Some Sample Problems concerning Tensor Products that a student enrolling in this course should be able to solve.

First some notation and facts about Tensor Products:

**Notation:**
Let $\mathbb{C}^n$ denote the $n$-dimensional vector space of column vectors of $n$ complex numbers. We will use the terms, *point* and *vector*, somewhat interchangeably.

We denote a column vector $x \in \mathbb{C}^n$ as follows:

$$\vec{x} = \begin{bmatrix} x_0 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

**Defn:** The tensor product of two vectors $\vec{x} \in \mathbb{C}^n$ and $\vec{y} \in \mathbb{C}^k$ is the vector

$$\vec{x} \otimes \vec{y} = \begin{bmatrix} x_0 \vec{y} \\ x_1 \vec{y} \\ \vdots \\ x_{n-1} \vec{y} \end{bmatrix} \in \mathbb{C}^{nk}$$

The tensor product is **bilinear**:

i.e.

$$\vec{x} \otimes (\vec{y} + \vec{z}) = \vec{x} \otimes \vec{y} + \vec{x} \otimes \vec{z} \quad \text{where} \quad \vec{x}, \vec{y}, \vec{z} \in \mathbb{C}^k$$

And

$$(\vec{x} + \vec{y}) \otimes \vec{z} = \vec{x} \otimes \vec{z} + \vec{y} \otimes \vec{z} \quad \text{where} \quad \vec{x}, \vec{y} \in \mathbb{C}^n, \vec{z} \in \mathbb{C}^k$$

**Defn:** Let $X$ denote a $M \times R$ matrix and let $Y$ denote a $L \times S$ matrix. Then the tensor product of $X$ and $Y$ is:

$$X \otimes Y = \begin{bmatrix} x_{0,0}Y & x_{0,1}Y & \cdots & x_{0,R-1}Y \\ x_{1,0}Y & x_{1,1}Y & \cdots & x_{1,R-1}Y \\ \vdots & \vdots & \ddots & \vdots \\ x_{M-1,0}Y & x_{M-1,1}Y & \cdots & x_{M-1,R-1}Y \end{bmatrix}$$

is the $N \times T$ matrix where $N = LM$ and $T = RS$
The tensor product acts on the vector space, $\mathbb{C}^T$ and the \textit{action} on tensor products of vectors $\tilde{x} \otimes \tilde{y}$ where $\tilde{x} \in \mathbb{C}^R$ and $\tilde{y} \in \mathbb{C}^S$ is defined by
\[
(X \otimes Y)(\tilde{x} \otimes \tilde{y}) = (X\tilde{x}) \otimes (Y\tilde{y})
\]

**Multiplication Theorem for Tensor Products:**
If $X$ is a $M \times R$ matrix, $Y$ is a $L \times S$ matrix, $A$ is a $R \times U$ matrix, and $B$ is a $S \times V$ matrix, then $\ (X \otimes Y)(A \otimes B) = (XA) \otimes (YB)$

**Associative Law:** $(A \otimes B) \otimes C = A \otimes (B \otimes C)$

**Distributive Laws:**
\[
(A + B) \otimes C = (A \otimes C) + (B \otimes C)
\]
\[
A \otimes (B + C) = (A \otimes B) + (A \otimes C)
\]

**Note:** The tensor product is \textbf{NOT} commutative.

**Defn:** The matrix direct sum is defined to be $A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$

**Left Distributive Law:** $(A \oplus B) \otimes C = (A \otimes C) \oplus (B \otimes C)$

**Problems:**

1. Show that the tensor product of vectors is bilinear.

2. Compute $(A \otimes B)(\vec{a} \otimes \vec{b})$ for
   \[
   A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}
   \]
   \[
   \vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}
   \]

3. For vectors all vectors $\vec{a}$ and $\vec{b}$ running over all vectors of sizes 2 and 3 respectively, show that the tensor products $\vec{a} \otimes \vec{b}$ span $\mathbb{C}^6$.
4. The canonical basis of \( \mathbb{C}_L \) is given by the vectors:

\[
\begin{align*}
    e^{(L)}_0 &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
    e^{(L)}_1 &= \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
    e^{(L)}_2 &= \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \\
    \cdots & \\
    e^{(L)}_{L-1} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}
\end{align*}
\]

Show that the \( LM \) tensor products

\[
e^{(L)}_r \otimes e^{(M)}_s, \text{ for } 0 \leq r < L, \ 0 \leq s < M \text{ describe the canonical basis of } \mathbb{C}^{LM}
\]