Chapter 10

Linear Quadratic Regulator Problem

Minimize the cost function $J$ given by

$$ J = \frac{1}{2} \int_{0}^{\infty} (x'Qx + u'Ru)dt $$

- $R > 0$ positive definite (symmetric with positive eigenvalues)
- $Q \geq 0$ positive semi definite (symmetric with nonnegative eigenvalues)

subject to

$$ \dot{x} = Ax + Bu $$

$$ (y = Cx) $$

LQR SOLUTION:
Find the positive-definite solution $P$ of the ARE (Algebraic Ricatti Equation)

$$ A'P + PA + Q - PBR^{-1}B'P = 0 $$

$$ u = -Kx \quad \text{where} \quad K = R^{-1}B'P $$

The positive-definite solution of the ARE results in an asymptotically stable closed-loop system if:

1) the system is controllable
2) $R > 0$
3) $Q = C_q'C_q$ where $(C_q,A)$ is observable

These conditions are necessary and sufficient

We can define another output $z$ where

$$ z = C_q'x \quad \Rightarrow \quad \text{controlled or regulated output} $$

therefore

$$ x'Qx = x'C_q'C_q = z'z $$

LQR design of double integrator

$$ A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C_q = \begin{bmatrix} 1 & 0 \end{bmatrix} $$

assume

$$ Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad R = 1 $$

$$ Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = C_q'C_q $$
(A, B) is controllable
(C(q, A) is observable

\[\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & p_1 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}
\]

\[P = \begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix}\]

\[K = R^{-1}B'P = \begin{bmatrix} 0 & 1 \\ 1 & \sqrt{2} \end{bmatrix}\]

The closed loop system matrix becomes

\[A - BK = \begin{bmatrix} 0 & 1 \\ -1 & -\sqrt{2} \end{bmatrix}\]

Closed loop roots are:

\[\lambda^2 + \sqrt{2}\lambda + 1 = 0\]

\[\lambda = \frac{\sqrt{2}}{2}(-1 \pm j)\]

damping ratio is 0.707
The loop transfer function is:

\[
K \phi(s) B \\
= K (sI - A)^{-1} B \\
= \sqrt{2} \left[ s + \frac{\sqrt{2}}{2} \right] \\
\frac{s^2}{\sqrt{2}}
\]

→ 65° phase margin
→ infinite gain margin

\[
L(s) = \frac{\sqrt{2} \left[ s + \frac{\sqrt{2}}{2} \right]}{s^2} \\
= \frac{\sqrt{2} \left[ \frac{\sqrt{2}}{2} \right]}{s^2} \left[ 1 + \frac{\sqrt{2}}{2} s \right] \\
= \frac{1 + \frac{s}{\sqrt{2}}}{s^2} \\
= \frac{1 + \frac{s}{0.7071}}{s^2}
\]

\[
|L(j\omega)|_{\omega>0.7071} \approx \frac{j\omega}{0.7071} \\
= \frac{1.4142}{\omega}
\]
\[ |L(j\omega_c)| = 1 = \frac{1.4142}{\omega_c} \]
\[ \omega_c = 1.4142 \]

\[ \text{Phase @ } \omega_c = -180 + \tan^{-1}\left(\frac{\omega_c}{0.7071}\right) \]

\[ \text{Phase margin} = \tan^{-1}\left(\frac{1.4142}{0.7071}\right) = \tan^{-1}(2) = 63.4^\circ \]

**USING MATLAB TO GET EXACT RESULTS**

Matlab

\[ \text{num} = \sqrt{2}*[1\; \sqrt{2}/2] \]
\[ \text{den} = [1\; 0\; 0] \]

**results:**

\[ \text{margin(num,den)} \]
\[ \text{gm}=\infty \]
\[ \text{pm}=65.53 \quad @ \; \omega=1.554 \]

**Properties of LQR design**

From the ARE we can derive the relation

\[ |1 + L(j\omega)|^2 = 1 + \frac{1}{\rho}|G_q(j\omega)|^2 \quad (*) \]

\[ \rho \text{ is a scalar} \]

where \( L(s) = K\phi(s)B \)

- loop gain

\[ \phi(s) \equiv (sI - A)^{-1} \]

and

\[ Q = C_q'C_q \]
\[ G_q(s) = C_q\phi(s)B \]

From (*) we see

\[ |1 + L(j\omega)| \geq 1 \]

This implies that the Nyquist plot of the loop transfer function of an LQR design always stays outside of a unit circle centered at (-1,0).
In SISO case, LQR design has > 60° phase margin, infinite gain margin and a gain reduction tolerance of -6dB (i.e. the gain can be reduced by a factor of $\frac{1}{2}$ before instability occurs).

Recall pole placement does not guarantee stability margins.

High-frequency roll-off rate

Closed loop transfer function $T(j\omega) = -K(j\omega I - A + BK)^{-1}B$

$$\lim_{\omega \to \infty} T(j\omega) = \frac{1}{j\omega} KB = \frac{1}{j\omega} R^{-1} B' PB < 0$$

$\Rightarrow$ -20dB/dec roll off rate at high frequencies

- not good for noise suppression

Optimal Observers – Kalman Filter

State estimation – plant represented as

$$\dot{x} = Ax + Bu + \omega \quad \leftarrow \text{process noise}$$
$$y = Cx + v \quad \leftarrow \text{measurement noise}$$

The optimal filter is given by

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$
where \( L = \Sigma C' R_0^{-1} \)

where \( \Sigma \) is the positive definite solution of

\[
A\Sigma + \Sigma A'Q_0 - \Sigma C' R_0^{-1} C\Sigma = 0
\]

\( Q_0 \) and \( R_0 \) are noise covariance matrices, which represent the intensity of the process and sensor noise inputs.

Require \( Q_0 \geq 0, R_0 > 0 \) and system to be observable.

If we combine the Kalman-Bucy Filter (optimal estimator) with LQR design, we have LQG (Linear Quadratic Gaussian). Let’s do a LQG design for double integrator plant. We already have the LQR design.

For Kalman filter, assume

\[
Q_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad R_0 = 1
\]

Solving Ricatti equation with \( \Sigma = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \)

\[ a^2 = 2b + 1 \quad \text{we find} \quad ab = c \quad b^2 = 1 \]

\[ \Rightarrow \quad \Sigma = \begin{bmatrix} \sqrt{3} & 1 \\ 1 & \sqrt{3} \end{bmatrix} \]

and \( L = \Sigma C' R_0^{-1} = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} \)

Transfer function of compensator is given by

\[
H(s) = K(L - A + BK + LC)^{-1} L
= \frac{3.14(s + 0.3)}{(s + 1.57 + j1.4)(s + 1.57 - j1.4)}
\]

Comparison of LQR and LQG

- LQR has guaranteed stability margins
- LQG has no guaranteed stability margins
- high freq. roll off in LQG can be > 20 dB/dec exhibited by LQR → greater noise filtering in LQG.

- LQG is not robust → uncertainty in plant may cause system to go unstable.

**Loop Transfer Recovery (LTR)**

LQR →

- > 60° phase margin
- infinite gain margin

LQG →

- no guaranteed margins

The properties of LQR can be recovered asymptotically by using Q₀ and R₀ as tuning parameters.

\[ R = 0 \quad + \quad \bar{U}(s) \quad - \quad U(s) \quad B \quad \Phi(s) \quad X(s) \quad C \quad Y(s) \]

**LQR**

- Loop gain, \( L(s) = K\phi(s)B \)
  
  \[ L(s) = K(sI - A)^{-1}B \]

**LQG**

\[ s\hat{X} = (-BK + A - LC)\hat{X} + LY(s) \]

\[ \frac{\hat{X}(s)}{Y(s)} = (sI - A + BK + LC)^{-1}L \]

- Loop gain, \( L_{LQG}(s) = K(sI - A + BK + LC)^{-1}LC\phi(s)B \)
If the following two conditions hold then LQR loop properties can be recovered if

1) \( G(s) \) is minimum phase
2) \( R_0=1 \) and \( Q_0=q^2BB' \)

Then it can be shown

\[
\lim_{q \to \infty} L_{LQR}(s) = L(s)
\]

The variable \( y \) that is recovered may be different from the variable \( z \) that is to be controlled

where \( y = Cx \) and \( z = C_q x \)

**Loop Shaping Steps**

1) Determine the controlled variable and set

\[
Q = C'C \quad \text{and} \quad Q = C_q' C_q
\]

2) Get a desired loop gain in LQR design. Use R as tuning parameter.

3) Select scalar \( q \) and solve the filter Ricatti equation

\[
A\Sigma + \Sigma A' + q^2 BB' - \Sigma C' C\Sigma = 0
\]

\[
L = \Sigma C'
\]

4) Increase \( q \) until the resulting loop transfer function is close to the LQR design

**Do not make \( q \) too high since**

1) large gains in \( L \) are required
2) the undesirable -20dB/dec high freq. roll-off of LQR will be recovered

**Example**

Double integrator system

\[
Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{R=1}
\]

\( \Rightarrow \) Gave 65° phase margin for LQR design
Figure 10.19  Step response, Bode plots, and filter poles for LTR using $q = (1, 10, 100, 1000)$. (a) Closed-loop step response. (b) and (c) Open-loop magnitude and phase Bode plots. (d) Filter poles.
\[ \begin{array}{|c|c|c|c|c|}
\hline
q & 1 & 10 & 100 & 1000 \\
\hline
PM & 32.6 & 41.9 & 55.0 & 61.7 \\
GM & 9.5 & 13.0 & 21.1 & 30.4 \\
L & 1.4 & 4.5 & 14.1 & 44.7 \\
Filter poles & \(-0.7 + 0.7j\) & \(-2.2 + 2.2j\) & \(-7.0 + 7.0j\) & \(-22.3 + 22.3j\) \\
poles & \(-0.7 - 0.7j\) & \(-2.2 - 2.2j\) & \(-7.0 - 7.0j\) & \(-22.3 - 22.3j\) \\
\hline
\end{array} \]
Robustness

1) Robust stability – stable in the face of plant uncertainties
2) Robust performance – performance met even in the face of plant uncertainties

Two important properties of feedback –
1) sensitivity reduction
2) disturbance rejection

General feedback system

![Feedback System Diagram]

\[ Y(s) = \frac{G(s)H(s)}{1 + G(s)H(s)} R(s) + \frac{1}{1 + G(s)H(s)} D(s) - \frac{G(s)H(s)}{1 + G(s)H(s)} N(s) \]

Tracking error \( e = r - y \)

\[ E(s) = \frac{1}{1 + G(s)H(s)} R(s) - \frac{1}{1 + G(s)H(s)} D(s) - \frac{1}{1 + G(s)H(s)} N(s) \]

Actuator output (i.e. plant input) is given by

\[ U(s) = \frac{H(s)}{1 + G(s)H(s)} \left[ R(s) - D(s) - N(s) \right] \quad \text{note: } U(s) = H(s)E(s) \Rightarrow E(s) = H^{-1}(s)U(s) \]

Define the following terms

\[ J(s) = 1 + GH \quad \text{return difference} \]
\[ S(s) = \frac{1}{1 + GH} \quad \text{sensitivity} \]
\[ T(s) = \frac{GH}{1 + GH} \quad \text{complementary sensitivity} \]

\[ \text{note: } S(s) + T(s) = 1 \]

Using these definitions

system output: \[ Y(s) = S(s)D(s) + T(s)\left[ R(s) - N(s) \right] \]

tracking error: \[ E(s) = S(s)\left[ R(s) - D(s) - N(s) \right] \]
plant input: \[ U(s) = H(s)S(s)[R(s) - D(s) - N(s)] \]

From these expressions we see that we need

1) Disturbance rejection: From \( Y(s) \) expression we see we require \( S \) small \( \Rightarrow \) \( GH \gg 1 \) (since SD)

2) Tracking: \( S \) small

3) Noise suppression: From \( Y(s) \) we have \( T(s)N(s) \) \( \Rightarrow \) require \( T \) small

4) Actuator limits: From \( U(s) \) expression want \( H(s)S(s) \) bounded

Tracking and Disturbance rejection require small \( S \)
Noise suppression requires small \( T \)

However \( S + T = 1 \)

However command inputs and disturbances are low frequency whereas measurement noise is high frequency signal

\( \Rightarrow \) keep \( S \) small in low frequency range and \( T \) small in high frequency range

Also \( H(s)S(s) = \frac{H(s)}{1 + G(s)H(s)} = \frac{T(s)}{G(s)} \)

\( \Rightarrow \) making \( T \) small we reduce control energy

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**Loop Gain Properties**

<table>
<thead>
<tr>
<th>Performance (R)</th>
<th>Low Frequency</th>
<th>Mid. Frequency</th>
<th>High Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>High Gain</td>
<td>Smooth Transition (for good margins)</td>
<td></td>
</tr>
<tr>
<td>Disturbance Rejection (D)</td>
<td>High Gain</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Noise Suppression (N)</td>
<td></td>
<td></td>
<td>Low Gain</td>
</tr>
</tbody>
</table>

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**Uncertainty Modeling**

Two categories --
1) structured uncertainty
2) unstructured uncertainty

We will deal with unstructured uncertainty
Additive uncertainty: actual model $\tilde{G}(s)$

$$\tilde{G}(s) = \frac{G(s)}{\text{model}} + \frac{\Delta_a(s)}{\text{uncertainty or error}}$$

Multiplicative uncertainty: $\tilde{G}(s) = [1 + \Delta_m(s)]G(s)$

Robust Stability

We say a compensator robustly stabilizes a system if the closed-loop system remains stable for the true plant $\tilde{G}(s)$.

Robustness results can be derived using the small gain theorem.

Small Gain Theorem

The closed-loop system will remain stable if

$$|G(s)H(s)| < 1$$

no since $|G(s)H(s)| \leq |G(s)||H(s)|$
then closed-loop stability is guaranteed if

$$|G(s)H(s)| < 1$$

There is no possibility of encirclements of (-1,0) point by Nyquist plot.

Two equations that the small gain theorem can help us to answer

1) Given that the uncertainty is stable and bounded, will the closed-loop system be stable for the given uncertainty?
2) For a given system, what is the smallest uncertainty that will destabilize the system?

To answer these questions we first do some block diagram manipulation

![Block diagram](image)

With multiplicative output uncertainty

![Block diagram with uncertainty](image)

where $M(s) = \frac{-G(s)H(s)}{1 + G(s)H(s)}$

Determine $M(s)$, the transfer function seen by $\Delta_m$
By small gain theorem, closed-loop system will be robustly stable if
\[ |\Delta_m| < \frac{1}{GH(1+GH)^{-1}} \]

i.e. \[ |\Delta_m| < \frac{1}{T} \]  
T – complementary sensitivity

If the uncertainty is bounded by \( \gamma \) so that
\[ |\Delta_m| < \gamma \]
then the closed-loop system will be stable if
\[ |T| \frac{1}{\gamma} \] or \[ |\gamma T| < 1 \]

This answers the first question

Second question: find the size of the smallest stable uncertainty that will destabilize the system

Because the uncertainty must be smaller that \( 1/T \), it must be smaller that the minimum of \( 1/T \). We must find the maximum of \( T \).

Define \[ M_r = \sup_{\omega} |T(j\omega)| \] \( \sup = \) supremum (least upper bound)

Then the smallest destabilizing uncertainty, we call this the **multiplicative stability margin** or MSM, is given by
\[ MSM = \frac{1}{M_r} \]

For additive uncertainty
\[ M(s) = \frac{-H(s)}{1+G(s)H(s)} \]

closed-loop will be robustly stable if
\[ |\Delta_s| < \frac{1}{H(1+GH)^{-1}} \] or \[ |\Delta_s| < \frac{1}{|HS|} \]
if uncertainty is stable and bounded by

$$|\Delta_a| < \gamma$$

then we guarantee closed-loop stability if

$$|HS| < \frac{1}{\gamma} \quad \text{or} \quad |\gamma HS| < 1$$

we can define additive stability margin (ASM) by

$$ASM = \frac{1}{\sup_\omega H(j\omega)S(j\omega)}$$

Example

$$G(s) = \frac{5 - s}{(s + 5)(s^2 + 0.2s + 1)}$$, \hspace{1cm} $$H(s) = \frac{5(s + 0.1)}{s} + \frac{0.2}{s + 5}$$

phase margin: 38°
gain margin: 2.8 (9dB)

Find MSM and ASM:

MSM

Find peak of $T$ (complementary sensitivity function)

peak = 1.52 \rightarrow MSM = 0.65

\rightarrow the system will be robustly stable against unmodelled multiplicative uncertainties with transfer function magnitude < 0.65
Problem 10.9

a.) \[ \tilde{G} = (1 + \Delta_m)G \quad \Rightarrow \quad \Delta_m = \frac{\tilde{G}}{G} - 1 \]

\[ \Delta_m = \frac{2(s + 1)}{s^2(s^2 + s + 1)} - 1 \]

\[ = \frac{2(s + 1)}{s^2(s^2 + s + 1)} - \frac{s^2 + s + 1}{s^2(s^2 + s + 1)} \]

\[ = \frac{-s^2 + s + 1}{s^2 + s + 1} \]

b.)

![Block diagram of control system](image)

\[ M(s) \frac{-GH}{(1 + GH)} = \frac{-20(s + 1)}{s^2(s + 10)} = \frac{-20(s + 1)}{s^2(s + 10) + 20(s + 1)} = \frac{20(s + 1)}{s^3 + 10s^2 + 20s + 20} \]

c.) SGT: \[ |\Delta_m| |M| < 1 \]

\[ \Rightarrow |\Delta_m| < \frac{1}{|M|} = \frac{1}{GH} \frac{1}{1 + GH} \]

\[ \Rightarrow |\Delta_m| < |1 + (GH)^{-1}| \]
Additive uncertainty

\[ \tilde{G} = G + \Delta_a \]

\[ \Rightarrow \Delta_a = \tilde{G} - G \]

\[ \Delta_a = \frac{2(s + 1)}{s^2 (s^2 + s + 1)} - \frac{1}{s^2} \]

\[ = \frac{1}{s^2} \left[ \frac{2s + 2 - s^2 - s - 1}{s^2 + s + 1} \right] \]

\[ = \frac{-s^2 + s + 1}{s^2 (s^2 + s + 1)} \]

\[ M(s) = \frac{-H}{(1 + GH)} \]

SGT: \[ |\Delta_a| |M| < 1 \]

\[ \Rightarrow |\Delta_a| < \frac{1}{|M|} \]

\[ \Rightarrow |\Delta_a| < |H^{-1} + G| \]
\[ |\Delta_a| < \left| \frac{s + 10}{20s + 20} + \frac{1}{s^2} \right| \]

\[ < \frac{s^2(s + 10) + 20s + 20}{s^2(20s + 20)} \]

\[ |\Delta_a| < \frac{s^3 + 10s^2 + 20s + 20}{s^2(20s + 20)} \]
Basic Bode Magnitude Plots

\[ G(s) = \frac{A}{1 + \frac{s}{\omega_0}} \Rightarrow |G| \]

\[ G(s) = A(1 + \frac{s}{\omega_0}) \Rightarrow |G| \]

\[ G(s) = \frac{A}{1 + \frac{1}{Q} \left( \frac{s}{\omega_0} \right) + \left( \frac{s}{\omega_0} \right)^2} \Rightarrow |G| \]

\[ G(s) = A \left[ 1 + \frac{1}{Q} \left( \frac{s}{\omega_0} \right) + \left( \frac{s}{\omega_0} \right)^2 \right] \Rightarrow |G| \]

If \( Q < \frac{1}{4} \) then roots are real. Factor the expression and use the resulting product of two first order transfer functions to find magnitude response.
Example

\[ G(s) = \frac{A}{(1 + \frac{s}{\omega_0})[1 + \frac{1}{Q} \left( \frac{s}{\omega_0} \right) + \left( \frac{s}{\omega_0} \right)^2]} \Rightarrow \]

\[ \omega_0 < \omega_1 \]
Example

The $\Delta M$ structure of a system has been determined to be given by

$$
\Delta = \frac{A_\Delta}{1 + \frac{1}{Q} \left( \frac{\omega}{\omega_1} \right) + \left( \frac{\omega}{\omega_2} \right)^2}
$$

$$
M = \frac{A_M (1 + \frac{\omega}{\omega_1})}{(1 + \frac{\omega}{\omega_1})(1 + \frac{\omega}{\omega_2})}
$$

where $\omega_1 << \omega_2 << \omega_3$ and $\omega_s = \sqrt{\omega_2 \omega_3}$

Determine the conditions under which robust stability is assured.

Answer

By SGT we require $|\Delta| |M| < 1$ or $|\Delta| < \frac{1}{|M|}$

$$
\frac{1}{M} = \frac{A^{-1}_M (1 + \frac{\omega}{\omega_1})(1 + \frac{s}{\omega_1})}{(1 + \frac{s}{\omega_2})}
$$

From the above diagram we can see that we require

- $A_\Delta < A^{-1}_M$ and $QA_\Delta < A^{-1}_M \frac{\omega_2}{\omega_1}$

or

- $A_\Delta < A^{-1}_M \frac{\omega_2}{Q\omega_1}$