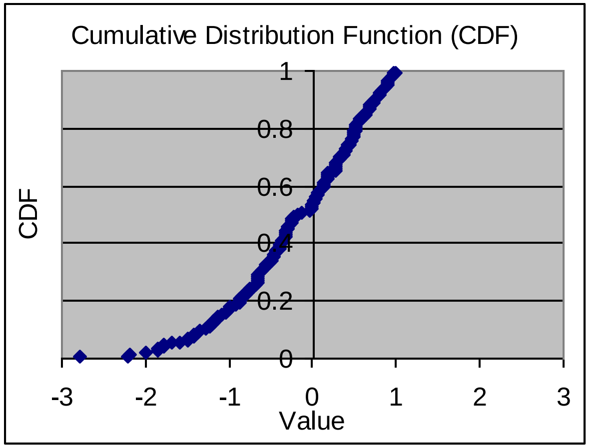
JMP Plots

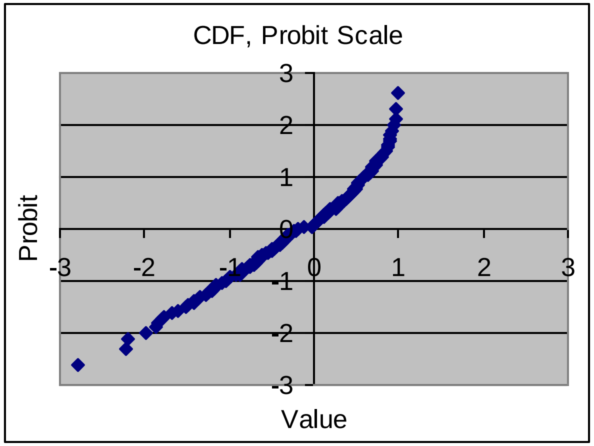


- JMP versions of probit, “exbit”, and Weibit plots

Truncated Distributions







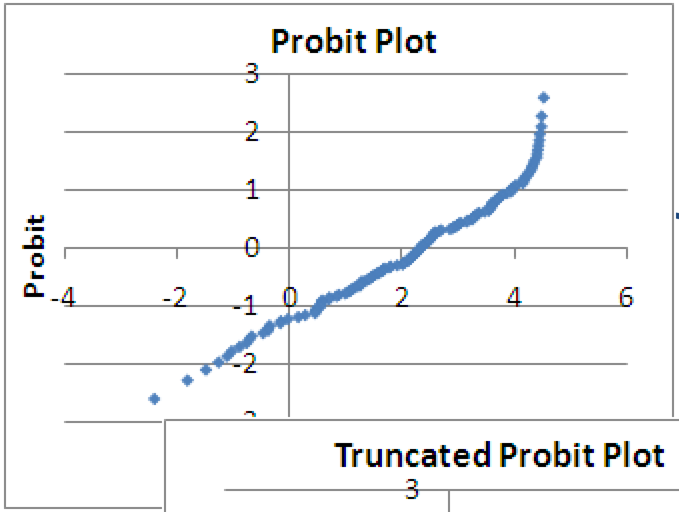
Top-Truncated Distributions

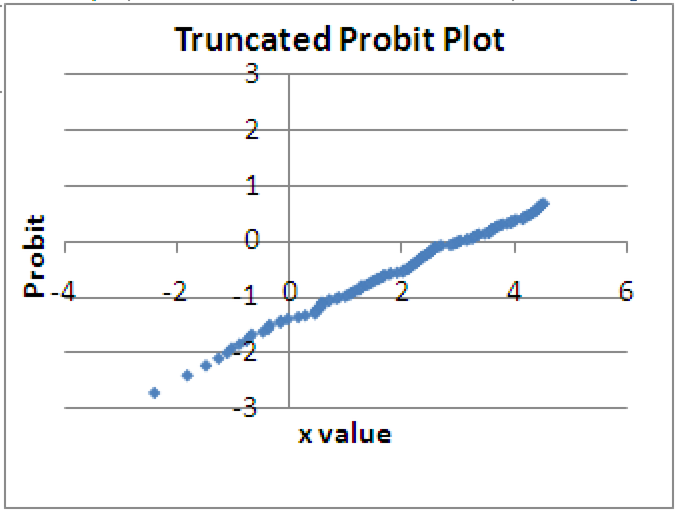
$$\frac{Rank - 0.3}{Count + 0.4}$$

$$\frac{Rank - 0.3}{Count + Missing + 0.4}$$

Missing	50			
Count	150			
Data	**CDF**	**Probit**	**Adj CDF**	**Adj Probit**
4.510582	0.995346	2.600519	0.747006	0.665098
4.473302	0.988697	2.280022	0.742016	0.649573
4.469389	0.982048	2.09801	0.737026	0.634203
4.438034	0.975399	1.966836	0.732036	0.618982
4.42984	0.96875	1.862732	0.727046	0.603903

Note Adj CDF doesn't reach 1

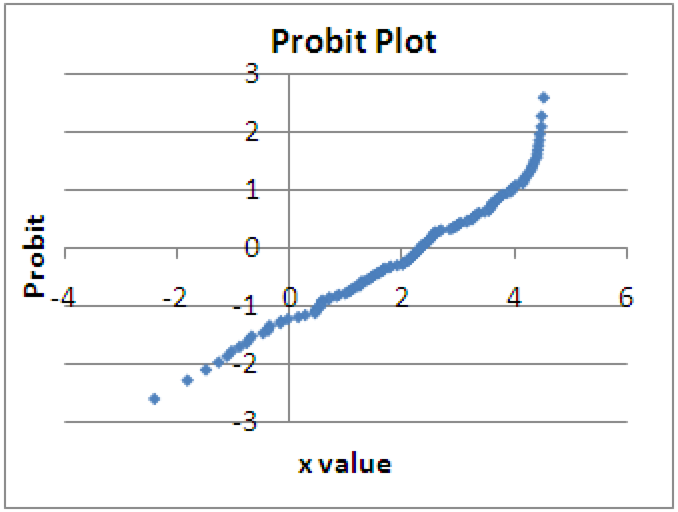




Exercise 4.4

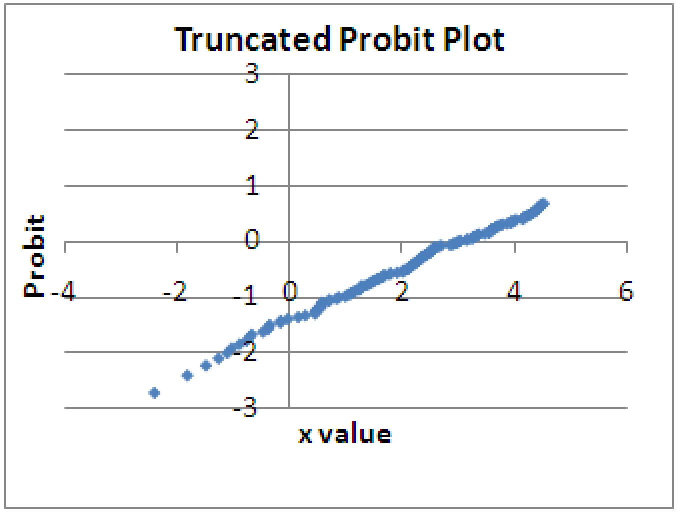
- Make a truncated probit plot of the data on tab Ex8.
- Find the mean and standard deviation of the original distribution as best you can.

Solution 4.4



mean (μ)	2.177593
std dev (σ)	1.630164

Original:	
mean (μ)	3.00
std dev (σ)	2.03



mean (μ)	3.097412
std dev (σ)	2.188841

Data Censoring

- Missing data is called "censored"
  - Type I, time censored
    - Exact times to fail up to time t; no data after
  - Type II, fail count censored
    - Exact times to fail for the first r units to fail; no data after
  - Multicensored or readout
    - Have a time interval within which each unit failed up to tmax; no data after

The End