4F3 - Predictive Control

Lecture 3 - Predictive Control with Constraints

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Constraints on System Variables

In practice, system variables are always constrained by:

- Physical limitations
  - Input constraints – e.g. actuator limits
  - State constraints – e.g. reservoir capacities
- Safety considerations (e.g. critical temperatures/pressures)
- Performance specifications (e.g. limit overshoot)

\[
1 \leq y_1 \leq 4 \\
1 \leq y_2 \leq 3 \\
1 \leq y_3 \leq 3 \\
1.5 \leq y_4 \leq 3 \\
1.5 \leq y_5 \leq 3
\]
Systems with Input Saturation

- A common system nonlinearity is *input saturation*.

\[
x(k + 1) = Ax(k) + B \text{sat}(u(k)) \quad \text{(nonlinear)}
\]

\[
y(k) = Cx(k)
\]

- Easily transformed into a constraint on a linear system:

\[
\underline{u}\{i\} \leq u\{i\} \leq \bar{u}\{i\}
\]

where \(v\{i\}\) is the \(i^{th}\) component (row) of a column vector \(v\).
**Constrained LQR Problem**

**Problem:** Given an initial state $x(0)$ at time $k = 0$, compute and implement an input sequence

$$\{u(0), u(1), \ldots, \}$$

that minimizes the infinite horizon cost function

$$\sum_{i=0}^{\infty} \left( x(k)^T Q x(k) + u(k)^T R u(k) \right)$$

while guaranteeing that constraints are satisfied for all time.

- It is usually impossible to solve this problem.
- Predictive control provides an approximate solution.
- RHC laws with constraints will be *nonlinear*. 
**Problem:** Given an initial state \( x = x(k) \), compute a finite horizon input sequence \( \{u_0, u_1, \ldots, u_{N-1}\} \) that minimizes the finite horizon cost function

\[
x_N^T P x_N + \sum_{i=0}^{N-1} \left( x_i^T Q x_i + u_i^T R u_i \right)
\]

where

\[
x_0 = x
\]

\[
x_{i+1} = Ax_i + Bu_i, \quad i = 0, 1, \ldots, N - 1
\]

while guaranteeing that all constraints are satisfied over the prediction horizon \( i \in 0, 1, \ldots, N \).
1. Obtain measurement of current output/state.
2. Compute optimal finite horizon input sequence subject to constraints.
3. Implement first part of optimal input sequence.
4. Return to step 1.
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Prediction Matrices

Recall that we previously solved for the sequence of predicted states $X$ in terms of the stacked inputs $U$:

$$
\begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_N
\end{pmatrix}
:=
\begin{pmatrix}
  A \\
  A^2 \\
  \vdots \\
  A^N
\end{pmatrix}
\begin{pmatrix}
  x_0 \\
  u_0 \\
  u_1 \\
  \vdots \\
  u_{N-1}
\end{pmatrix}
+ 
\begin{pmatrix}
  B & 0 & \cdots & 0 \\
  AB & B & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  A^{N-1}B & A^{N-2}B & \cdots & B
\end{pmatrix}
\begin{pmatrix}
  u_0 \\
  u_1 \\
  \vdots \\
  u_{N-1}
\end{pmatrix}
$$

or, defining $x := x_0$,

$$
X := \Phi x + \Gamma U.
$$

The matrices $\Phi$ and $\Gamma$ are the prediction matrices.
Incorporating Constraints

Now incorporate a set of linear inequality constraints on the predicted states $x_i$ and inputs $u_i$.

$$M_i x_i + E_i u_i \leq b_i, \quad \text{for all } i = 0, 1, \ldots, N - 1$$

$$M_N x_N \leq b_N.$$

Many constraints take this form:

- $M_s = 0 \Rightarrow$ Input constraints only
- $E_s = 0 \Rightarrow$ State constraints only
- Can include constraints on outputs or controlled variables

For simplicity, assume that

$$E_i = E, \quad M_i = M \quad \text{and} \quad b_i = b \quad \text{for } i = 0, 1, \ldots, N - 1$$
Writing Constraints in Standard Form

Suppose we have the following input and output constraints:

\[ u_{low} \leq u_i \leq u_{high}, \quad i = 0, 1, \ldots, N - 1 \]
\[ y_{low} \leq y_i \leq y_{high}, \quad i = 0, 1, \ldots, N \]

Recalling that \( y_i = Cx_i \), this is equivalent to:

\[
\begin{pmatrix}
0 \\
0 \\
-C \\
+C
\end{pmatrix} x_i + 
\begin{pmatrix}
-I \\
+I \\
0 \\
0
\end{pmatrix} u_i \leq 
\begin{pmatrix}
-u_{low} \\
u_{high} \\
-y_{low} \\
y_{high}
\end{pmatrix} \quad \text{for } i = 0, 1, \ldots, N - 1
\]

Similar expression for terminal constraint (in terms of \( x_N \) only)
Writing Constraints in Standard Form

From the previous example, we can write the constraints in the form:

\[ M_i x_i + E_i u_i \leq b_i, \quad \text{for all } i = 0, 1, \ldots, N - 1 \]

\[ M_N x_N \leq b_N. \]

by defining:

\[
M_i := \begin{pmatrix} 0 \\ 0 \\ -C \\ +C \end{pmatrix}, \quad E_i := \begin{pmatrix} -I \\ +I \\ 0 \\ 0 \end{pmatrix}, \quad b_i := \begin{pmatrix} -u_{low} \\ +u_{high} \\ -y_{low} \\ +y_{high} \end{pmatrix}, \quad \text{for } i = 0, 1, \ldots, N - 1
\]

and

\[
M_N := \begin{pmatrix} -C \\ +C \end{pmatrix}, \quad b_N := \begin{pmatrix} -y_{low} \\ +y_{high} \end{pmatrix}
\]
Writing Constraints in Terms of $x$, $X$ and $U$

- Taking all of the constraints together:

$$
\begin{pmatrix}
M_0 \\
0 \\
\vdots \\
0
\end{pmatrix}
x_0 +
\begin{pmatrix}
0 & \cdots & 0 \\
M_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & M_N
\end{pmatrix}
\begin{pmatrix}
x_1 \\
\vdots \\
x_N
\end{pmatrix}
+
\begin{pmatrix}
E_0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & E_{N-1}
\end{pmatrix}
\begin{pmatrix}
u_0 \\
\vdots \\
u_{N-1}
\end{pmatrix}
\leq
\begin{pmatrix}
b_0 \\
b_1 \\
\vdots \\
u_N
\end{pmatrix}
$$

- By appropriately defining $D$, $M$, $E$ and $c$ (recalling $x := x_0$):

$$
Dx + MX + EU \leq c.
$$

- Next, will eliminate $X$ using prediction matrices.
Writing Constraints in Terms of $x$ and $U$

Substitute $X = \Phi x + \Gamma U$ into

$$Dx + MX + EU \leq c.$$  

and collect terms. The constraints can be written in the form:

$$JU \leq c + Wx$$

where

$$J := M\Gamma + E$$  

and

$$W := -D - M\Phi$$

Our constraints are now in terms of the input sequence $U$ and the initial state $x = x_0 = x(k)$.  

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Writing Constraints in Terms of $x$ and $U$

In summary, the basic procedure is:

- Define linear inequalities in $u_i$, $x_i$, $y_i$ and $z_i$
- Write the constraints in the form:

$$M_i x_i + E_i u_i \leq b_i, \quad \text{for all } i = 0, 1, \ldots, N - 1$$
$$M_N x_N \leq b_N.$$

- Stack the constraints to get them in the form:

$$D x + MX + EU \leq c.$$  

- Substitute $X = \Phi x + \Gamma U$ and rearrange to the form:

$$J U \leq c + W x.$$
Cost Function

Recall that the cost function

\[ V(x, U) := x_N^T P x_N + \sum_{i=0}^{N-1} \left( x_i^T Q x_i + u_i^T R u_i \right) \]

can be rewritten (with \( x := x_0 \)) as

\[ V(x, U) = \frac{1}{2} U^T G U + U^T F x + x^T (Q + \Phi^T \Omega \Phi) x \]

for some matrices \( F, G \) and \( \Omega \) (defined in lecture 2).

Remember that \( G \succ 0 \) if \( P \succeq 0, Q \succeq 0 \) and \( R \succ 0 \).
Quadratic Programming

**Definition** *Quadratic Program (QP)*

*Given matrices $Q$ and $A$ and vectors $c$ and $b$, the optimization problem:*

$$\min_{\theta} \frac{1}{2} \theta^T Q \theta + c^T \theta$$

*subject to: $A \theta \leq b$*

*is called a quadratic program (QP).*

**Proposition** *If $Q \succ 0$ in the above quadratic program, then*

1) *The optimization problem is strictly convex.*

2) *A global minimizer can always be found.*

3) *The global minimizer is unique.*
Nature of Solutions to QPs

\[ A \theta \leq b \]

- Constrained minimum
- Level set of cost function
- Unconstrained minimum
Writing Problems as QPs

Many problem types can be cast as QPs:

**Example:** Nonlinear constraint.

\[
\min_{\theta} \frac{1}{2} \theta^T Q \theta + c^T \theta \quad \Rightarrow \quad \min_{\theta} \frac{1}{2} \theta^T Q \theta + c^T \theta
\]

subject to: \( \max\{e^T \theta, f^T \theta\} \leq b \)  

subject to: \( e^T \theta \leq b, f^T \theta \leq b \)

**Example:** Nonlinear cost function.

\[
\min_{\theta} |c^T \theta| \quad \Rightarrow \quad \min_{\theta, \delta} \delta
\]

subject to: \( A \theta \leq b \)  

subject to: \( A \theta \leq b, c^T \theta - \delta \leq 0, -c^T \theta - \delta \leq 0 \)
Constrained Optimal Control as a QP

Our constrained optimal control problem is:

$$
\min_U \frac{1}{2} U^T GU + U^T F x
$$

subject to: $JU \leq c + W x$

This is a quadratic program in standard form. Substitute

$$
\theta \rightarrow U \quad Q \rightarrow G \quad c \rightarrow F x
$$

$$
A \rightarrow J \quad b \rightarrow (c + W x)
$$

The optimal solution is

1. A global minimum (when $G \succeq 0$).
2. Unique (when $G \succ 0$).
Solution via Quadratic Programming

- Some QP parameters are functions of the current state $x$
  - Without constraints, $U^*(x)$ is a linear function of $x$.
  - With constraints, $U^*(x)$ is nonlinear.

- Must calculate $U^*(x)$ by solving a QP online for each $x$.

- See JMM §3.2 and §3.3 for an introduction to solving QPs.

- In Matlab, our QP can be solved using

$$U = \text{quadprog}(G, F^*x, J, c+Wx)$$
Implementing the RHC Law

- The RHC input is the first part of the optimal input sequence:

\[ \kappa_{rhc}(x) := u_0^*(x) = \left( I_m \ 0 \ \ldots \ 0 \right) U^*(x) \]

- Since \( U^*(x) \) is no longer linear, \( \kappa_{rhc} : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is a nonlinear control law.

- The dynamics of the closed loop system are nonlinear:

\[ x(k+1) = Ax(k) + B\kappa_{rhc}(x) \]
Complexity of Solutions

**Example:** Double integrator

\[ x(k + 1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(k) \]

Constraints:

\[ |u| \leq 1, \|y\|_\infty \leq 12 \]

Horizon length = 12

Quadratic cost with

\[ Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R = 1, \quad P = 1 \]

**Solution:** Controller with 57 regions. Each region \( i \) has \( u^*_i(x) = v_i + K_i x \)