

Sample solution for Pierce 23.5.1.

The goal is to show Preservation for System F as shown in Figure 23-1.

Be sure to note the erratum for p. 342, which explains that the context $\Gamma, x : T$ is considered well-formed only if every type variable free in T is bound in Γ .

Following the hint in the answers, we begin by proving the following Lemma about preservation under type substitutions.

Lemma 1 *If $\Gamma, x, \Delta \vdash t : T$, then $\Gamma, [x \mapsto S]\Delta \vdash [x \mapsto S]t : [x \mapsto S]T$.*

Proof. By induction on the depth of the derivation of $\Gamma, x, \Delta \vdash t : T$, then by cases on the final typing rule.

- (T-VAR) Here $t = z$ and $z : T \in \Gamma, x, \Delta$. Clearly $[x \mapsto S]z = z$. There are two sub-cases.
 - i. If $z : T \in \Gamma$, then $z : T \in \Gamma, [x \mapsto S]\Delta$, so by T-VAR, $\Gamma, [x \mapsto S]\Delta \vdash z : T$. By the well-formedness condition on contexts, $x \notin FV(T)$, so $T = [x \mapsto S]T$, giving the desired result.
 - ii. If $z : T \in \Delta$, then $([x \mapsto S]\Delta)(z) = [x \mapsto S]T$, so by T-VAR, $\Gamma, [x \mapsto S]\Delta \vdash z : [x \mapsto S]T$, as desired.
- (T-ABS) Here $t = \lambda x : T_1. t_2$ where $\Gamma, x, \Delta, x : T_1 \vdash t_2 : T_2$ and $T = T_1 \rightarrow T_2$. By induction, $\Gamma, [x \mapsto S]\Delta, x : [x \mapsto S]T_1 \vdash [x \mapsto S]t_2 : [x \mapsto S]T_2$. So by T-ABS, $\Gamma, [x \mapsto S]\Delta \vdash \lambda x : [x \mapsto S]T_1. [x \mapsto S]t_2 : [x \mapsto S]T_1 \rightarrow [x \mapsto S]T_2$, i.e. $\Gamma, [x \mapsto S]\Delta \vdash [x \mapsto S](\lambda x : T_1. t_2) : [x \mapsto S](T_1 \rightarrow T_2)$, as desired.
- (T-APP) Here $t = t_1 t_2$ where $\Gamma, x, \Delta \vdash t_1 : T_{11} \rightarrow T_{12}$, $\Gamma, x, \Delta \vdash t_2 : T_{11}$, and $T = T_{12}$. By induction $\Gamma, [x \mapsto S]\Delta \vdash [x \mapsto S]t_1 : [x \mapsto S](T_{11} \rightarrow T_{12})$, and $\Gamma, [x \mapsto S]\Delta \vdash [x \mapsto S]t_2 : [x \mapsto S]T_{11}$. Distributing the substitution in the first of these derivations and applying T-APP, we get $\Gamma, [x \mapsto S]\Delta \vdash ([x \mapsto S]t_1)([x \mapsto S]t_2) : [x \mapsto S]T_{12}$; undistributing the substitution on the term gives us the desired result.
- (T-TABS) Here $t = \lambda Y. t_2$ where $\Gamma, x, \Delta, Y \vdash t_2 : T_2$ and $T = \forall Y. T_2$. By induction, $\Gamma, [x \mapsto S]\Delta, Y \vdash [x \mapsto S]t_2 : [x \mapsto S]T_2$. So by T-TABS, $\Gamma, [x \mapsto S]\Delta \vdash \lambda Y. [x \mapsto S]t_2 : \forall Y. [x \mapsto S]T_2$. Pulling the substitutions out of the quantifiers gives us the desired result.
- (T-TAPP) Here $t = t_1[T_2]$, where $\Gamma, x, \Delta \vdash t_1 : \forall Y. T_{12}$ and $T = [Y \mapsto T_2]T_{12}$. By induction, $\Gamma, [x \mapsto S]\Delta \vdash [x \mapsto S]t_1 : [x \mapsto S](\forall Y. T_{12})$. Pushing the substitution inside the quantifier and applying T-TAPP gives $\Gamma, [x \mapsto S]\Delta \vdash ([x \mapsto S]t_1)[x \mapsto S]T_2 : [Y \mapsto ([x \mapsto S]T_2)]([x \mapsto S]T_{12})$. Pulling the substitutions to the left gives $\Gamma, [x \mapsto S]\Delta \vdash [x \mapsto S](t_1[T_2]) : [x \mapsto S]([Y \mapsto T_2]T_{12})$, as desired.

Next, we need a revised version of Lemma 9.3.8.

Lemma 2 *If $\Gamma, x : S \vdash t : T$ and $\Gamma \vdash s : S$, then $\Gamma \vdash [x \mapsto s]t : T$.*

Proof. By induction on the depth of the derivation of $\Gamma, x : S \vdash t : T$, and then by cases on the final typing rule. Cases T-VAR, T-ABS, and T-APP are exactly as in the proof of Lemma 9.3.8 on pp. 106-107. The remaining cases are:

- (T-TABS) Here $t = \lambda X. t_2$ where $\Gamma, x : S, X \vdash t_2 : T_2$ and $T = \forall X. T_2$. By induction, $\Gamma, X \vdash [x \mapsto s]t_2 : T_2$. So by T-TABS, $\Gamma \vdash \lambda X. [x \mapsto s]t_2 : \forall X. [x \mapsto s]T_2$, i.e., $\Gamma \vdash [x \mapsto s](\lambda X. t_2) : [x \mapsto s](\forall X. T_2)$, as desired.
- (T-TAPP) Here $t = t_1[T_2]$, where $\Gamma, x : S \vdash t_1 : \forall X. T_{12}$ and $T = [X \mapsto T_2]T_{12}$. By induction $\Gamma \vdash [x \mapsto s]t_1 : \forall X. t_{12}$. So, by T-TAPP, $\Gamma \vdash ([x \mapsto s]t_1)[T_2] : [X \mapsto T_2]T_{12}$, i.e., $\Gamma \vdash [x \mapsto s](t_1[T_2]) : [X \mapsto T_2]T_{12}$, as desired.

Finally, we can prove the preservation theorem itself.

Theorem 1 *If $\Gamma \vdash t : T$ and $t \rightarrow t'$, then $\Gamma \vdash t' : T$.*

Proof. By induction on the derivation of $\Gamma \vdash t : T$, then case analysis on the final step. The cases for T-VAR, T-ABS, and T-TABS can't occur, as no E-rule applies to them. The T-APP case is exactly as in the proof of Thm 9.39 on p. 507 (where the Substitution Lemma now refers to our Lemma 2).

The remaining case is T-TAPP. Here $t = t_1[T_2]$, $\Gamma \vdash t_1 : \forall X. T_{12}$ and $T = [X \mapsto T_2]T_{12}$. There are two possible evaluation rules

- (E-TAPP) Here $t_1 \rightarrow t'_1$ and $t' = t'_1[T_2]$. By induction, $\Gamma \vdash t'_1 : \forall X. T_{12}$. Hence, by T-TAPP, $\Gamma \vdash t'_1[T_2] : [X \mapsto T_2]T_{12}$, as desired.
- (E-TAPPTABS) Here $t_1 = \lambda X. t_{12}$ and $t' = [X \mapsto T_2]t_{12}$. By inversion, $\Gamma, X \vdash t_{12} : T_{12}$. So by Lemma 1, $\Gamma \vdash [X \mapsto T_2]t_{12} : [X \mapsto T_2]T_{12}$, as desired.