

Sample solution for Pierce 11.4.1(1).

We show that ascription can be formulated as a derived form by proving a theorem with the same structure as Thm. 11.3.1 (with errata corrected).

We write λ^E for the simply-typed λ -calculus with the additional rules given in Figure 11-3, and λ^I for pure simply-typed λ -calculus as given in Figure 9-1. We describe an elaboration function $e \in \lambda^E \rightarrow \lambda^I$, as follows:

$$\begin{aligned} e(\mathbf{t} \text{ as } \mathbf{T}) &= (\lambda \mathbf{x}:\mathbf{T}.\mathbf{x}) \ e(\mathbf{t}) \\ e(\mathbf{t}) &= \mathbf{t} \text{ with each subterm } \mathbf{t}' \text{ replaced by } e(\mathbf{t}') \text{ for any other } \mathbf{t} \end{aligned}$$

For convenience we will write $\lceil \mathbf{t} \rceil$ for $e(\mathbf{t})$.

Theorem 1 *For each term \mathbf{t} of λ^E ,*

- i. If $\mathbf{t} \rightarrow_E \mathbf{t}'$, then $\lceil \mathbf{t} \rceil \rightarrow_I \lceil \mathbf{t}' \rceil$.*
- ii. If $\lceil \mathbf{t} \rceil \rightarrow_I \mathbf{u}$, then \exists a λ^E term \mathbf{t}' such that $\mathbf{t} \rightarrow_E \mathbf{t}'$ and $\lceil \mathbf{t}' \rceil = \mathbf{u}$.*
- iii. If $\Gamma \vdash^E \mathbf{t} : \mathbf{T}$, then $\Gamma \vdash^I \lceil \mathbf{t} \rceil : \mathbf{T}$.*
- iv. If $\Gamma \vdash^I \lceil \mathbf{t} \rceil : \mathbf{T}$, then $\Gamma \vdash^E \mathbf{t} : \mathbf{T}$.*

Proof.

The proof of each result is by structural induction on \mathbf{t} . In each instance, it is obvious that the only case of interest is $\mathbf{t} = \mathbf{t}_1 \text{ as } \mathbf{T}$, so $\lceil \mathbf{t} \rceil = (\lambda \mathbf{x}:\mathbf{T}.\mathbf{x}) \ \lceil \mathbf{t}_1 \rceil$. To shorten the proof, we consider only this case in the various sections below.

- i. By hypothesis, $\mathbf{t} \rightarrow_E \mathbf{t}'$. There are two cases, corresponding to the one-step rules that might apply to \mathbf{t} :
 - (E-ASCRIBE) In this case, $\mathbf{t}_1 = \mathbf{v}_1$ for some value \mathbf{v}_1 , and $\mathbf{t}' = \mathbf{v}_1$. So $\lceil \mathbf{t} \rceil = (\lambda \mathbf{x}:\mathbf{T}.\mathbf{x}) (\lceil \mathbf{v}_1 \rceil) = (\lambda \mathbf{x}:\mathbf{T}.\mathbf{x}) (\mathbf{v}_1)$. By (E-APPABS), $(\lambda \mathbf{x}:\mathbf{T}.\mathbf{x}) (\mathbf{v}_1) \rightarrow_I \mathbf{v}_1 = \lceil \mathbf{v}_1 \rceil = \lceil \mathbf{t}' \rceil$.
 - (E-ASCRIBE1) In this case, $\mathbf{t}_1 \rightarrow_E \mathbf{t}'_1$. We have $\lceil \mathbf{t} \rceil = (\lambda \mathbf{x}:\mathbf{T}.\mathbf{x}) (\lceil \mathbf{t}_1 \rceil)$. By induction, $\lceil \mathbf{t}_1 \rceil \rightarrow_E \lceil \mathbf{t}'_1 \rceil$, so by (E-APP2), $(\lambda \mathbf{x}:\mathbf{T}.\mathbf{x}) (\lceil \mathbf{t}_1 \rceil) \rightarrow_I (\lambda \mathbf{x}:\mathbf{T}.\mathbf{x}) (\lceil \mathbf{t}'_1 \rceil) = \lceil \mathbf{t}' \rceil$.
- ii. By hypothesis, $\exists \mathbf{u}$ such that $\lceil \mathbf{t} \rceil \rightarrow_I \mathbf{u}$. There are two cases, corresponding to the one-step rules that might apply to $\lceil \mathbf{t} \rceil$:
 - (E-APPABS) In this case, $\lceil \mathbf{t}_1 \rceil = \mathbf{v}_1$ for some value \mathbf{v}_1 , and $\mathbf{u} = \mathbf{v}_1$. It is easy to see that $\mathbf{t}_1 = \mathbf{v}_1$ as well, since only a variable can elaborate to a variable. Hence, by (E-ASCRIBE), we have $\mathbf{t} \rightarrow_E \mathbf{v}_1$ and $\lceil \mathbf{v}_1 \rceil = \mathbf{u}$.

- (E-APP2) In this case, $u = (\lambda x:T.x) u_1$, where $[\tau_1] \rightarrow_I u_1$. By induction, $\exists \tau'_1$ such that $\tau_1 \rightarrow_E \tau'_1$ and $[\tau'_1] = u_1$. Hence, by (E-ASCRIE1), we have $\tau \rightarrow_E \tau'_1$ as T and $[\tau'_1 \text{ as } T] = (\lambda x:T.x) [\tau'_1] = (\lambda x:T.x) u_1 = u$.

iii. By hypothesis, $\Gamma \vdash^E \tau : T$. The only applicable rule is (T-ASCRIE), so we must have $\Gamma \vdash^E \tau_1 : T$. By induction, $\Gamma \vdash^I [\tau_1] : T$. So we can build the following deduction:

$$\frac{\frac{\frac{x : T \in \Gamma, x : T}{\Gamma, x : T \vdash^I x : T} \text{T-VAR}}{\Gamma \vdash^I (\lambda x:T.x) : T \rightarrow T} \text{T-ABS} \quad \Gamma \vdash^I [\tau_1] : T}{\Gamma \vdash^I (\lambda x:T.x) [\tau_1] : T} \text{T-APP}$$

iv. By hypothesis, $\Gamma \vdash^I [\tau] : T$. The only applicable rule is (T-APP), which gives $\Gamma \vdash^I (\lambda x:T.x) : T' \rightarrow T$ and $\Gamma \vdash^I [\tau_1] : T'$. By the Inversion Lemma, $T' = T$. By induction, $\Gamma \vdash^E \tau_1 : T$. Finally, by (T-ASCRIE), $\Gamma \vdash^E T_1 \text{ as } T : T$. \square