# ECE 510 Lecture 3
# Functions

## Reliability Functions, T&T 2.1-6, 9
## Distributions, T&T 3.1-4, 4.1-4, 5.1-3

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# Reliability Functions

# Reliability Functions

- Functions of time
  - CDF(x) → F(t)
- Survival function S(t) = 1-F(t)
- PDF(x) → f(t)

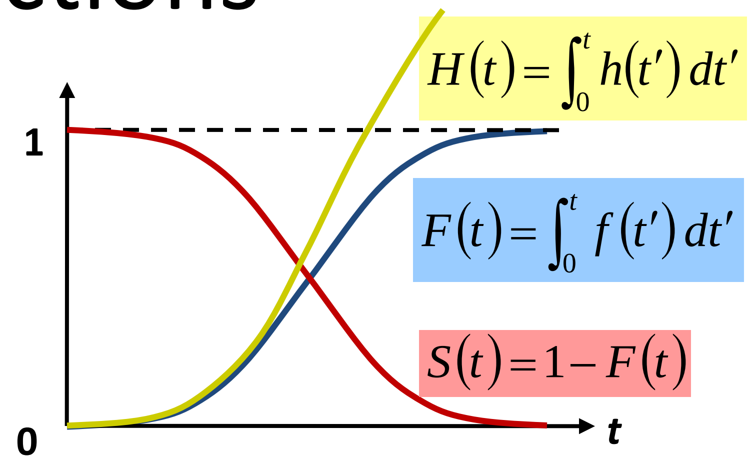
$$f(t) = \frac{\text{fraction of ORIGINAL population that fails in } dt}{dt}$$
$$= \frac{dF(t)}{dt} = -\frac{dS(t)}{dt}$$

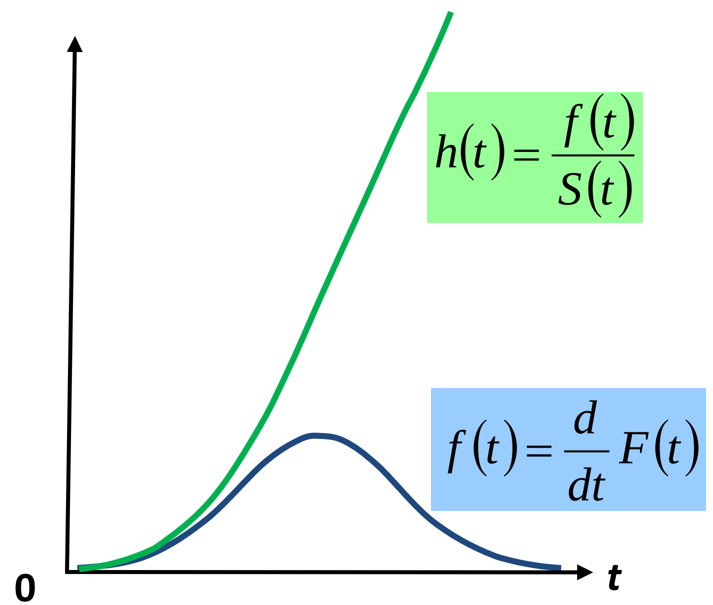
- Hazard function h(t)

$$h(t) = \frac{\text{fraction of CURRENT population that fails in } dt}{dt}$$
$$= \frac{f(t)}{S(t)} = -\frac{dS(t)}{dt}\frac{1}{S(t)} = -\frac{d \ln S(t)}{dt}$$

- Cum hazard function H(t)

$$H(t) = \int_0^t h(t)\,dt$$
$$S(t) = \exp[-H(t)]$$
$$F(t) = 1 - \exp[-H(t)]$$

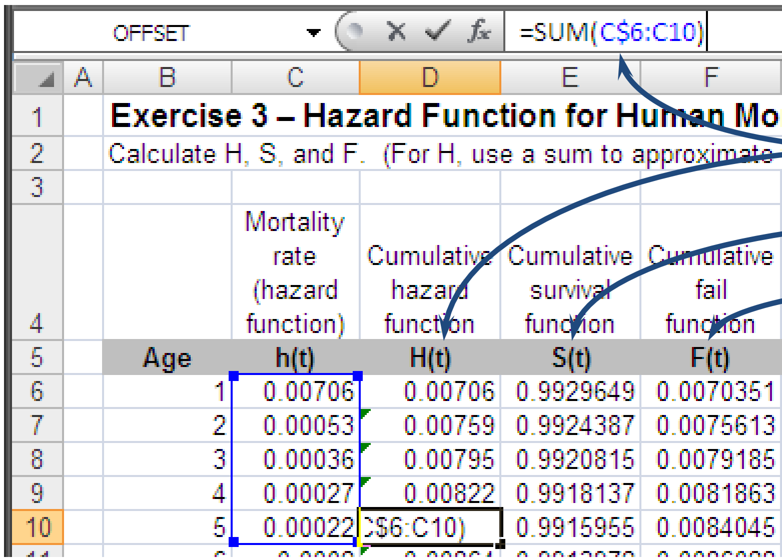


# Exercise 3.1a

- Calculate H(t), S(t), and F(t) for the given human mortality data, and plot h(t), S(t), and F(t). The data is given as h(t) for each age, that is, the probability of a living person dying at the given age. Use a sum to approximate the integral for H(t).

# Exercise 3.1a Solution, Part 1



OFFSET =SUM(C$6:C10)

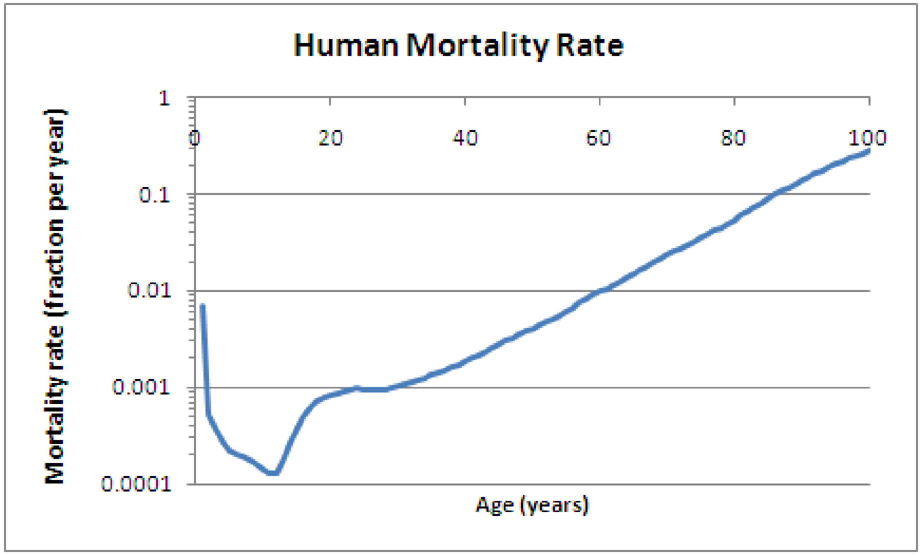
| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | | Exercise 3 – Hazard Function for Human Mor | | | | |
| 2 | | Calculate H, S, and F. (For H, use a sum to approximate t | | | | |
| 3 | | | | | | |
| 4 | | | Mortality rate (hazard function) | Cumulative hazard function | Cumulative survival function | Cumulative fail function |
| 5 | | Age | h(t) | H(t) | S(t) | F(t) |
| 6 | | 1 | 0.00706 | 0.00706 | 0.9929649 | 0.0070351 |
| 7 | | 2 | 0.00053 | 0.00759 | 0.9924387 | 0.0075613 |
| 8 | | 3 | 0.00036 | 0.00795 | 0.9920815 | 0.0079185 |
| 9 | | 4 | 0.00027 | 0.00822 | 0.9918137 | 0.0081863 |
| 10 | | 5 | 0.00022 | C$6:C10) | 0.9915955 | 0.0084045 |

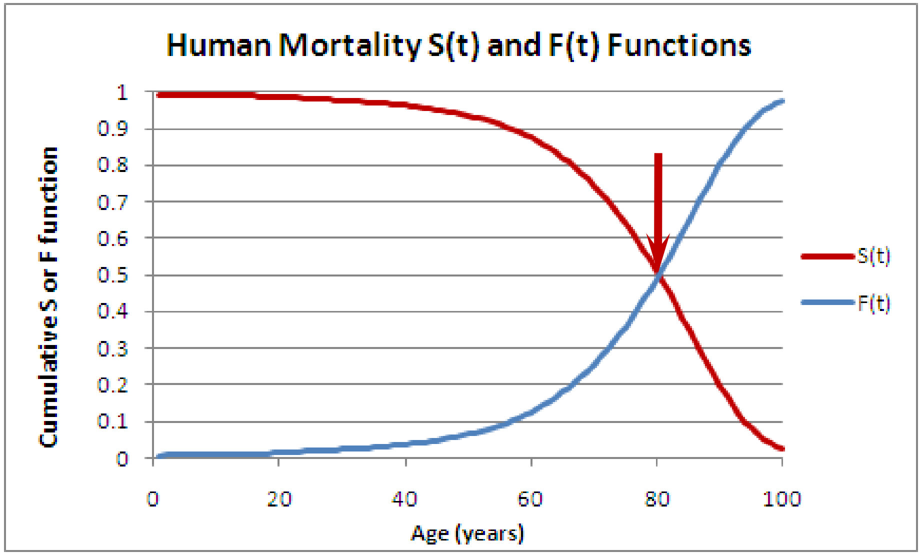
$$H(t) = \int_0^t h(t)\, dt$$

$$S(t) = \exp[-H(t)]$$

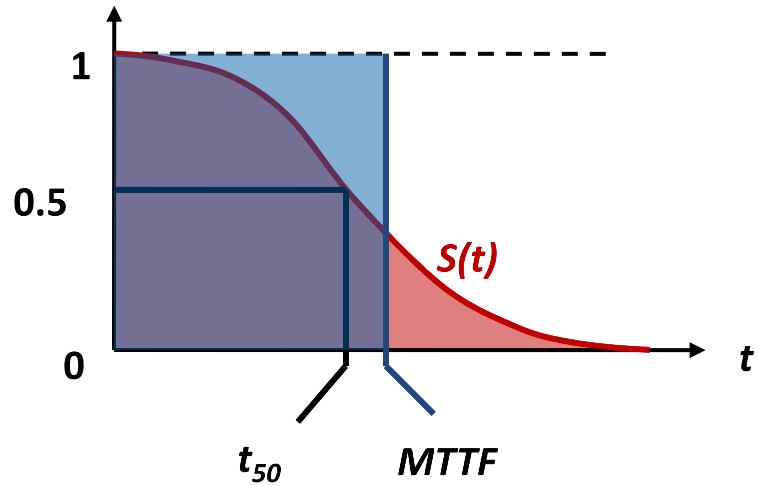
$$F(t) = 1 - \exp[-H(t)]$$

# Human Mortality Graphs



# Reliability Indicators




- Mean time to failure (MTTF)

$$MTTF = \int_0^\infty t\ f(t)\,dt = \frac{1}{N}\sum_{j=1}^{N} t_N = \int_0^\infty S(t)\,dt$$

- Median time to failure ($t_{50}$) is the solution of

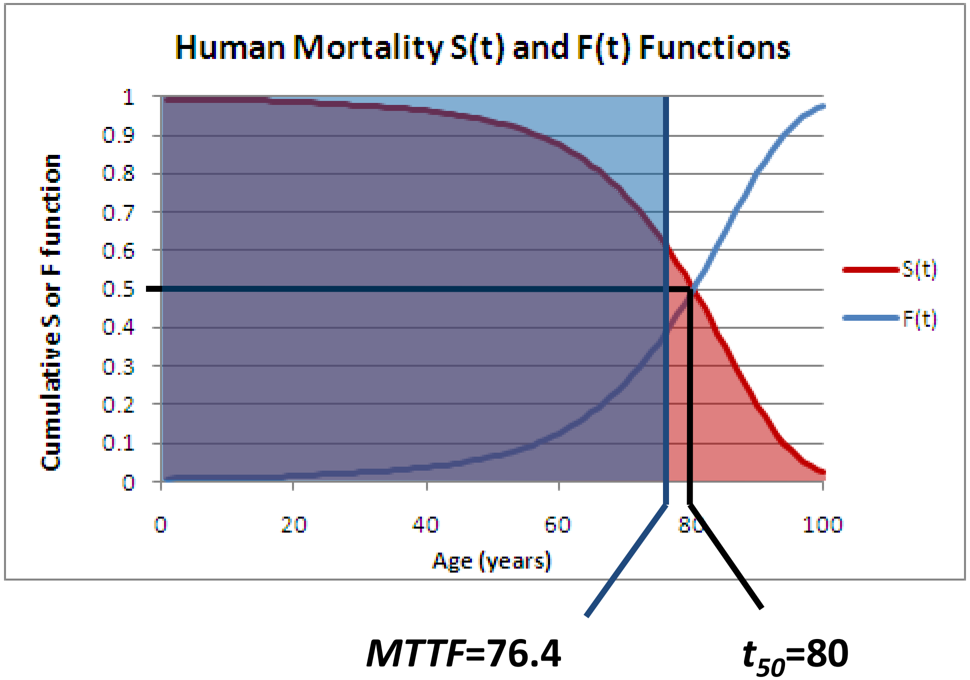
$$S(t_{50}) = 0.5$$

  - Time at which half of the initial population fails

# Exercise 3.1b

- Find the mean and median times to failure for the human mortality data set from the last exercise

# Exercise 3.1b Solution




- Sum S(t) to get MTTF

# Reliability Measures: DPM

- Metric designed for low fail rates
- DPM = **D**efects **P**er **M**illion

| % pass | % fail | DPM |
|---|---|---|
| 99 | 1 | 10,000 |
| 99.9 | 0.1 | 1000 |
| 99.95 | 0.05 | 500 |
| 99.99 | 0.01 | 100 |
| 99.999 | 0.001 | 10 |

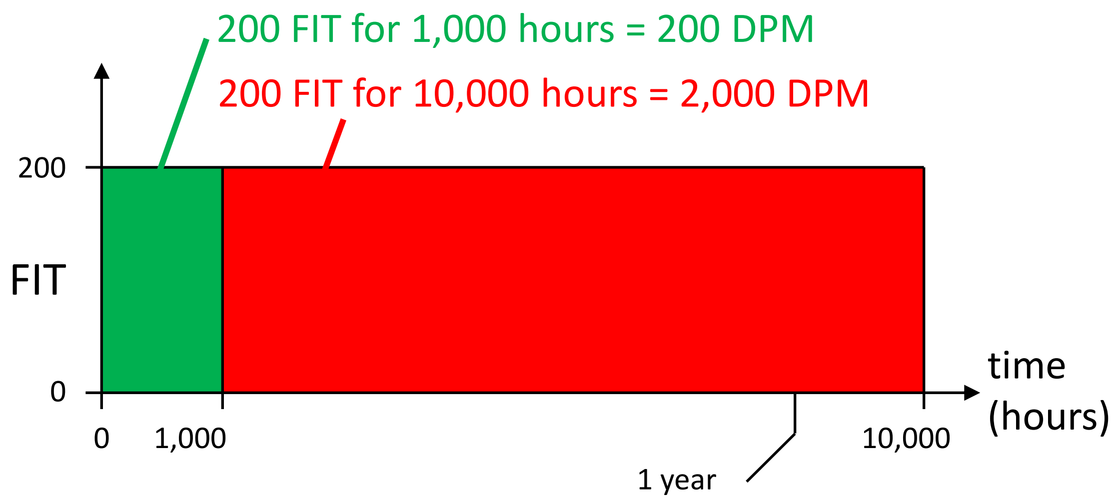
Typical target at end of life (10,000)

Typical target at t=0 (500)

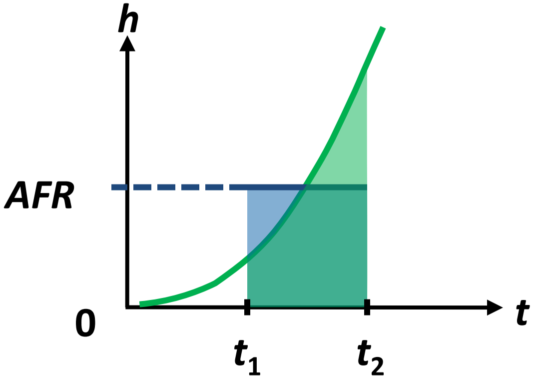
Typical range for semiconductor reliability

# Reliability Measures: FIT

- FIT = **F**ailures **I**n **T**ime
- FIT is a fail *rate*, fails per **b**illion device hours
  - FIT = DPM per 1,000 hours
- DPM is a fail total, fails per **m**illion total devices
  - DPM = FIT * hours / 1,000



# Reliability Indicators: AFR




- AFR, Average Fail Rate

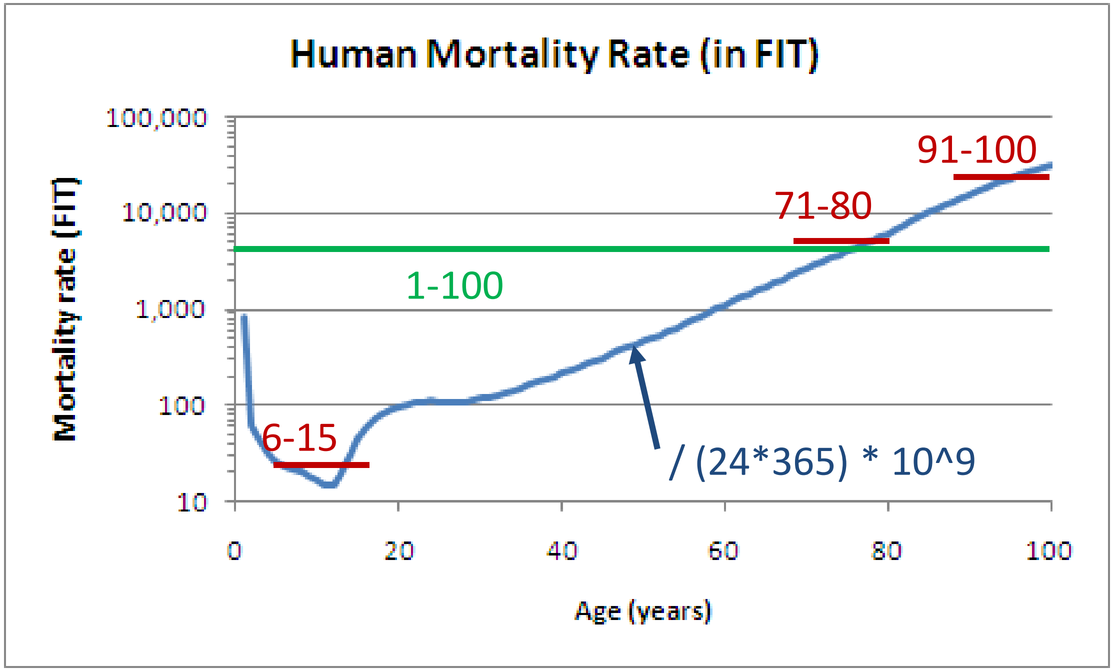
$$AFR(t_1, t_2) = \frac{\int_{t_1}^{t_2} h(t)\, dt}{t_2 - t_1} = \frac{H(t_2) - H(t_1)}{t_2 - t_1} = \frac{\ln S(t_1) - \ln S(t_2)}{t_2 - t_1}$$

  - If t in hours, units are fail fraction per hour
  - Multiply by $10^9$ for units of FIT

# Exercise 3.1c

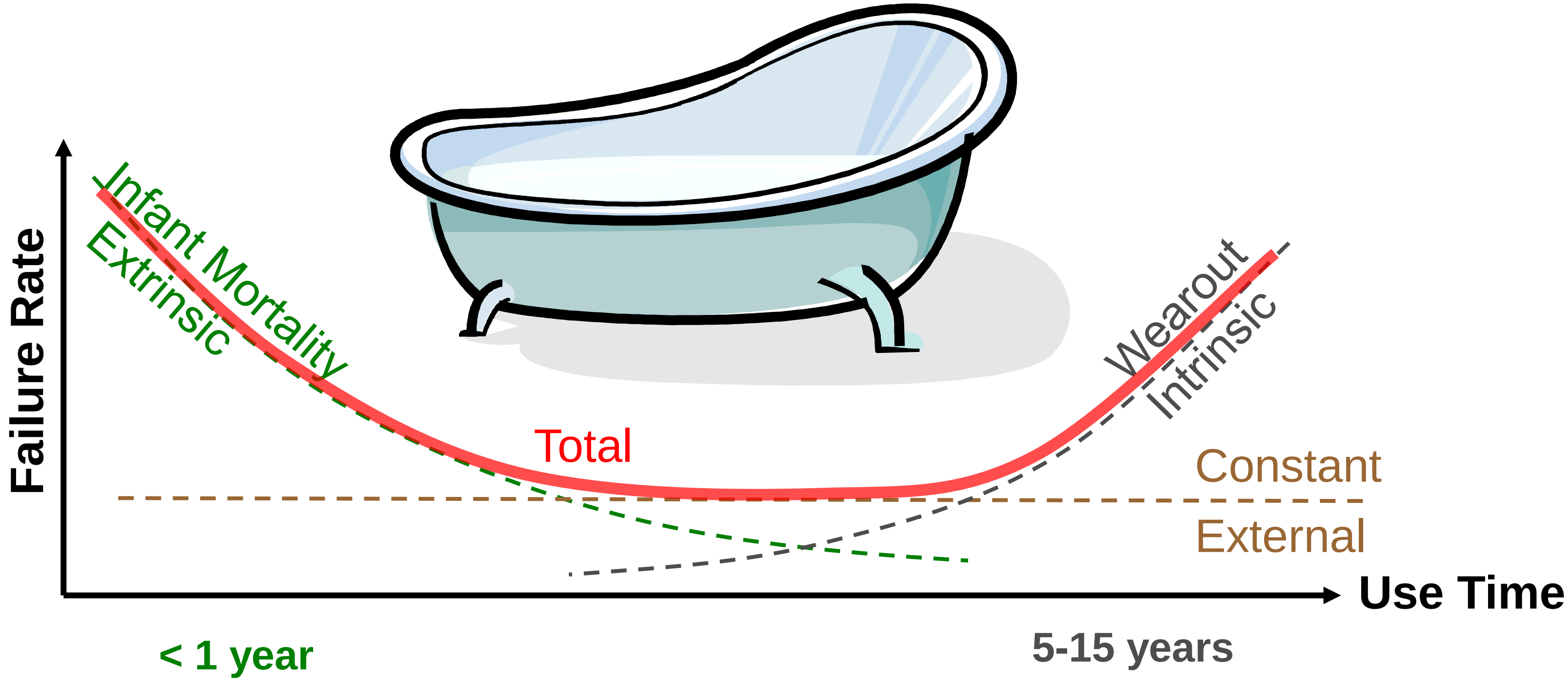
1. Plot the hazard function in FIT
2. Find the AFR (in FIT) for:
   - The 10-year range from ages 6 to 15
   - The 10-year range from ages 71 to 80
   - The 10-year range from ages 91 to 100
   - The entire 100-year range from ages 1 to 100

# Exercise 3.1c Solution




| Age Range | AFR (FIT) |
|---|---|
| 6-15 | 22 |
| 71-80 | 4,311 |
| 91-100 | 24,116 |
| 1-100 | 4,270 |

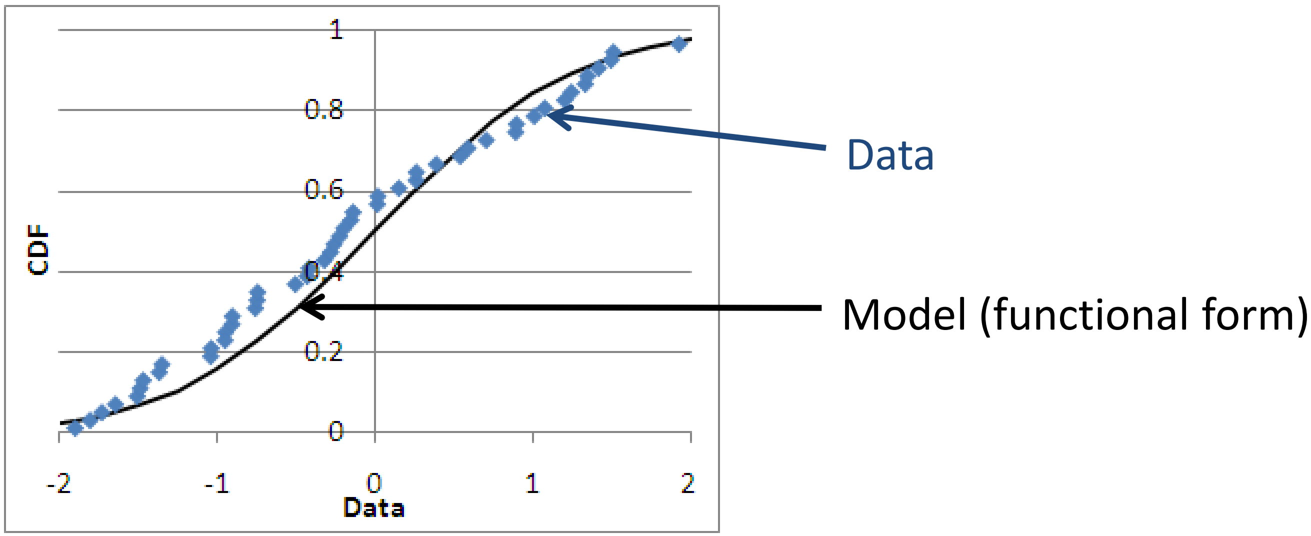
# The Bathtub Curve




- Infant Mortality (IM) from latent reliability defects
- Wearout from reliability mechs like oxide wearout
- Constant from external effects like radiation
- Many versions of this graph – it is a very important concept

# Functional Forms

# Reliability Functional Forms




- Choose functional form for model to fit data

# A Function Bestiary

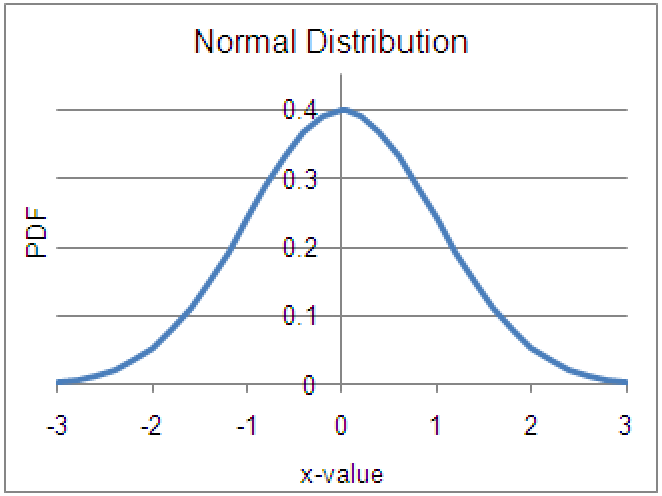
– *Bestiary: A medieval collection of stories providing physical and allegorical descriptions of real or imaginary animals*

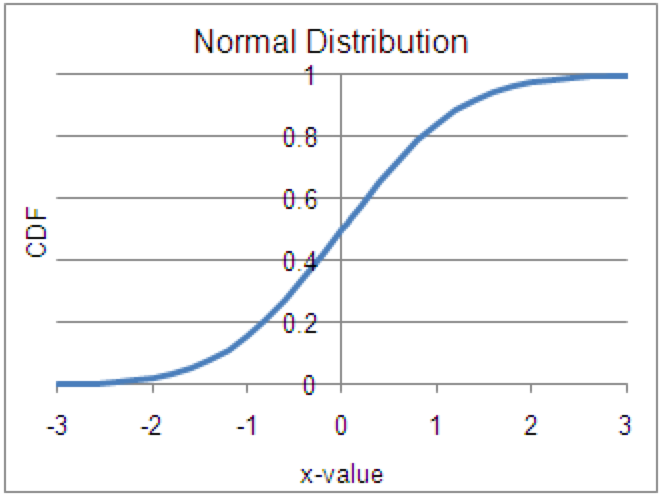
- Continuous distributions
  - Normal
  - Exponential
  - Lognormal
  - Weibull
  - Gamma
  - Beta

  Most common for reliability (Normal, Exponential, Lognormal, Weibull)

- Discrete distributions
  - Hypergeometric
  - Binomial
  - Poisson
- Statistical distributions
  - Chi-square
  - Student's t
  - F

# Normal Distribution




μ = mean

σ = standard deviation

$\sigma^2$ = variance

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} \leftarrow e^{-x^2}$$

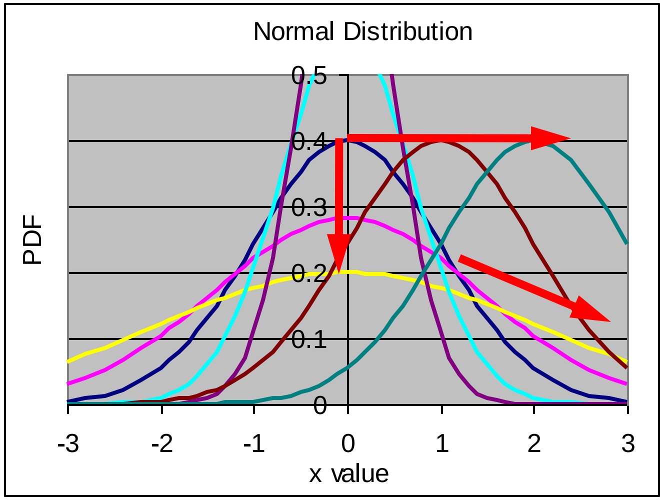
$$F(x) = \int_{-\infty}^{x} dx' \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x'-\mu}{\sigma}\right]^2}$$

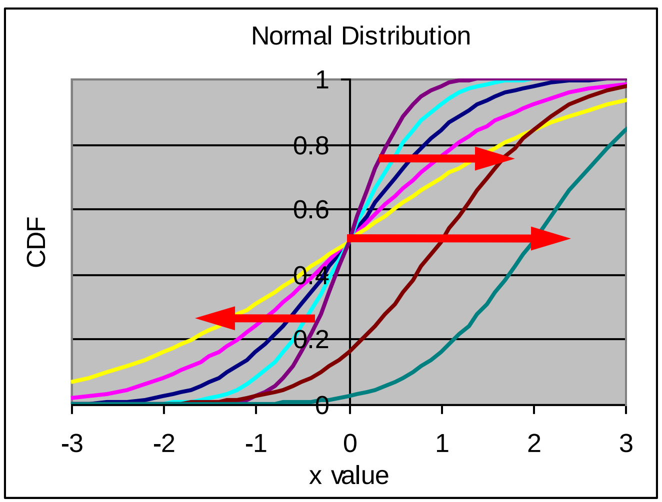
$$\text{rand normal} = NORMSINV(CDF)$$

where CDF is rand uniform

- Using Excel:
  - PDF = NORMDIST(x,μ,σ,FALSE)
  - CDF = NORMDIST(x,μ,σ,TRUE)
- Plot using:
  - y-axis = probit = NORMSINV(CDF)
  - x-axis = x
  - σ = 1/slope
  - μ = x-intercept = – (y-intercept) / slope

# Normal Distribution




| | | | | | | | |
|---|---|---|---|---|---|---|---|
| mean | 0 | 0 | 0 | 0 | 0 | 1 | 2 |
| std | 1 | 1.41 | 2 | 0.71 | 0.5 | 1 | 1 |

$\mu$ = mean

$\sigma$ = standard deviation

$\sigma^2$ = variance

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}$$

$$F(x) = \int_{-\infty}^{x} dx' \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x'-\mu}{\sigma}\right]^2}$$

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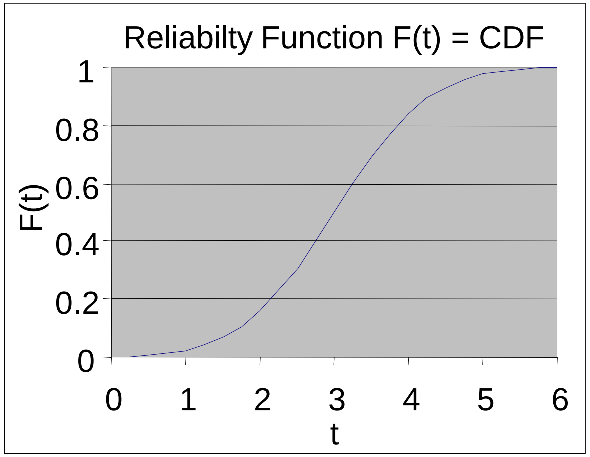
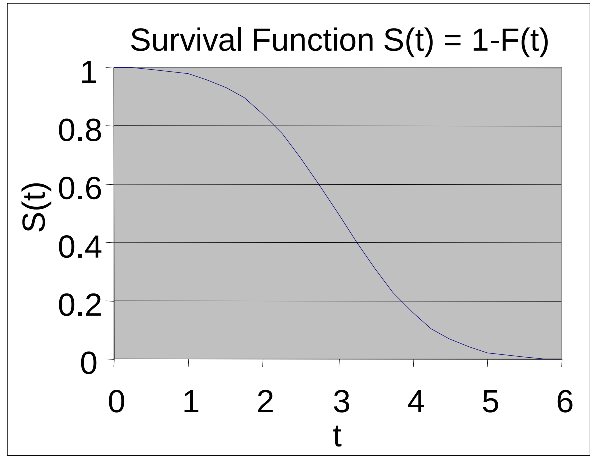
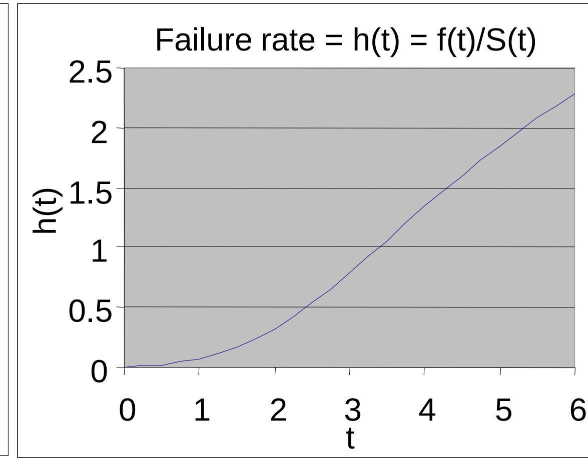
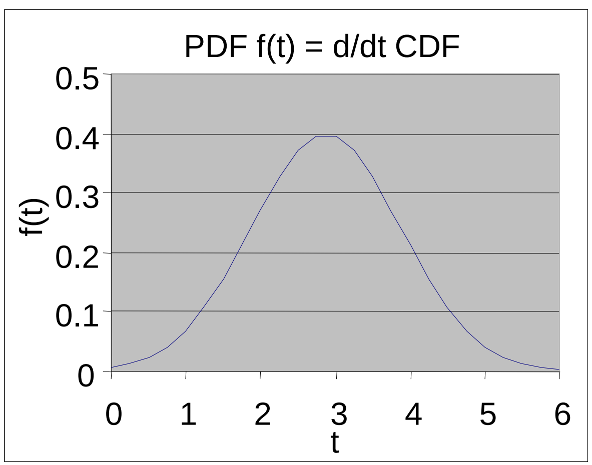
  - x-axis = x
  - σ = 1/slope
  - μ = x-intercept

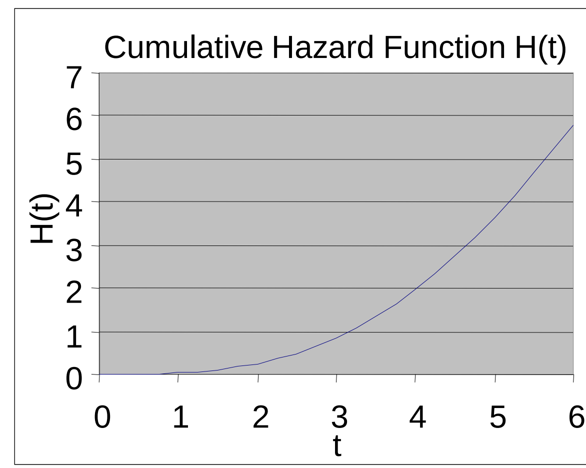
# Normal Distribution Reliability Plots



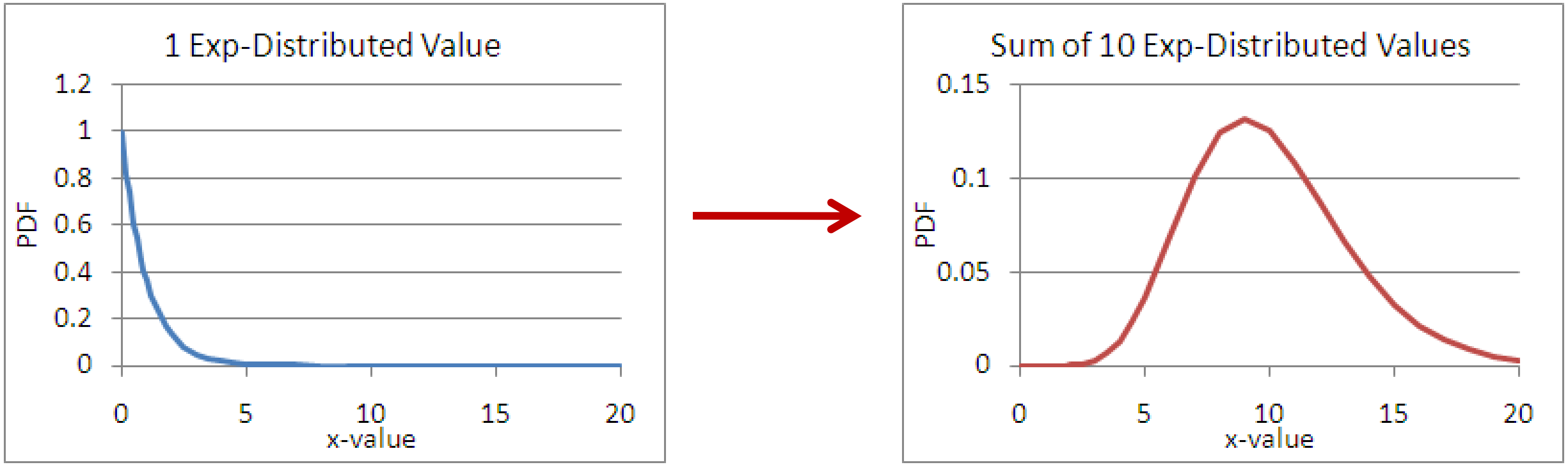



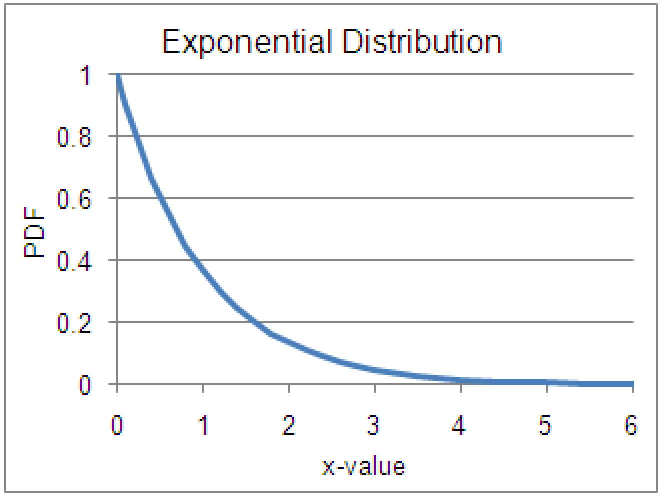



# Use of Normal Distributions




- Most measurement error
- Sum of random things is normal

# Exponential Distribution




$\lambda$ = scale factor

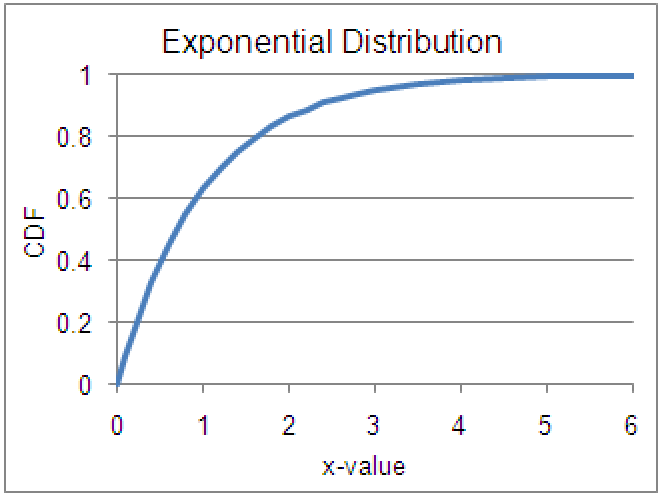
$$f(x) = \lambda e^{-\lambda x}$$

$$F(x) = 1 - e^{-\lambda x}$$

$$e^{-t/\tau}$$

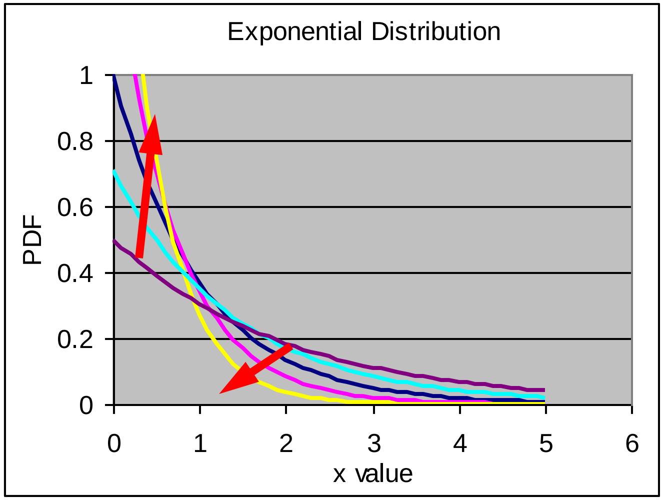
$$\text{rand exponential} = -\frac{\ln(1-CDF)}{\lambda}$$

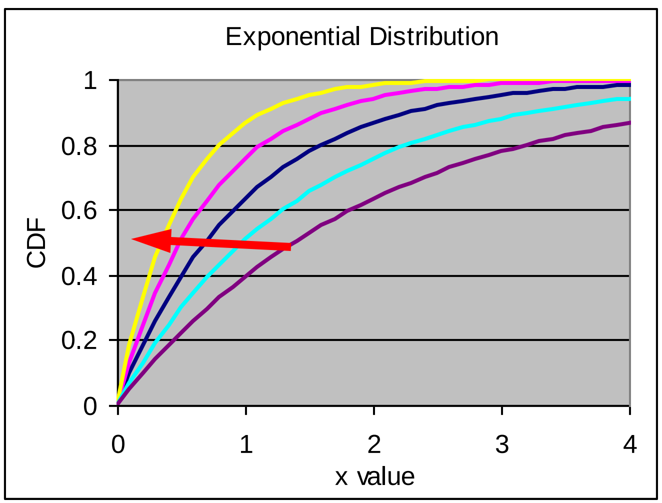
where CDF is rand uniform




- Using Excel:
  - PDF = λ*EXP(-λx)
  - CDF = 1-EXP(-λx)
- Plot using:
  - y-axis = “exbit” = -LN(1-CDF)
  - x-axis = x
  - λ = slope

# Exponential Distribution




| | | | | | |
|---|---|---|---|---|---|
| lambda | 1 | 1.41 | 2 | 0.71 | 0.5 |

λ = scale factor

$$f(x) = \lambda e^{-\lambda x}$$

$$F(x) = 1 - e^{-\lambda x}$$

$$\text{rand exponential} = -\frac{\ln(1-CDF)}{\lambda}$$

where CDF is rand uniform

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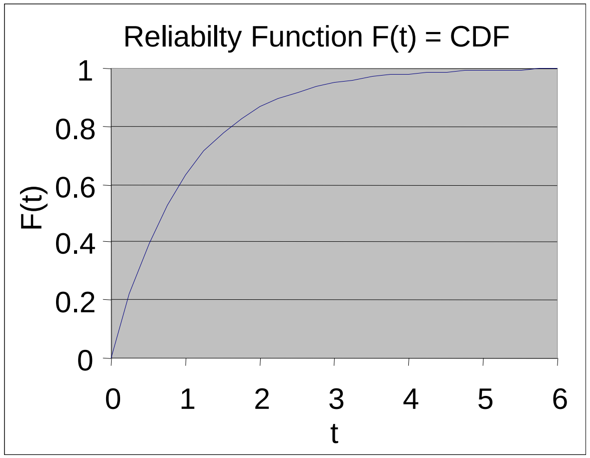
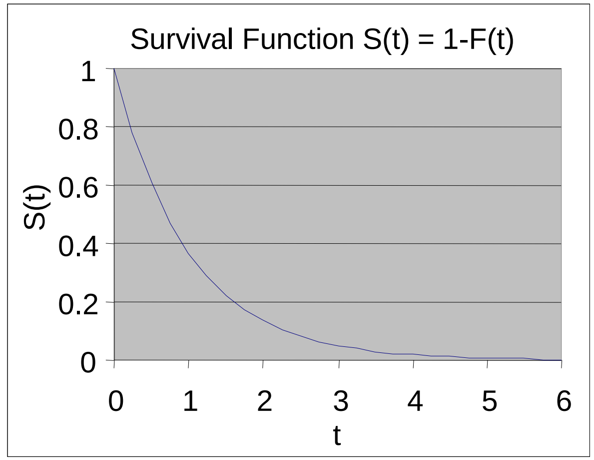
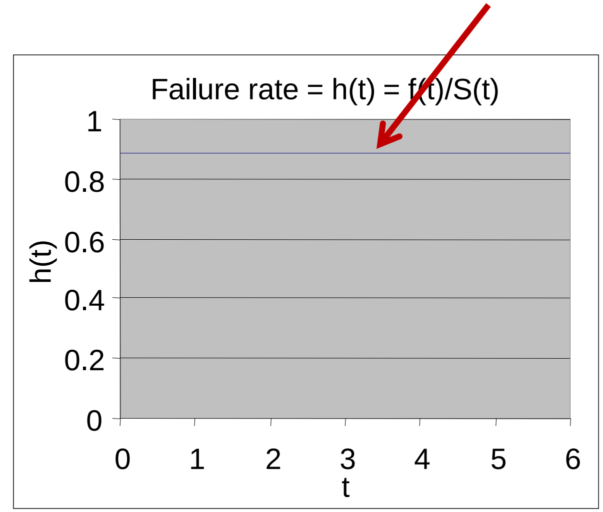
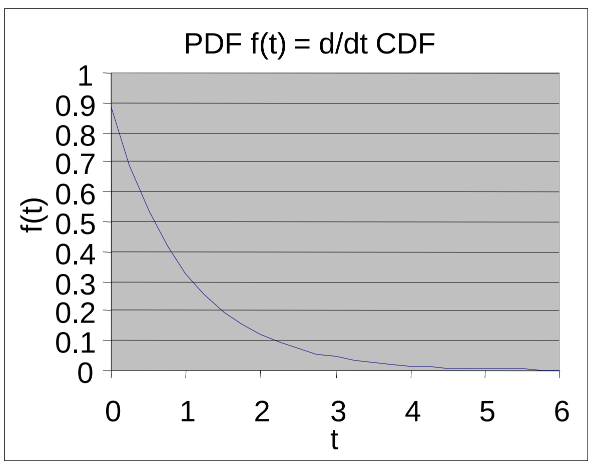
  - x-axis = x
  - λ = slope

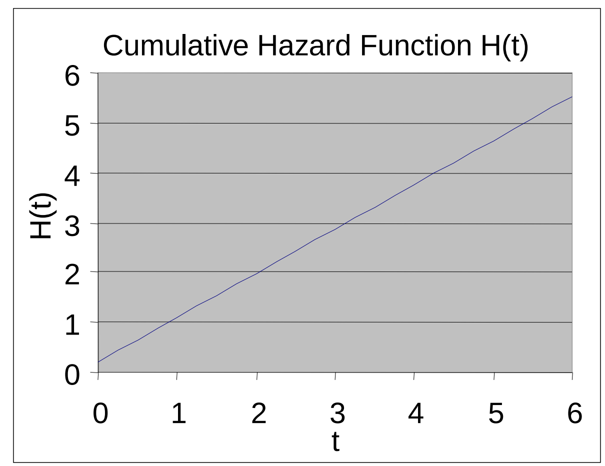
# Exponential Reliability Plots



# Use of Exponential Distributions

- Constant fail rate
  - No "memory" of the past; no age
  - Radioactive decay
  - Soft errors, external environment
- Easy to calculate
  - MTTF = $1/\lambda$
  - Median time to fail from $F(t_{50}) = 1 - e^{-\lambda t_{50}} = 0.5$ so $t_{50} = \dfrac{\ln 2}{\lambda}$

# Exercise 3.2

- Given an exponential fail distribution with

$$\lambda = \frac{0.04\%}{\text{khr}}$$

what is the probability of failure within 15,000 hours of use?
What is the MTTF?

# Solution 3.2

- Convert to “pure” units

$$\lambda = \frac{0.04\%}{\text{khr}} = 0.000\,000\,4\ \frac{\text{fails}}{\text{hour}}$$

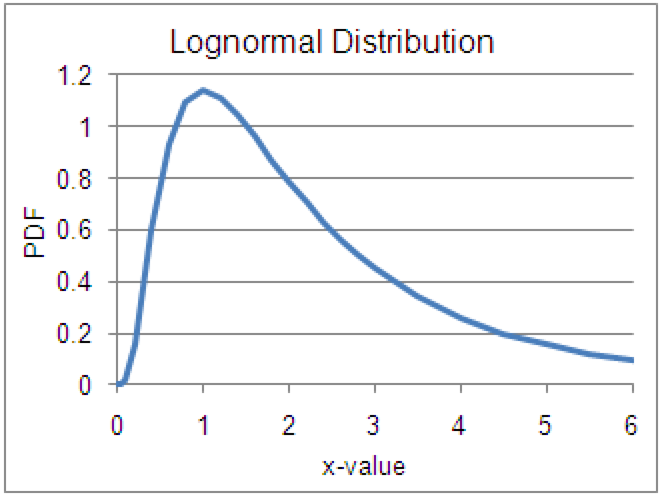
then evaluate the fail function at 15,000 hours

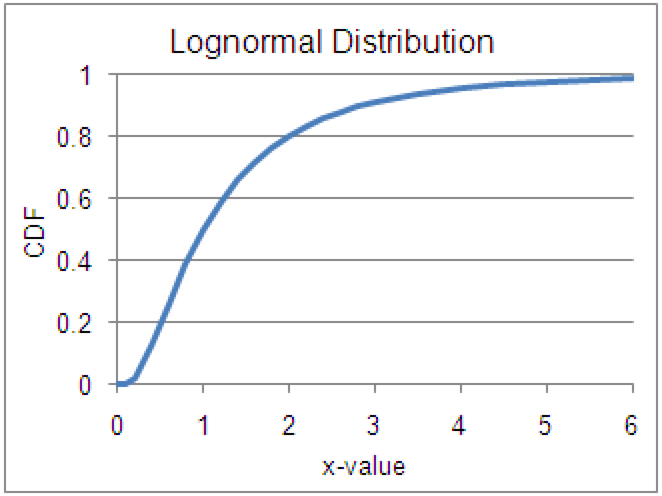
$$F(t) = 1 - e^{-\lambda t} = 1 - e^{-0.000\,000\,4 \times 15{,}000} = 0.006 = 0.6\%$$

The MTTF is even easier

$$MTTF = \frac{1}{\lambda} = 2{,}500{,}000 \text{ hours}$$

# LogNormal Distribution




t50 = median time to fail
σ = standard deviation

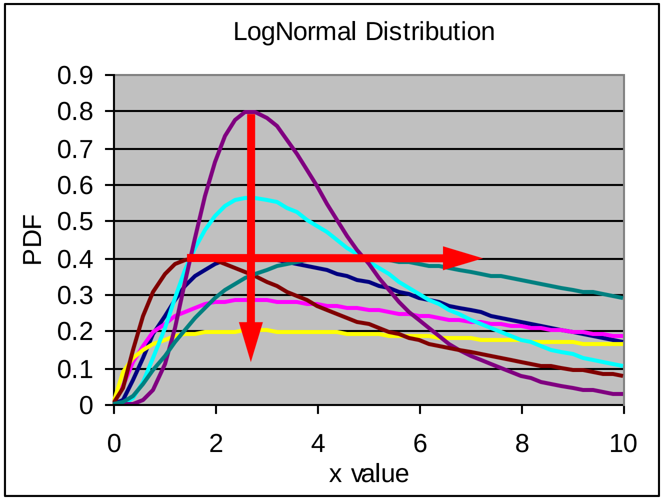
$$f(t) = \frac{1}{\sigma\, t\sqrt{2\pi}}\, e^{-\frac{1}{2}\left[\frac{\ln(t)-\ln(t50)}{\sigma}\right]^2}$$

$$F(t) = \int_{-\infty}^{t} dt' \frac{1}{\sigma\, t'\sqrt{2\pi}}\, e^{-\frac{1}{2}\left[\frac{\ln(t')-\ln(t50)}{\sigma}\right]^2}$$

$$\text{rand normal} = \exp\left(NORMSINV\left(CDF\right)\right)$$
where CDF is rand uniform

- Using Excel:
  - PDF = NORMDIST(ln(t),ln(t50),σ,FALSE)/t
  - CDF = NORMDIST(ln(t),ln(t50),σ,TRUE)
- Plot using:
  - y-axis = probit = NORMSINV(CDF)
  - x-axis = ln(t)
  - σ = 1/slope
  - ln(t50) = x-intercept

# LogNormal Distribution




| | | | | | | | |
|---|---|---|---|---|---|---|---|
| t50 | 1 | 1 | 1 | 1 | 1 | 0.5 | 1.5 |
| std | 1 | 1.41 | 2 | 0.71 | 0.5 | 1 | 1 |

t50 = median time to fail
σ = standard deviation

$$f(t)=\frac{1}{\sigma\, t\sqrt{2\pi}}e^{-\frac{1}{2}\left[\frac{\ln(t)-\ln(t50)}{\sigma}\right]^2}$$

$$F(t)=\int_{-\infty}^{t}dt'\frac{1}{\sigma\, t'\sqrt{2\pi}}e^{-\frac{1}{2}\left[\frac{\ln(t')-\ln(t50)}{\sigma}\right]^2}$$

$$\text{rand normal} = \exp\left(NORMSINV\left(CDF\right)\right)$$

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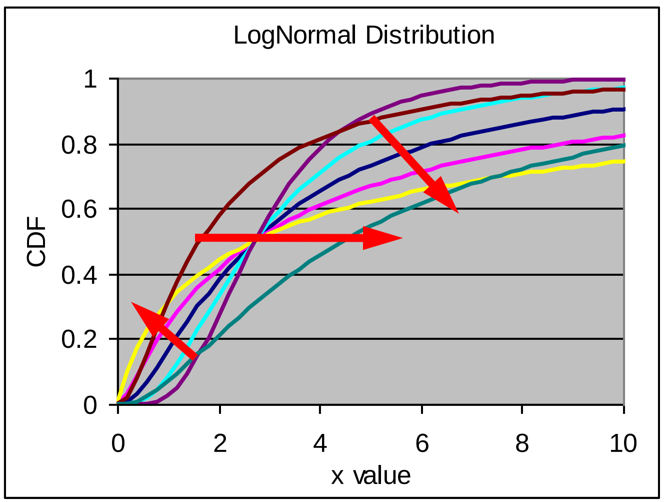


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- Plot using:
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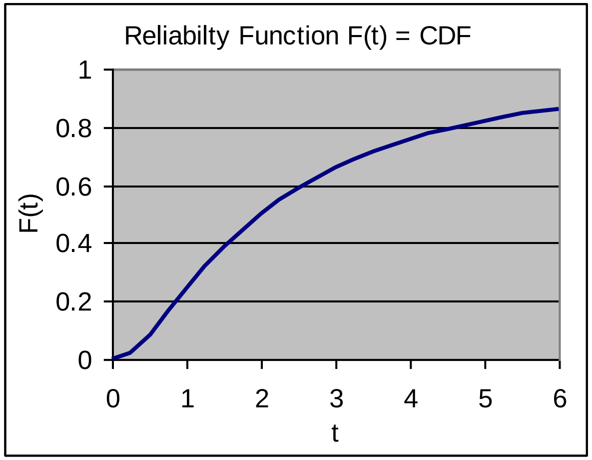
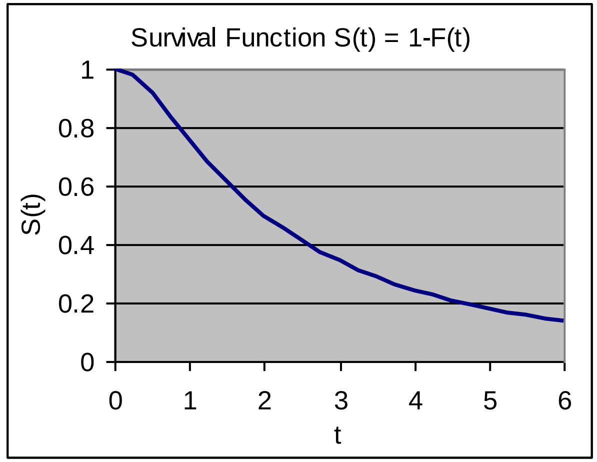
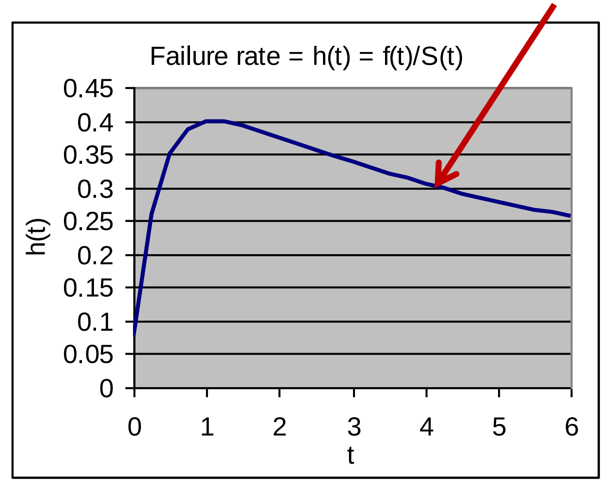
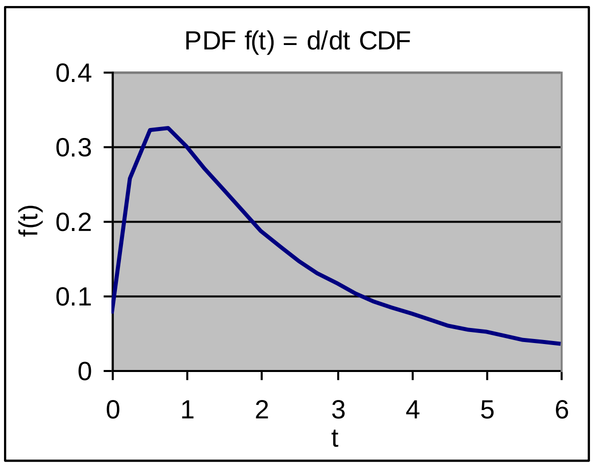
  - x-axis = ln(t)
  - σ = 1/slope
  - ln(t50) = x-intercept

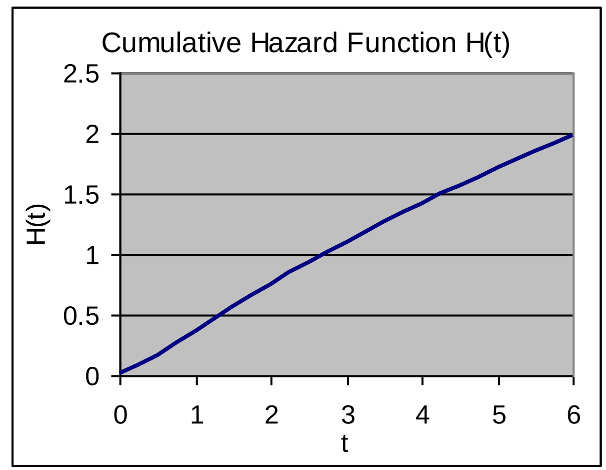
# Lognormal Reliability Plots



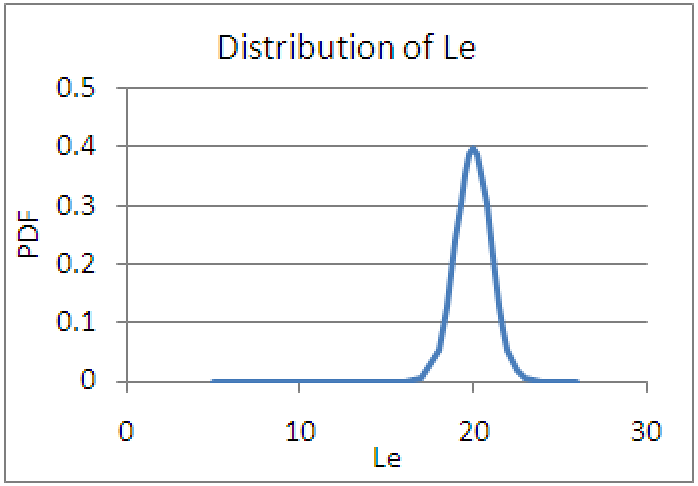



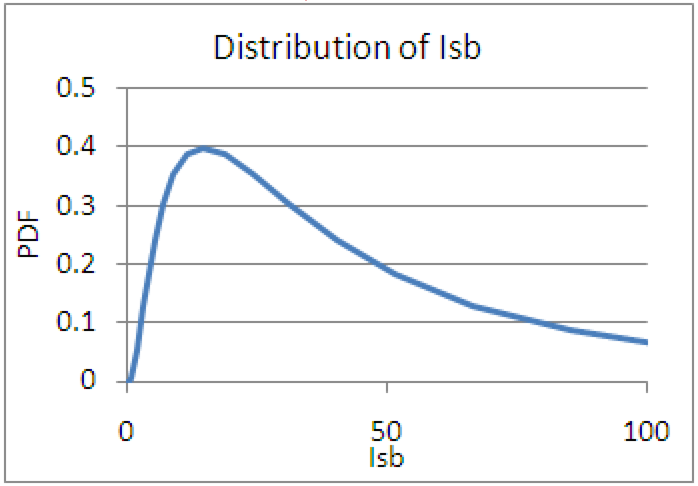



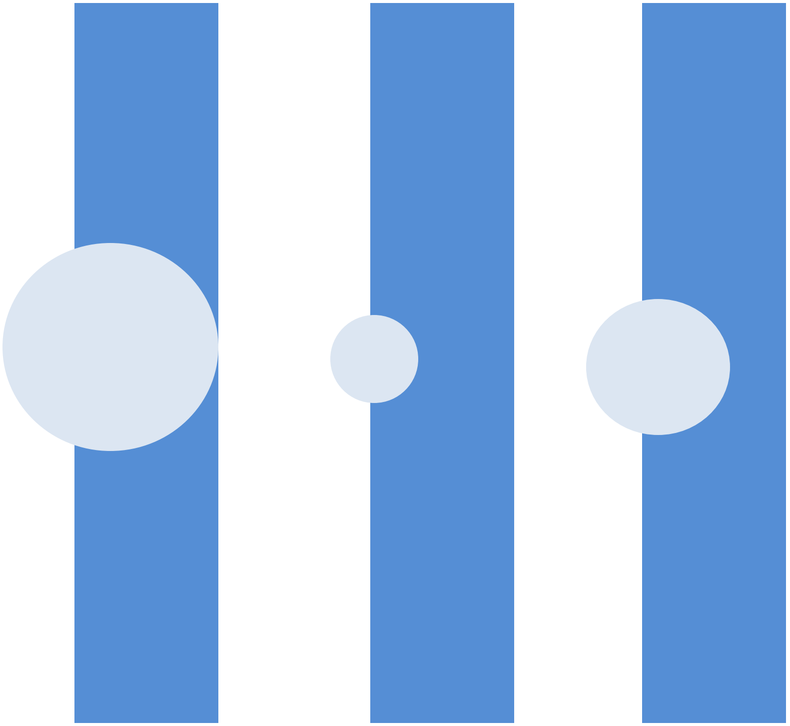

Mostly decreasing failure rate:
IM-type mechanism

# Use of Lognormal Distributions




$$I_{SB} \sim e^{Le}$$

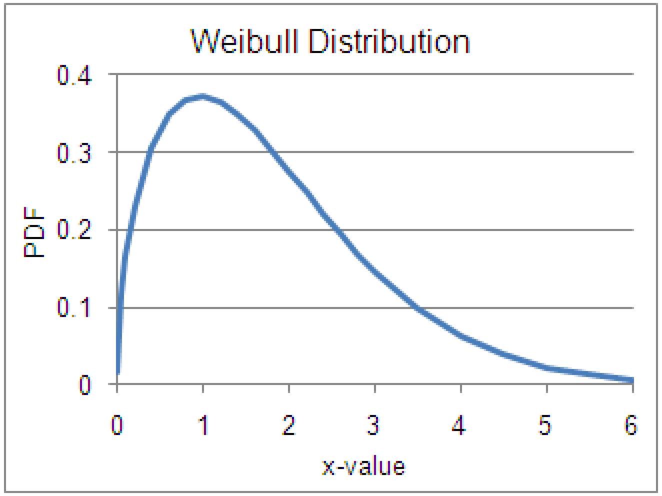


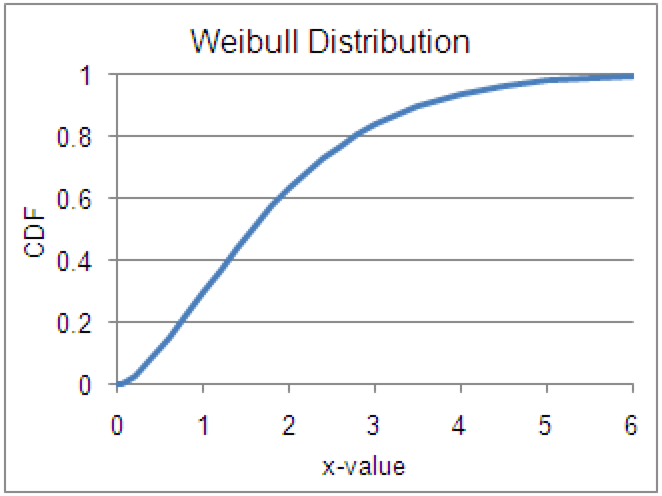



$$R_t = (1+\delta) \times R_{t-1}$$

# Weibull Distribution

$$e^{-\left(\frac{t}{\alpha}\right)^{\beta}}$$




$$f(x) = \frac{\beta}{\alpha}\left(\frac{x-\gamma}{\alpha}\right)^{\beta-1} \exp\left[-\left(\frac{x-\gamma}{\alpha}\right)^{\beta}\right]$$

β = shape parameter
α = scale parameter
γ = location parameter

$$F(x) = 1 - \exp\left[-\left(\frac{x-\gamma}{\alpha}\right)^{\beta}\right]$$

Note: α and β are often swapped in meaning!

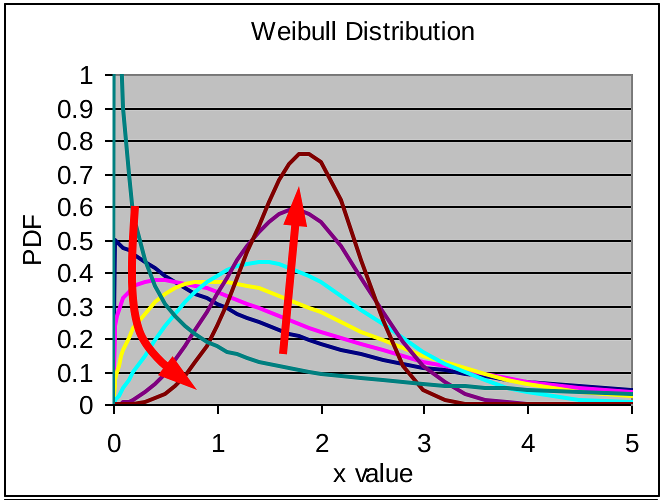
Excel swaps them (below).
T&T use β→m and α→c.

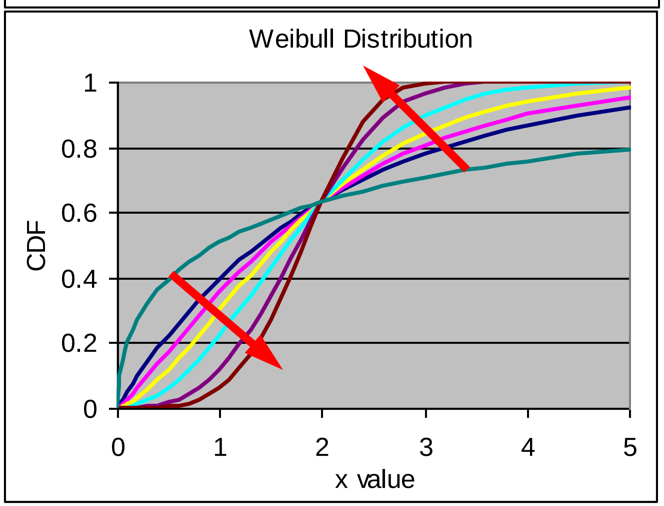
$$\text{rand Weibull} = \alpha\left[-\ln(1-CDF)\right]^{1/\beta}$$

where CDF is rand uniform

- Using Excel:
  - PDF = WEIBULL(x,β,α,FALSE)
  - CDF = WEIBULL(x,β,α,TRUE) =1-EXP(-((x/α)^β))
  - Note γ=0 in Excel
- Plot using:
  - y-axis = weibit = ln(-ln(1-CDF))
  - x-axis = ln(x)
  - β = slope
  - α = exp(-intercept/slope)

# Weibull Distribution




| | | | | | | | |
|---|---|---|---|---|---|---|---|
| beta | 1 | 1.2 | 1.5 | 2 | 3 | 4 | 0.5 |
| alpha | 2 | 2 | 2 | 2 | 2 | 2 | 2 |

$$f(x) = \frac{\beta}{\alpha}\left(\frac{x-\gamma}{\alpha}\right)^{\beta-1} \exp\left[-\left(\frac{x-\gamma}{\alpha}\right)^{\beta}\right]$$

β = shape parameter
α = scale parameter
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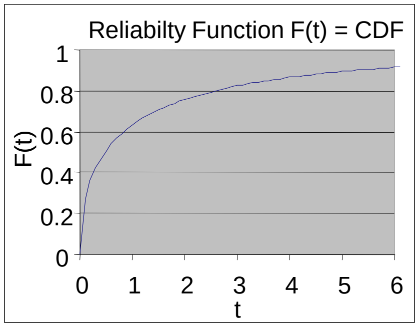
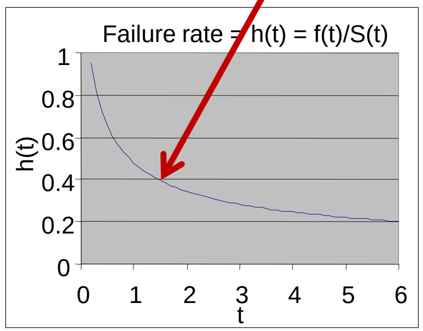
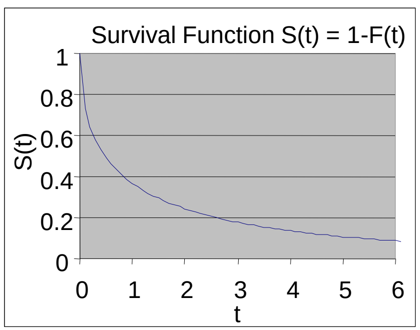
- Plot using:
  - y-axis = weibit = ln(-ln(1-CDF))
  - x-axis = ln(x)
  - β = slope
  - α = exp(-intercept/slope)

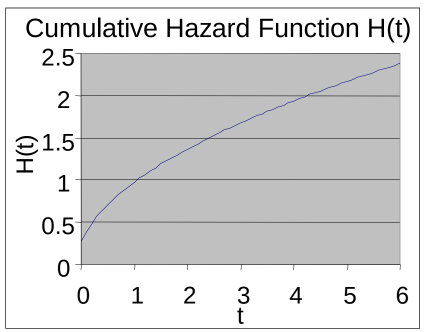
# Weibull Reliability Plots

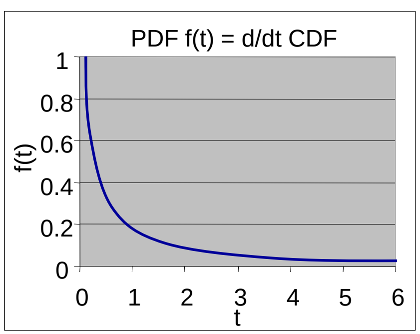
## Weibull, $\beta$=0.5 (<1)



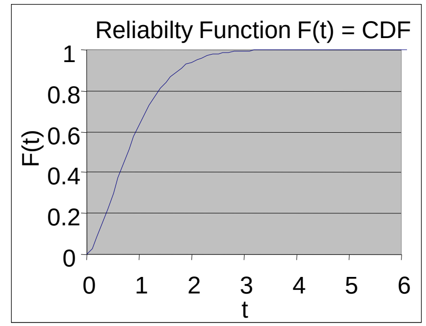



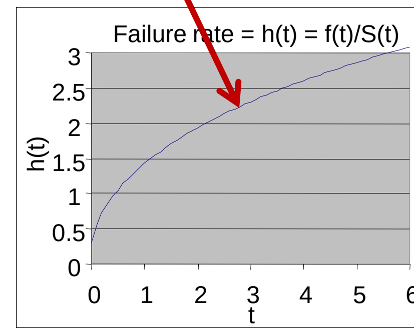



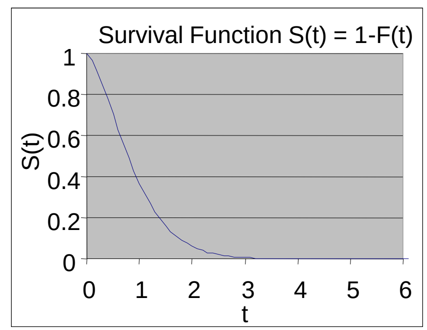



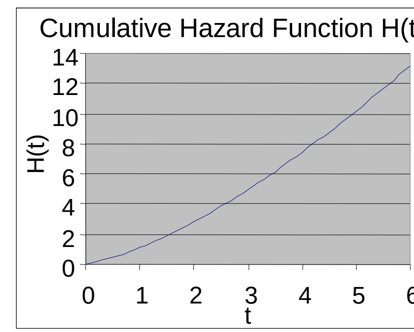

Decreasing failure rate:
Infant Mortality (IM) type mechanism

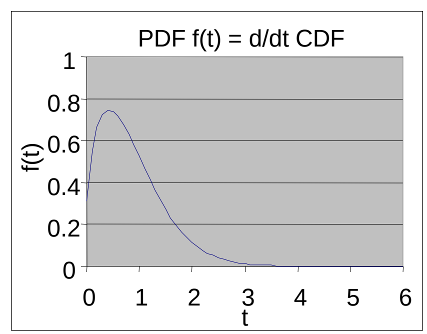
## Weibull, $\beta$=1.5 (>1)



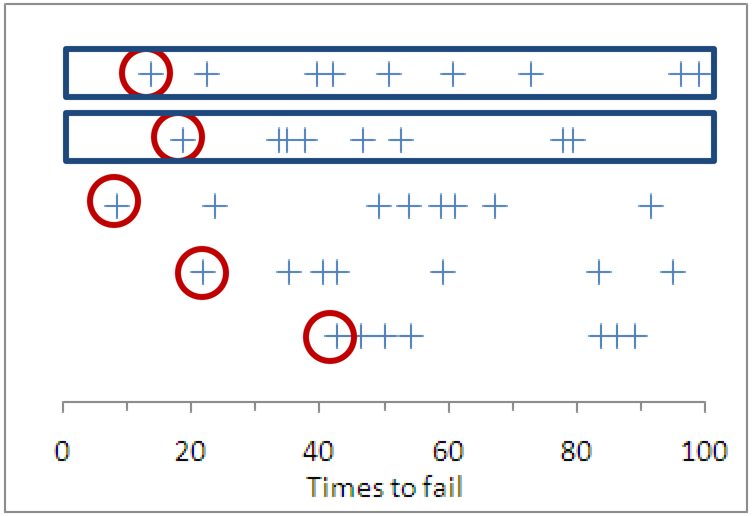



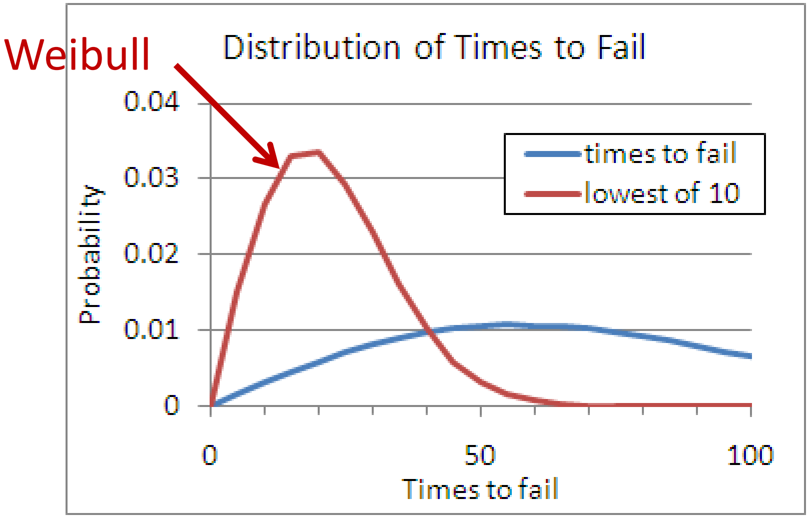




Increasing failure rate:
Wearout (WO) type mechanism

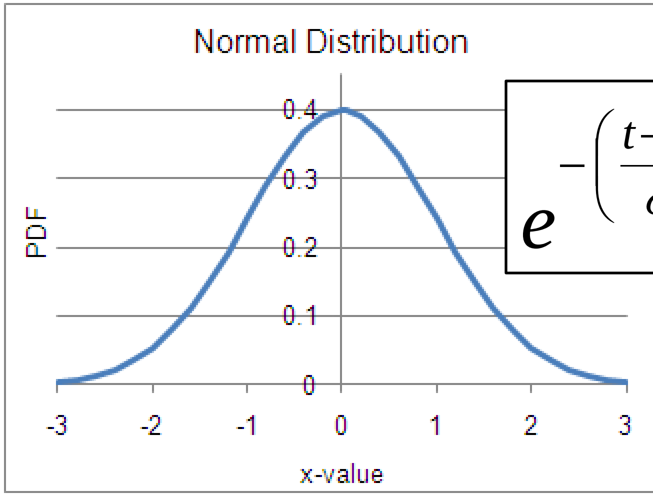
# Use of Weibull Distributions

- When fail is caused by the worst of many items
- When it fits the data well

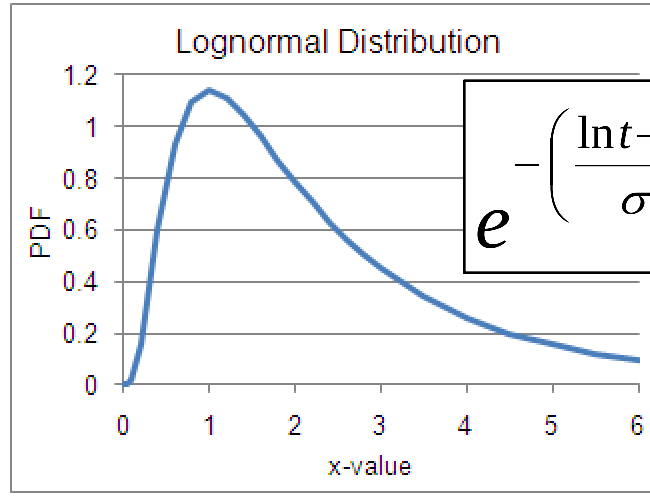


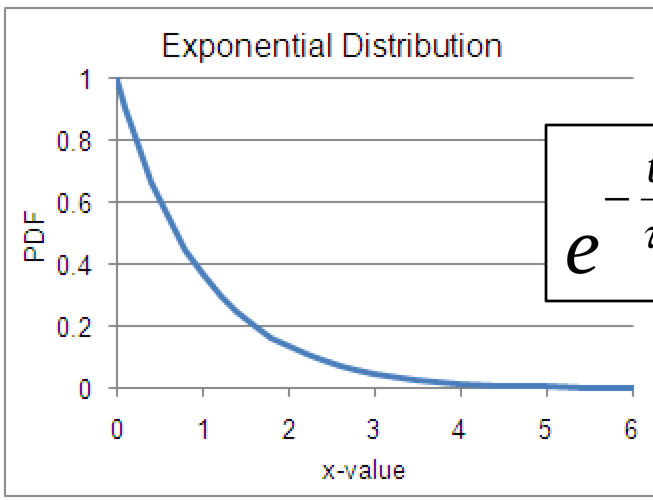
# Main Reliability Functions



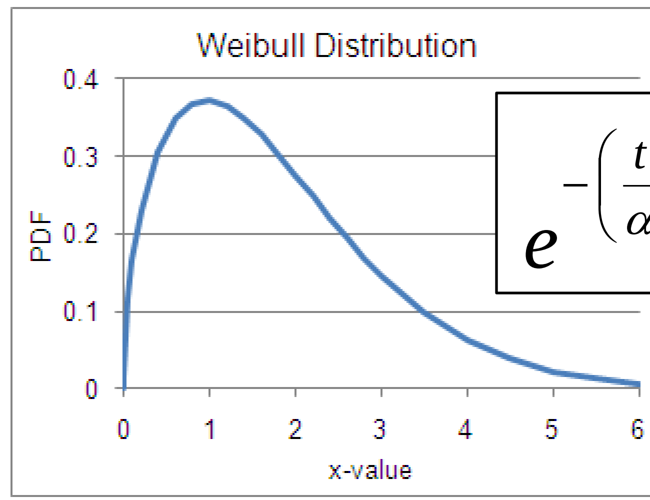

$$e^{-\left(\frac{t-\mu}{\sigma}\right)^2}$$




$$e^{-\left(\frac{\ln t-\mu}{\sigma}\right)^2}$$



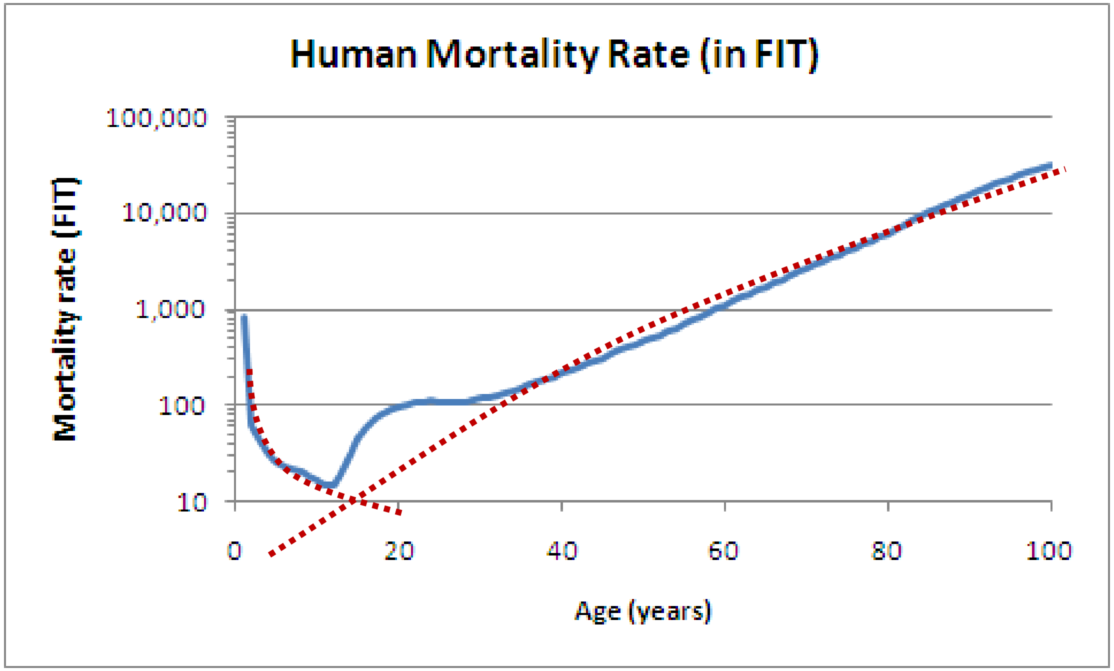

$$e^{-\frac{t}{\tau}}$$




$$e^{-\left(\frac{t}{\alpha}\right)^\beta}$$

# Multiple Mechanisms

# Multiple Mechanisms




Survivals multiply, hazard rates add:

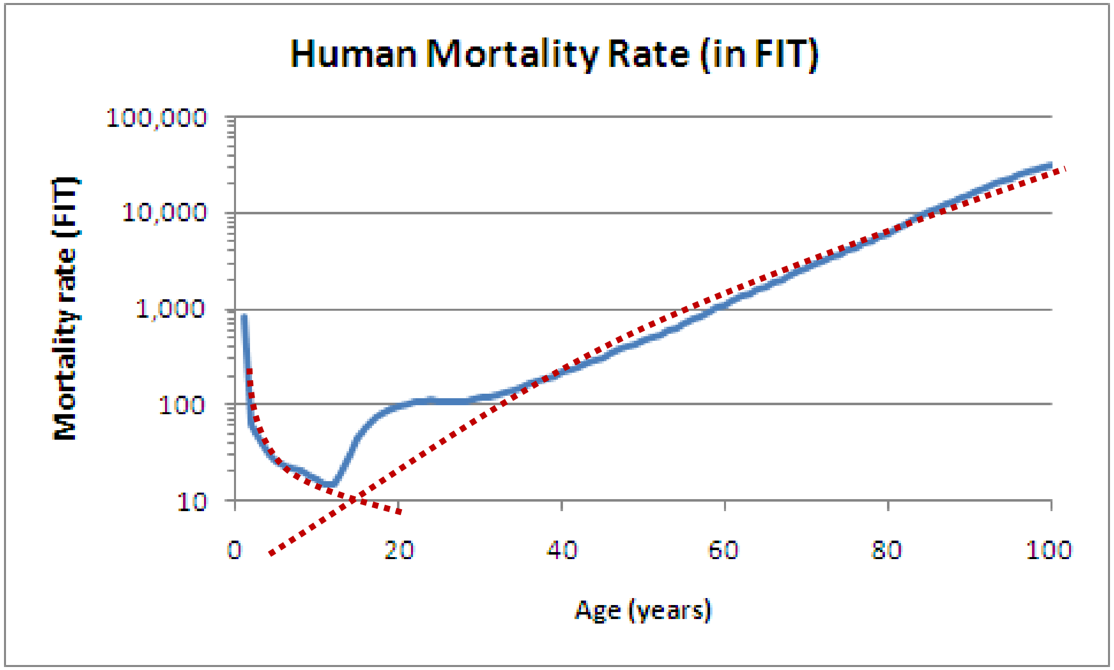
$$S_{tot}(t) = S_1(t)\,S_2(t)$$

$$F_{tot}(t) = 1 - S_1(t)\,S_2(t) \approx F_1(t) + F_2(t)$$

$$h_{tot}(t) \approx h_1(t) + h_2(t)$$

# Exercise 3.3

Hand fit 2 Weibull distributions to the human mortality data like this:




Plot both the hazard rate h(t) (like above) and the fail function F(t).

Useful: for the Weibull, from T&T table 4.3:

$$h(t) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta}$$

# Solution 3.3

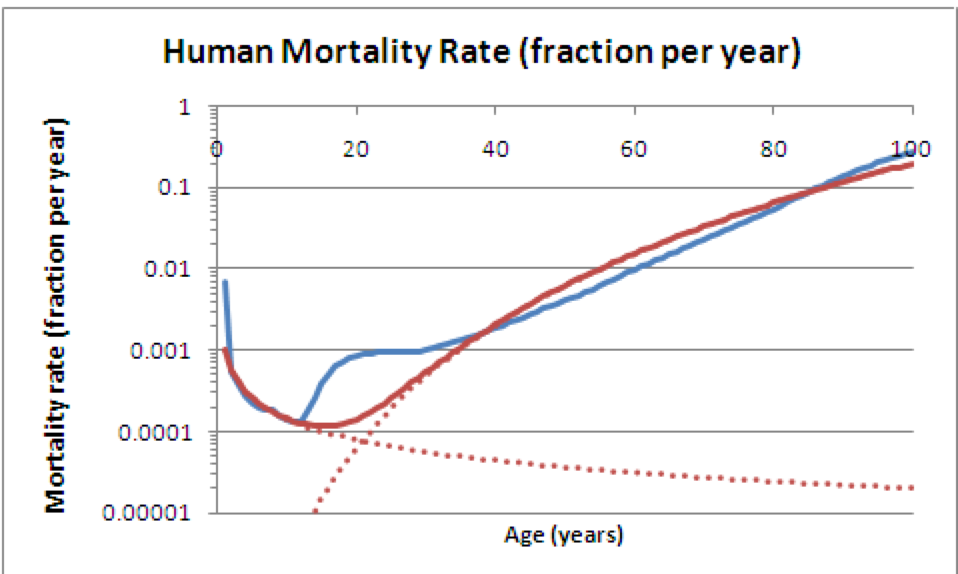
$$h(t)=\frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta}$$

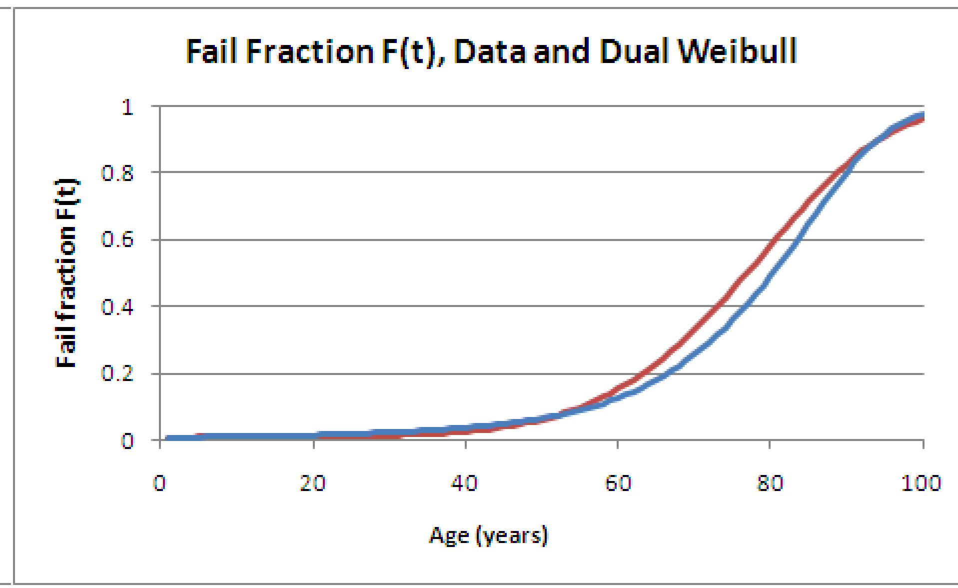
$$h_1(t)+h_2(t)$$

$$F(t)=1-e^{\left(\frac{t}{\alpha_1}\right)^{\beta_1}}e^{\left(\frac{t}{\alpha_2}\right)^{\beta_2}}$$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | alpha | 3E+14 | 82 | | |
| | | | beta | 0.15 | 6 | | |
| | | | | | | | |
| Age | data h(t) | data H(t) | data F(t) | Weib1 h(t) | Weib2 h(t) | Weib h(t) | Weib F(t) |
| 1 | 0.00706 | 0.00706 | 0.007035 | 0.0010105 | 1.974E-11 | 0.00101 | 0.006714 |
| 2 | 0.00053 | 0.00759 | 0.007561 | 0.0005606 | 6.316E-10 | 0.000561 | 0.007447 |
| 3 | 0.00036 | 0.00795 | 0.007918 | 0.0003972 | 4.796E-09 | 0.000397 | 0.007912 |
| 4 | 0.00027 | 0.00822 | 0.008186 | 0.000311 | 2.021E-08 | 0.000311 | 0.008259 |

| | | |
|---|---|---|
| alpha | 3E+14 | 82 |
| beta | 0.15 | 6 |



# The End