Lecture 9: Divide & Conquer
The Master Theorem, Mergesort & Quicksort

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Thursday, 29 October 2015
What is Divide-and-Conquer?
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Solves a problem instance of size \( n \) by:
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Solves a problem instance of size $n$ by:

1. splitting it into $b$ smaller instances, of size $\sim n/b$
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Solves a problem instance of size $n$ by:

1. splitting it into $b$ smaller instances, of size $\sim n/b$
2. solving some or all of them (in general, solving $a$ of them), using the same algorithm recursively.
What is Divide-and-Conquer?

Solves a problem instance of size \( n \) by:

1. splitting it into \( b \) smaller instances, of size \( \sim \frac{n}{b} \)
2. solving some or all of them (in general, solving \( a \) of them), using the same algorithm recursively.
3. combining the solutions to the \( a \) smaller problems to get the solution to the original problem.
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Time taken:
What is Divide-and-Conquer?

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2. solving some or all of them (in general, solving $a$ of them), using the same algorithm recursively.
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Time taken:

$$T(n) = aT\left(\frac{n}{b}\right)$$
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True or False?
What is Divide-and-Conquer?

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Time taken:

$$T(n) = aT\left(\frac{n}{b}\right) + T_{\text{split+combine}}(n)$$
Algorithm MergeSort(A[0..n − 1])

// Sorts array A[0..n − 1] by recursive mergesort
// Input: An array A[0..n − 1] of orderable elements
// Output: Array A[0..n − 1] sorted in nondecreasing order
if n > 1
    copy A[0..[n/2] − 1] to B[0..[n/2] − 1]
copy A[[n/2]..n − 1] to C[0..[n/2] − 1]
MergeSort(B[0..[n/2] − 1])
MergeSort(C[0..[n/2] − 1])
Merge(B, C, A)
**Merge**

**ALGORITHM**  
\( \text{Merge}(B[0..p-1], C[0..q-1], A[0..p+q-1]) \)

//Merges two sorted arrays into one sorted array  
//Input: Arrays \( B[0..p-1] \) and \( C[0..q-1] \) both sorted  
//Output: Sorted array \( A[0..p+q-1] \) of the elements of \( B \) and \( C \)  
\( i \leftarrow 0; \; j \leftarrow 0; \; k \leftarrow 0 \)

**while** \( i < p \) **and** \( j < q \) **do**

**if** \( B[i] \leq C[j] \)

\( A[k] \leftarrow B[i]; \; i \leftarrow i + 1 \)

**else** \( A[k] \leftarrow C[j]; \; j \leftarrow j + 1 \)

\( k \leftarrow k + 1 \)

**if** \( i = p \)

copy \( C[j..q-1] \) to \( A[k..p+q-1] \)

**else** copy \( B[i..p-1] \) to \( A[k..p+q-1] \)
Time Complexity of Mergesort
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Suppose that we are sorting an array of $n = 2^k$ elements.
Time Complexity of Mergesort

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° Let \( C(n) \) = number of comparisons when sorting \( n \) elements, then:
Time Complexity of Mergesort

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✧ Let $C(n) =$ number of comparisons when sorting $n$ elements, then:

```
ALGORITHM Mergesort(A[0..n−1])
//Sorts array A[0..n−1] by recursive mergesort
//Input: An array A[0..n−1] of orderable elements
//Output: Array A[0..n−1] sorted in nondecreasing order
if n > 1
    copy A[0..n/2 − 1] to B[0..n/2 − 1]
    copy A[n/2..n − 1] to C[0..n/2 − 1]
    Mergesort(B[0..n/2 − 1])
    Mergesort(C[0..n/2 − 1])
    Merge(B, C, A)
```
Time Complexity of Mergesort

- Suppose that we are sorting an array of \( n=2^k \) elements
- Let \( C(n) \) = number of comparisons when sorting \( n \) elements, then:
  - \( C(1) = 0 \)

```
ALGORITHM Mergesort(A[0..n - 1])
//Sorts array A[0..n - 1] by recursive mergesort
//Input: An array A[0..n - 1] of orderable elements
//Output: Array A[0..n - 1] sorted in nondecreasing order
if n > 1
  copy A[0..\[n/2] - 1] to B[0..\[n/2] - 1]
  copy A[\[n/2]..n - 1] to C[0..\[n/2] - 1]
  Mergesort(B[0..\[n/2] - 1])
  Mergesort(C[0..\[n/2] - 1])
  Merge(B, C, A)
```
Time Complexity of Mergesort

ặ Suppose that we are sorting an array of $n=2^k$ elements

ặ Let $C(n)$ = number of comparisons when sorting $n$ elements, then:

- $C(1) = 0$
- $C(n) = 2 \cdot C(n/2) + C_{merge}(n)$

<table>
<thead>
<tr>
<th>ALGORITHM</th>
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</tr>
</thead>
<tbody>
<tr>
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<td></td>
</tr>
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<td></td>
</tr>
<tr>
<td>//Output: Array $A[0..n-1]$ sorted in nondecreasing order</td>
<td></td>
</tr>
<tr>
<td>if $n &gt; 1$</td>
<td></td>
</tr>
<tr>
<td>copy $A[0..\lfloor n/2 \rfloor - 1]$ to $B[0..\lfloor n/2 \rfloor - 1]$</td>
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Time Complexity of Mergesort

- Suppose that we are sorting an array of $n=2^k$ elements
- Let $C(n) =$ number of comparisons when sorting $n$ elements, then:
  
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ALGORITHM Mergesort(A[0..n − 1])

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if n > 1
    copy A[0..[n/2] − 1] to B[0..[n/2] − 1]
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    Mergesort(B[0..[n/2] − 1])
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    Merge(B, C, A)
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- Suppose that we are sorting an array of $n=2^k$ elements
- Let $C(n)$ = number of comparisons when sorting $n$ elements, then:
  
  - $C(1) = 0$
  - $C(n) = 2 \cdot C(n/2) + C_{merge}(n)$
  - $C_{merge}(n) = n - 1$ in the worst case ($n/2$ in the best)
  - $C_{worst}(n) = 2 \cdot C_{worst}(n/2) + n - 1$

**Algorithm Mergesort**

```plaintext
Mergesort(A[0..n − 1])
// Sorts array A[0..n − 1] by recursive mergesort
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    Mergesort(B[0..[n/2] − 1])
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```
Solve the recurrence:
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\[ C(n) = 2 \, C(n/2) + n - 1 \] (omitting worst)
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\[ C(n) = 2 \ C(n/2) + n - 1 \quad \text{(omitting worst)} \]

\[ C(n)/n = 2 \ C(n/2)/n + 1 - 1/n \quad \text{(divide by \( n \))} \]

\[ = C(n/2)/(n/2) + 1 - 1/n \quad \text{(algebra)} \]
Solve the recurrence:

\[ C(n) = 2 \cdot C(n/2) + n - 1 \]  
\[ C(n)/n = 2 \cdot C(n/2)/n + 1 - 1/n \]  
\[ = C(n/2)/(n/2) + 1 - 1/n \]  
\[ = C(n/4)/(n/4) + 1 + 1 - 1/n - 2/n \]  

(omitting \textit{worst})  
(divide by \(n\))  
(algebra)  
(subst. \(n \Rightarrow n/2\))
Solve the recurrence:

\[ C(n) = 2 \frac{C(n)}{n} - 1 \quad \text{(omitting worst)} \]

\[ \frac{C(n)}{n} = \frac{2 C(n/2)}{n/2} + 1 - \frac{1}{n} \quad \text{(divide by n)} \]

\[ = \frac{C(n/2)}{(n/2)} + 1 - \frac{1}{n} \quad \text{(algebra)} \]

\[ = \frac{C(n/4)}{(n/4)} + 1 + 1 - \frac{1}{n} - \frac{2}{n} \quad \text{(subst. } n \Rightarrow n/2) \]

\[ = \frac{C(n/8)}{(n/8)} + 1 + 1 + 1 - \frac{1}{n} - \frac{2}{n} - \frac{4}{n} \]
Solve the recurrence:

\[ C(n) = 2 \frac{C(n/2)}{n} + n - 1 \]  
\[ \text{omitting worst} \]

\[ C(n)/n = 2 \frac{C(n/2)/n}{n} + 1 - 1/n \]  
\[ \text{divide by } n \]

\[ = \frac{C(n/2)/(n/2)}{n} + 1 - 1/n \]  
\[ \text{algebra} \]

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\[ \text{subst. } n \Rightarrow n/2 \]

\[ = \frac{C(n/8)/(n/8)}{n} + 1 + 1 + 1 - 1/n - 2/n - 4/n \]  

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\[ \cdots \]

\[ = C(n/n)/(n/n) + 1 + 1 + 1 + \ldots - 1/n - 2/n - 4/n - \ldots \]
Solve the recurrence:

\[ C(n) = 2 \ C(n/2) + n - 1 \]  \hspace{2cm} \text{(omitting worst)}

\[ C(n)/n = 2 \ C(n/2)/n + 1 - 1/n \]  \hspace{2cm} \text{(divide by } n)\]

\[ = C(n/2)/(n/2) + 1 - 1/n \]  \hspace{2cm} \text{(algebra)}

\[ = C(n/4)/(n/4) + 1+1- 1/n - 2/n \]  \hspace{2cm} \text{(subst. } n \Rightarrow n/2)\]

\[ = C(n/8)/(n/8) + 1+1+1- 1/n - 2/n - 4/n \]

\[ \cdots \]

\[ = C(n/n)/(n/n) + 1+1+1+\ldots- 1/n - 2/n - 4/n - \ldots \]

\[ C(n)/n = C(1) + 1+1+1+\ldots- 1/n - 2/n - 4/n - \ldots \]
Solve the recurrence:

\[ C(n) = 2 \times C(n/2) + n - 1 \]  
\[ C(n)/n = 2 \times C(n/2)/n + 1 - 1/n \]  
\[ = C(n/2)/(n/2) + 1 - 1/n \]  
\[ = C(n/4)/(n/4) + 1 + 1 - 1/n - 2/n \]  
\[ = C(n/8)/(n/8) + 1 + 1 + 1 - 1/n - 2/n - 4/n \]  
\[ \vdots \]  
\[ = C(n/n)/(n/n) + 1 + 1 + 1 + \ldots - 1/n - 2/n - 4/n - \ldots \]  
\[ C(n)/n = C(1) + 1 + 1 + 1 + \ldots - 1/n - 2/n - 4/n - \ldots \]  
\[ \cong \log n \]  

(neglecting small terms)
Solve the recurrence:

\[ C(n) = 2 \frac{C(n)}{2} + n - 1 \]  
(omitting worst)

\[ \frac{C(n)}{n} = 2 \frac{C(n)}{2}/n + 1 - 1/n \]  
(divide by n)

\[ = \frac{C(n)}{2}/(n/2) + 1 - 1/n \]  
(algebra)

\[ = \frac{C(n)}{4}/(n/4) + 1 + 1 - 1/n - 2/n \]  
(subst. n ⇒ n/2)

\[ = \frac{C(n)}{8}/(n/8) + 1 + 1 + 1 - 1/n - 2/n - 4/n \]  

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\[ C(n) = n \log n \]
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\[ C(n)/n = C(1) + 1 + 1 + 1 + \ldots - 1/n - 2/n - 4/n - \ldots \]  
\[ \approx \lg n \quad \text{(neglecting small terms)} \]  
\[ C(n) = n \cdot \lg n \]
Look at the recursion tree:

\[ T(N) = 2 \cdot T(N/2) + N \]

for \( N > 1 \), with \( T(1) = 0 \)

(assume that \( N \) is a power of 2)

\[ T(N) = N \lg N \]

Divide & Conquer

When designing a divide and conquer algorithm, the time for the “split” and “combine” phases contributes to the cost:

A. not at all
B. in a small way
C. in a major way
D. in a way that dominates the total cost
E. in a way that depends on the algorithm
Divide & Conquer

In mergesort, the time for the “split” and “combine” phases contributes to the cost

A. not at all
B. in a small way
C. in a major way
D. in a way that dominates the total cost
Divide & Conquer

In Quicksort, the time for the “split” and “combine” phases contributes to the cost

A. not at all
B. in a small way
C. in a major way
D. in a way that dominates the total cost
The Master Theorem:
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\[ T(n) = aT(n/b) + T_{\text{combine}}(n) \]
The Master Theorem:

\[ T(n) = aT(n/b) + T_{\text{combine}}(n) \]

Suppose that \( T_{\text{combine}}(n) \in \Theta(n^d) \)

Then:

\[ T(n) \in \begin{cases} 
\Theta(n^d) & \text{when } a < b^d \\
\Theta(n^d \log n) & \text{when } a = b^d \\
\Theta(n^{\log_b a}) & \text{when } a > b^d 
\end{cases} \]
Application to Mergesort:

✧ We divide the problem into 2 (roughly equal) parts, so \( b = 2 \)

✧ We have to solve \textit{both} of them, so \( a = 2 \)

✧ Combining the parts is linear, i.e., in \( \Theta(n^1) \), so \( d = 1 \)

✧ Hence, in the Master Theorem, \( a = 2 = 2^1 = b^d \)

\[
T(n) \in \begin{cases} 
\Theta(n^d) & \text{when } a < b^d \\
\Theta(n^d \log n) & \text{when } a = b^d \\
\Theta(n^{ \log_b a } ) & \text{when } a > b^d 
\end{cases}
\]

So \( T(n) = \Theta(n \log n) \)
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Does the base of this log matter?  A: yes  B: no
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\end{cases} \]
Problem:

Why are there three cases in the Master Theorem?

- Look at the recursion tree:

At level $i$: $a^i$ problems of size $\frac{n}{b^i}$, each requiring combining work in $\Theta \left( \left( \frac{n}{b^i} \right)^d \right)$.

So total work is at level $i$ is: $\Theta \left( a^i \left( \frac{n}{b^i} \right)^d \right) = \Theta \left( n^d \left( \frac{a}{b^d} \right)^i \right)$.
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Problem:

Why are there three cases in the Master Theorem?

Look at the recursion tree:

\[ \text{instance of size } n \]

\[ a \text{ times: } \]

\[ \text{instance of size } n/b \]
\[ \text{instance of size } n/b \]
\[ \text{instance of size } n/b \]
\[ \text{...} \]
\[ \text{instance of size } n/b \]

\[ \text{size } n/b^2 \]
\[ \text{size } n/b^2 \]
\[ \text{...} \]
\[ \text{size } n/b^2 \]
\[ \text{size } n/b^2 \]
\[ \text{...} \]

At level \( i \): \( a^i \) problems of size \( \frac{n}{b^i} \), each requiring combining work in \( \Theta \left( \left( \frac{n}{b^i} \right)^d \right) \)

So total work is at level \( i \) is: \( \Theta \left( a^i \left( \frac{n}{b^i} \right)^d \right) = \Theta \left( n^d \left( \frac{a}{b^d} \right)^i \right) \)
Problem:

Why are there three cases in the Master Theorem?

- Look at the recursion tree:

\[
\begin{align*}
\text{instance of size } n \\
\text{a times:} & \quad \text{instance of size } n/b & \text{instance of size } n/b & \ldots & \text{instance of size } n/b \\
\text{a}^2 \text{ times:} & \quad \text{size } n/b^2 & \text{size } n/b^2 & \ldots & \text{size } n/b^2 & \ldots & \text{size } n/b^2 & \text{size } n/b^2 & \ldots \\
\end{align*}
\]

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Thursday, 29 October 2015
Problem:

Why are there three cases in the Master Theorem?

- Look at the recursion tree:

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So total work is at level $i$ is: $\Theta \left( a^i \left( \frac{n}{b^i} \right)^d \right) = \Theta \left( n^d \left( \frac{a}{b^d} \right)^i \right)$
Problem:

Why are there three cases in the Master Theorem?
- Look at the recursion tree:

```
instance of size n

a times: instance of size n/b instance of size n/b ... instance of size n/b

a^2 times: size n/b^2 size n/b^2 ... size n/b^2 ... size n/b^2 size n/b^2 ...

a^3 times: ... size n/b^3 size n/b^3 ... size n/b^3 ... size n/b^3 ...
```

At level i: \( a^i \) problems of size \( \frac{n}{b^i} \), each requiring combining work in \( \Theta \left( \left( \frac{n}{b^i} \right)^d \right) \)

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Why are there 3 cases?
Why are there 3 cases?

Combining work at level $i$ is in $\Theta \left( n^d \left( \frac{a}{b^d} \right)^i \right)$

Does this increase with $i$, stay constant, or decrease?
Why are there 3 cases?

Combining work at level $i$ is in $\Theta \left( n^d \left( \frac{a}{b^d} \right)^i \right)$

Does this increase with $i$, stay constant, or decrease?

- It depends on whether $\frac{a}{b^d}$ is less than, equal to, or greater than 1
Why are there 3 cases?

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A. $a < b^d$
Why are there 3 cases?

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Does this increase with $i$, stay constant, or decrease?

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A. $a < b^d$

B. $a = b^d$
Why are there 3 cases?

Combining work at level $i$ is in $\Theta \left( n^d \left( \frac{a}{b^d} \right)^i \right)$

Does this increase with $i$, stay constant, or decrease?

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A. $a < b^d$

B. $a = b^d$

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Why are there 3 cases?

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Does this increase with $i$, stay constant, or decrease?

- It depends on whether $\frac{a}{b^d}$ is less than, equal to, or greater than 1

A. $a < b^d$  
B. $a = b^d$  
C. $a > b^d$

1. increases  
2. decreases  
3. constant
Master Theorem
Master Theorem

Total work at level $i$ is:

$$\Theta \left( n^d \left( \frac{a}{b^d} \right)^i \right)$$
Master Theorem

✧ Total work at level $i$ is:

$$\Theta \left( n^d \left( \frac{a}{b^d} \right)^i \right)$$

✧ But there are $\log_b n + 1$ levels
Master Theorem

✧ Total work at level $i$ is:

\[\Theta \left( n^d \left( \frac{a}{b^d} \right)^i \right)\]

✧ But there are $\log_b n + 1$ levels

✧ So, total work is

\[\Theta \left( \sum_{i=0}^{\log_b n} n^d \left( \frac{a}{b^d} \right)^i \right)\]
Master Theorem: Case $a < b^d$
Master Theorem: Case \( a < b^d \)
🐤 Total work is

\[
\Theta \left( \sum_{i=0}^{\log_b n} n^d \left( \frac{a}{b^d} \right)^i \right)
\]
Master Theorem: Case $a < b^d$

Total work is

$$\Theta \left( \log_b n \sum_{i=0}^{\log_b n} n^d \left( \frac{a}{b^d} \right)^i \right) = \Theta \left( n^d \sum_{i=0}^{\log_b n} \left( \frac{a}{b^d} \right)^i \right)$$
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✧ Total work is

$$\Theta \left( \sum_{i=0}^{\log_b n} n^d \left( \frac{a}{b^d} \right)^i \right) = \Theta \left( n^d \sum_{i=0}^{\log_b n} \left( \frac{a}{b^d} \right)^i \right)$$

✧ This is $n^d$ times the sum of a geometric series with ratio $r < 1$. (See Summation formula 5, p 476)
Master Theorem: Case $a < b^d$

✧ Total work is

$$\Theta \left( \sum_{i=0}^{\log_b n} n^d \left( \frac{a}{b^d} \right)^i \right) = \Theta \left( n^d \sum_{i=0}^{\log_b n} \left( \frac{a}{b^d} \right)^i \right)$$

✧ This is $n^d$ times the sum of a geometric series with ratio $r < 1$. (See Summation formula 5, p 476)

$$\sum_{i=0}^{n-1} r^i = \frac{1 - r^n}{1 - r} \approx \frac{1}{1 - r} \text{ as } n \to \infty$$
MergeSort (Example) — 1

Example from Ján Maňuch, Simon Fraser University

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MergeSort (Example) — 3

Example from Ján Maňuch, Simon Fraser University

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MergeSort (Example) — 4

Example from Ján Maňuch, Simon Fraser University
Thursday, 29 October 2015
MergeSort (Example) — 6

Example from Ján Maňuch, Simon Fraser University
Thursday, 29 October 2015
MergeSort (Example) — 8
MergeSort (Example) — 9

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MergeSort (Example) — 10
MergeSort (Example) — 11

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MergeSort (Example) — 12
MergeSort (Example) — 13

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MergeSort (Example) — 14
MergeSort (Example) — 15

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MergeSort (Example) — 16
MergeSort (Example) — 17
MergeSort (Example) — 19
MergeSort (Example) — 21
Merge Phase (Example)
Merge Phase (Example)
Merge Phase (Example)

Example from Ján Maňuch, Simon Fraser University

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Merge Phase (Example)
Merge Phase (Example)
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Merge Phase (Example)

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Merge Phase (Example)

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What’s wrong with this Algorithm?

**ALGORITHM Mergesort**(A[0..n - 1])

//Sorts array A[0..n - 1] by recursive mergesort
//Input: An array A[0..n - 1] of orderable elements
//Output: Array A[0..n - 1] sorted in nondecreasing order

if n > 1
  copy A[0..[n/2] - 1] to B[0..[n/2] - 1]
  copy A[[n/2]..n - 1] to C[0..[n/2] - 1]
  Mergesort(B[0..[n/2] - 1])
  Mergesort(C[0..[n/2] - 1])
  Merge(B, C, A)

**ALGORITHM Merge**(B[0..p - 1], C[0..q - 1], A[0..p + q - 1])

//Merges two sorted arrays into one sorted array
//Input: Arrays B[0..p - 1] and C[0..q - 1] both sorted
//Output: Sorted array A[0..p + q - 1] of the elements of B and C
i ← 0; j ← 0; k ← 0

while i < p and j < q do
  if B[i] ≤ C[j]
    A[k] ← B[i]; i ← i + 1
  else
    A[k] ← C[j]; j ← j + 1
    k ← k + 1

if i = p
  copy C[j..q - 1] to A[k..p + q - 1]
else
  copy B[i..p - 1] to A[k..p + q - 1]
Running time estimates:
- Home pc executes $10^8$ comparisons/second.
- Supercomputer executes $10^{12}$ comparisons/second.

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**Lesson 1.** Good algorithms are better than supercomputers.

Problem

✨ Is Mergesort a stable sorting algorithm?

- A yes
- B no
Problem

✧ Design an algorithm to rearrange elements of a given array of \( n \) real numbers so that all its negative elements precede all its non-negative elements. Your algorithm should be both time-efficient and space-efficient.

\[ \begin{array}{cccccccc}
  l & & & & & & & r \\
  -4 & 17 & 6 & -8 & 3 & -5 & 0 & 25 & -19 \\
\end{array} \]
Problem

Design an algorithm to rearrange elements of a given array of $n$ real numbers so that all its negative elements precede all its non-negative elements. Your algorithm should be both time-efficient and space-efficient.

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$< 0$  $\geq 0$
Partition based on Lomuto’s idea

Examine $A[i]$: 

$\geq 0$: just increment $i$

$< 0$: make room in the segment $A[i..s]$ by increment $s$; swap $A[i]$ and $A[s]$; increment $i$
Partition based on Lomuto’s idea

实习生 Examine $A[i]$: 

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✧ Examine \( A[i] \):

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  - increment \( i \)
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\[
\begin{array}{cccc}
  l & i & j & r \\
  \text{< 0} & -5 & \text{unknown} & 6 & \geq 0 \\
\end{array}
\]

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\begin{array}{cccc}
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Partition based on Hoare’s idea

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Quicksort

✧ Select a pivot (partitioning element) – a random element
✧ Rearrange the list so that all the elements in the first $j$ positions are smaller than or equal to the pivot and all the elements in the last $n-i$ positions are larger than or equal to the pivot.

✧ Exchange the pivot with the element in its partition closest to the center — the pivot is now in its final position
✧ Sort the two subarrays recursively
method partition(A, lo, hi) {
    def pivotIndex = randomBetween(lo) and (hi)
    def pivot = A[pivotIndex]
    var i := lo - 1
    var j := hi + 1
    while {
        do { i := i + 1 } while { (i ≤ hi).andAlso {A[i] ≤ pivot} }
        do { j := j - 1 } while { (j ≥ lo).andAlso {A[j] ≥ pivot} }
        i < j
    } do { exchange(A, i, j) }
    if (i < pivotIndex) then {
        exchange(A, i, pivotIndex) ; i := i + 1
    } elseif {j > pivotIndex} then {
        exchange(A, pivotIndex, j) ; j := j - 1
    }
    list.with(i, j)
}
Which Algorithmic Paradigm?

✨ **Partition:**

A. Brute force
B. Decrease by a constant
C. Decrease by a Variable Amount
D. Divide and Conquer
Which Algorithmic Paradigm?

 sokar Median Finding using Partition:

A. Brute force
B. Decrease by a constant
C. Decrease by a Variable Amount
D. Divide and Conquer
Which Algorithmic Paradigm?

-topic: Sorting using Partition (Quicksort):

A. Brute force
B. Decrease by a constant
C. Decrease by a Variable Amount
D. Divide and Conquer
Analysis of Quicksort

✧ Best case: split in the middle — $\Theta(n \log n)$
✧ Worst case: choose 1st element from sorted array! — $\Theta(n^2)$
✧ Average case: random arrays — $\Theta(n \log n)$

✧ Improvements:
  ▪ better pivot selection: median-of-three partitioning
  ▪ separate partition for keys equal to pivot
  ▪ switch to insertion sort on small sub-problems
  ▪ elimination of recursion
  ▪ These combine to give 20–25% improvement

✧ Considered the method of choice for internal sorting of large files ($n \geq 10000$)
Dual-pivot Quicksort

Vladimir Yaroslavskiy  |  11 Sep 12:35 2009

Replacement of Quicksort in java.util.Arrays with new Dual-Pivot Quicksort

Hello All,

I'd like to share with you new Dual-Pivot Quicksort which is faster than the known implementations (theoretically and experimental). I'd like to propose to replace the JDK's Quicksort implementation by new one.

... 

It is proved that for the Dual-Pivot Quicksort the average number of comparisons is $2n \ln(n)$, the average number of swaps is $0.8n \ln(n)$, whereas classical Quicksort algorithm has $2n \ln(n)$ and $1n \ln(n)$ respectively.

- Read for yourself: http://permalink.gmane.org/gmane.comp.java.openjdk.core-libs.devel/2628
Vladimir, Josh,

I *finally* feel like I understand what is going on. Now that I (think that) I see it, it seems straightforward and obvious.

Tony Hoare developed Quicksort in the early 1960s. I was very proud to make minor contributions to a particularly clean (binary) quicksort in the mid 1980s, to a relatively straightforward, industrial-strength Quicksort with McIlroy in the early 1990s, and then to algorithms and data structures for strings with Sedgewick in the mid 1990s.

I think that Vladimir's contributions to Quicksort go way beyond anything that I've ever done, and rank up there with Hoare's original design and Sedgewick's analysis. I feel so privileged to play a very, very minor role in helping Vladimir with the most excellent work!
Problem:

Is Quicksort stable?

A. Yes
B. No
Problem:

Should we stop scanning when we find an element = pivot?

Levitin: repeat $i \leftarrow i + 1$ until $A[i] \geq pivot$

Hoare: if $pivot < A[I]$ then goto down
Problem:

Should we stop scanning when we find an element = pivot?

Levitin: repeat $i \leftarrow i + 1$ until $A[i] \geq pivot$

Hoare: if $pivot < A[I]$ then goto down

Levitin claims: “Why is it worth stopping the scans after encountering an element equal to the pivot? Because doing this tends to yield more-even splits for arrays with a lot of duplicates, which makes the algorithm run faster.”
QUICKSORT IS OPTIMAL

Robert Sedgewick
Jon Bentley

  

Examines an interesting detail: How to handle keys equal to the pivot
Partitioning with equal keys

How to handle keys equal to the partitioning element?

**METHOD A:** Put equal keys all on one side?

1. 4 4 4 4 4 4 4 4 4 4 4 4
2. 4 4 4 4 4 4 4 4 4 4 4 4

NO: quadratic for n=1 (all keys equal)

**METHOD B:** Scan over equal keys? (linear for n=1)

1. 1 4 1 1 4 4 4 1 4 1 1 4 4
2. 1 1 1 4 4 4 1 4 1 4 4 4

NO: quadratic for n=2

**METHOD C:** Stop both pointers on equal keys?

1. 4 9 4 4 1 4 4 4 9 4 4 1 4
2. 1 4 4 4 1 4 4 4 9 4 9 4 4

YES: NlgN guarantee for small n, no overhead if no equal keys
Partitioning with equal keys

How to handle keys equal to the partitioning element?

**METHOD C**: Stop both pointers on equal keys?

| 4 9 4 4 1 4 4 4 9 4 4 1 4 |
| 1 4 4 4 1 4 4 4 9 4 4 4 |

YES: $N \lg N$ guarantee for small $n$, no overhead if no equal keys

**METHOD D** *(3-way partitioning)*: Put all equal keys into position?

| 4 9 4 4 1 4 4 4 9 4 4 1 4 |
| 1 1 4 4 4 4 4 4 4 4 4 9 |

yes, BUT: early implementations cumbersome and/or expensive
**Quicksort common wisdom (last millennium)**

1. **Method of choice in practice**
   - tiny inner loop, with locality of reference
   - $N\log N$ worst-case “guarantee” (randomized)
   - but use a radix sort for small number of key values

2. **Equal keys can be handled (with care)**
   - $N\log N$ worst-case guarantee, using proper implementation

3. **Three-way partitioning adds too much overhead**
   - “Dutch National Flag” problem

4. **Average case analysis with equal keys is intractable**
   - keys equal to partitioning element end up in both subfiles
Changes in Quicksort common wisdom

1. Equal keys abound in practice.
   - never can anticipate how clients will use library
   - linear time required for huge files with few key values

2. 3-way partitioning is the method of choice.
   - greatly expands applicability, with little overhead
   - easy to adapt to multikey sort
   - no need for separate radix sort

3. Average case analysis already done!
   - Burge, 1975
   - Sedgewick, 1978
   - Allen, Munro, Melhorn, 1978
Bentley-McIlroy 3-way partitioning

Partitioning invariant

<table>
<thead>
<tr>
<th>equal</th>
<th>less</th>
<th>greater</th>
<th>equal</th>
</tr>
</thead>
</table>

- move from left to find an element that is not less
- move from right to find an element that is not greater
- stop if pointers have crossed
- exchange
- if left element equal, exchange to left end
- if right element equal, exchange to right end

Swap equals to center after partition

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**KEY FEATURES**

- always uses N-1 (three-way) compares
- no extra overhead if no equal keys
- only one “extra” exchange per equal key
void quicksort(Item a[], int l, int r)
{
    int i = l-1, j = r, p = l-1, q = r; Item v = a[r];
    if (r <= l) return;
    for (;;)
    {
        while (a[++i] < v); 
        while (v < a[--j]) if (j == l) break;
        if (i >= j) break;
        exch(a[i], a[j]);
        if (a[i] == v) { p++; exch(a[p], a[i]); }
        if (v == a[j]) { q--; exch(a[j], a[q]); }
    }
    exch(a[i], a[r]); j = i-1; i = i+1;
    for (k = l; k < p; k++, j--) exch(a[k], a[j]);
    for (k = r-1; k > q; k--, i++) exch(a[i], a[k]);
    quicksort(a, l, j);
    quicksort(a, i, r);
}
Running time estimates:
- Home pc executes $10^8$ comparisons/second.
- Supercomputer executes $10^{12}$ comparisons/second.

<table>
<thead>
<tr>
<th>Computer</th>
<th>Thousand</th>
<th>Million</th>
<th>Billion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Home</td>
<td>Instant</td>
<td>2.8 hours</td>
<td>317 years</td>
</tr>
<tr>
<td>Super</td>
<td>Instant</td>
<td>1 second</td>
<td>1.6 weeks</td>
</tr>
</tbody>
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<td>0.3 sec</td>
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**Lesson 1.** Good algorithms are better than supercomputers.  
**Lesson 2.** Great algorithms are better than good ones.
The Problem of the Dutch National Flag

Levitin §5.2, Q9

The Dutch flag problem is to rearrange an array of characters $R$, $W$, and $B$ (red, white, and blue are the colors of the Dutch national flag) so that all the $R$’s come first, the $W$’s come next, and the $B$’s come last. Design a linear in-place algorithm for this problem.
Quickhull

Divide-and-conquer algorithm for Convex Hull of a set of points P

- Find two points $A, B \in P$ that are both on the convex hull
- divide the set $P$ into two subsets, $P_l$ (left) and $P_r$ (right) of chord $AB$.
  - Find point $C$ in $P_l$ furthest from $AB$.
  - discard the points inside $\triangle ABC$
  - recurse with chords $AC$ and $CB$
  ...
- repeat with chord $AB$ and $P_r$
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Thursday, 29 October 2015
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Animation:

◊ **Quickhull** at Princeton