2.6 ADDITION OF CARTESIAN VECTORS

**Important Points**

- Cartesian vector analysis is often used to solve problems in three dimensions.
- The positive directions of the $x$, $y$, $z$ axes are defined by the Cartesian unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, respectively.
- The magnitude of a Cartesian vector is $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$.
- The direction of a Cartesian vector is specified using coordinate direction angles $\alpha$, $\beta$, $\gamma$ which the tail of the vector makes with the positive $x$, $y$, $z$ axes, respectively. The components of the unit vector $\mathbf{u}_i = A/A$ represent the direction cosines of $\alpha$, $\beta$, $\gamma$. Only two of the angles $\alpha$, $\beta$, $\gamma$ have to be specified. The third angle is determined from the relationship $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$.
- Sometimes the direction of a vector is defined using the two angles $\theta$ and $\phi$ as in Fig. 2–28. In this case the vector components are obtained by vector resolution using trigonometry.
- To find the resultant of a concurrent force system, express each force as a Cartesian vector and add the $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ components of all the forces in the system.

**EXEMPLARY 2.8**

Express the force $\mathbf{F}$ shown in Fig. 2–30 as a Cartesian vector.

**SOLUTION**

Since only two coordinate direction angles are specified, the third angle $\alpha$ must be determined from Eq. 2–8; i.e.,

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$
$$\cos^2 \alpha + \cos^2 60^\circ + \cos^2 45^\circ = 1$$

Thus $\cos \alpha = \sqrt{1 - (0.5)^2 - (0.707)^2} = \pm 0.5$

Hence, two possibilities exist, namely,

$\alpha = \cos^{-1}(0.5) = 60^\circ$ or $\alpha = \cos^{-1}(-0.5) = 120^\circ$

By inspection it is necessary that $\alpha = 60^\circ$, since $\mathbf{F_1}$ must be in the $+x$ direction.

Using Eq. 2–9, with $F = 200$ N, we have

$$\mathbf{F} = F \cos \alpha \mathbf{i} + F \cos \beta \mathbf{j} + F \cos \gamma \mathbf{k}$$
$$= (200 \cos 60^\circ \mathbf{i} + (200 \cos 60^\circ \mathbf{j} + (200 \cos 45^\circ \mathbf{k}$$
$$= \begin{bmatrix} 100.0 \mathbf{i} + 100.0 \mathbf{j} + 141.4 \mathbf{k} \end{bmatrix} N$$

$F = 200$ N.

Show that indeed the magnitude of $F = 200$ N.

$$\sqrt{(100)^2 + (100)^2 + (141.4)^2} = 200$$
EXAMPLE 2.9

Determine the magnitude and the coordinate direction angles of the resultant force acting on the ring in Fig. 2–31a.

\[ \mathbf{F}_1 = \{50\mathbf{i} + 100\mathbf{j} + 100\mathbf{k}\} \text{ lb} \]
\[ \mathbf{F}_2 = \{60\mathbf{j} + 80\mathbf{k}\} \text{ lb} \]
\[ \mathbf{F}_R = \{50\mathbf{i} - 40\mathbf{j} + 180\mathbf{k}\} \text{ lb} \]
\[ \gamma = 19.6^\circ \]
\[ \alpha = 74.8^\circ \]
\[ \beta = 102^\circ \]

Fig. 2–31

SOLUTION

Since each force is represented in Cartesian vector form, the resultant force, shown in Fig. 2–31b, is

\[ \mathbf{F}_R = \sum \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = \{60\mathbf{j} + 80\mathbf{k}\} \text{ lb} + \{50\mathbf{i} - 100\mathbf{j} + 100\mathbf{k}\} \text{ lb} \]
\[ = \{50\mathbf{i} - 40\mathbf{j} + 180\mathbf{k}\} \text{ lb} \]

The magnitude of \( \mathbf{F}_R \) is

\[ F_R = \sqrt{(50 \text{ lb})^2 + (-40 \text{ lb})^2 + (180 \text{ lb})^2} = 191.0 \text{ lb} \]
\[ = 191 \text{ lb} \quad \text{Ans.} \]

The coordinate direction angles \( \alpha, \beta, \gamma \) are determined from the components of the unit vector acting in the direction of \( \mathbf{F}_R \).

\[ \mathbf{u}_{F_y} = \frac{\mathbf{F}_y}{F_R} = \frac{50}{191.0} \mathbf{i} - \frac{40}{191.0} \mathbf{j} + \frac{180}{191.0} \mathbf{k} \]
\[ = 0.2617 \mathbf{i} - 0.2094 \mathbf{j} + 0.9422 \mathbf{k} \]

so that

\[ \cos \alpha = 0.2617 \quad \alpha = 74.8^\circ \quad \text{Ans.} \]
\[ \cos \beta = -0.2094 \quad \beta = 102^\circ \quad \text{Ans.} \]
\[ \cos \gamma = 0.9422 \quad \gamma = 19.6^\circ \quad \text{Ans.} \]

These angles are shown in Fig. 2–31b:

NOTE: In particular, notice that \( \beta > 90^\circ \) since the \( \mathbf{j} \) component of \( \mathbf{u}_{F_y} \) is negative. This seems reasonable considering how \( \mathbf{F}_1 \) and \( \mathbf{F}_2 \) add according to the parallelogram law.
Important Points

- The dot product is used to determine the angle between two vectors or the projection of a vector in a specified direction.
- If vectors $\mathbf{A}$ and $\mathbf{B}$ are expressed in Cartesian vector form, the dot product is determined by multiplying the respective $x$, $y$, $z$ scalar components and algebraically adding the results, i.e.,
  $$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z.$$
- From the definition of the dot product, the angle formed between the tails of vectors $\mathbf{A}$ and $\mathbf{B}$ is
  $$\theta = \cos^{-1}(\mathbf{A} \cdot \mathbf{B} / AB).$$
- The magnitude of the projection of vector $\mathbf{A}$ along a line $aa$ whose direction is specified by $\mathbf{u}$ is determined from the dot product $A_u = \mathbf{A} \cdot \mathbf{u}.$

EXAMPLE 2.16

Determine the magnitudes of the projection of the force $\mathbf{F}$ in Fig. 2-44 onto the $u$ and $v$ axes.

![Diagram](image)

Fig. 2-44

SOLUTION

Projections of Force. The graphical representation of the projections is shown in Fig. 2-44. From this figure, the magnitudes of the projections of $\mathbf{F}$ onto the $u$ and $v$ axes can be obtained by trigonometry:

- $(F_u)_{proj} = (100 \text{ N})\cos 45^\circ = 70.7 \text{ N}$ \hspace{1cm} \text{Ans.}
- $(F_v)_{proj} = (100 \text{ N})\cos 15^\circ = 96.6 \text{ N}$ \hspace{1cm} \text{Ans.}

NOTE: These projections are not equal to the magnitudes of the components of force $\mathbf{F}$ along the $u$ and $v$ axes found from the parallelogram law. They will only be equal if the $u$ and $v$ axes are perpendicular to one another.

\text{NOT COMPONENTS OF } \mathbf{F} \; \text{B/C } U \not\parallel V \; \text{not } \perp
EXAMPLE 2.18

The pipe in Fig. 2-46a is subjected to the force of \( F = 80 \text{ lb} \). Determine the angle \( \theta \) between \( \textbf{F} \) and the pipe segment \( BA \) and the projection of \( \textbf{F} \) along this segment.

![Diagram showing pipe with forces](image)

**SOLUTION**

**Angle \( \theta \).** First we will establish position vectors from \( B \) to \( A \) and \( B \) to \( C \); Fig. 2-46b. Then we will determine the angle \( \theta \) between the tails of these two vectors. 

\[
\begin{align*}
\textbf{r}_{BA} &= \begin{pmatrix} -2i - 2j + 1k \end{pmatrix} \text{ ft}, \quad r_{BA} = 3 \text{ ft} \\
\textbf{r}_{BC} &= \begin{pmatrix} -3j + 1k \end{pmatrix} \text{ ft}, \quad r_{BC} = \sqrt{10} \text{ ft} \\
\end{align*}
\]

Thus,

\[
\cos \theta = \frac{\textbf{r}_{BA} \cdot \textbf{r}_{BC}}{r_{BA}r_{BC}} = \frac{(-2)(0) + (-2)(-3) + (1)(1)}{3 \sqrt{10}} = 0.7379
\]

\[
\theta = 42.5^\circ \quad \text{Ans.}
\]

**Components of \( \textbf{F} \).** The component of \( \textbf{F} \) along \( BA \) is shown in Fig. 2-46c. We must first formulate the unit vector along \( BA \) and force \( \textbf{F} \) as Cartesian vectors.

\[
\begin{align*}
\textbf{u}_{BA} &= \frac{\textbf{r}_{BA}}{r_{BA}} = \frac{\begin{pmatrix} -2i - 2j + 1k \end{pmatrix}}{3} = -\frac{2}{3}i - \frac{2}{3}j + \frac{1}{3}k \\
\textbf{F} &= 80 \text{ lb} \begin{pmatrix} \textbf{r}_{BC} \end{pmatrix} = 80 \begin{pmatrix} -3j + 1k \end{pmatrix} = -75.89j + 25.30k
\end{align*}
\]

Thus,

\[
\begin{align*}
\textbf{F}_{BA} &= \textbf{F} \cdot \textbf{u}_{BA} = (-75.89j + 25.30k) \cdot \left( -\frac{2}{3}i - \frac{2}{3}j + \frac{1}{3}k \right) \\
&= 0 \left( -\frac{2}{3} \right) + (-75.89) \left( -\frac{2}{3} \right) + (25.30) \left( \frac{1}{3} \right) \\
&= 59.0 \text{ lb} \quad \text{Ans.}
\end{align*}
\]

**NOTE:** Since \( \theta \) has been calculated, then also, \( F_{BA} = F \cos \theta = 80 \text{ lb} \cos 42.5^\circ = 59.0 \text{ lb} \).