Some Sample Problems concerning Complex Numbers that a student enrolling in this course should be able to solve.

First some facts about Complex Numbers:

**Defn:** A complex number is a number of the form \( z = x + iy \) where \( x, y \in \mathbb{R} \) and \( i^2 = -1 \). The real part of \( z \) is \( \text{Re}(z) = x \) and the imaginary part of \( z \) is \( \text{Im}(z) = y \).

**Defn:** The complex conjugate of a complex number \( z = x + iy \) is \( \bar{z} = x - iy \).

Note:

- \( z_1 \pm z_2 = \bar{z}_1 \pm \bar{z}_2 \)
- \( z_1 \bar{z}_2 = \overline{z_1 \cdot z_2} \)
- \( \frac{z_1}{z_2} = \frac{\bar{z}_1}{\bar{z}_2} \) \( (z_2 \neq 0) \)
- \( \arg z \) is the angle between the positive real axis and the line from the origin to point \( z \) (also called the phase)
- \( |z| = \sqrt{x^2 + y^2} \) (called the length or magnitude)
- \( |z|^2 = z\bar{z} \)
- \( |z_1 + z_2| \leq |z_1| + |z_2| \) (Triangle inequality)
- \( |z_1| - |z_2| \leq |z_1 + z_2| \)
- \( |z_1z_2| = |z_1| \cdot |z_2| \)
- \( |z_1 \cdots z_n| = |z_1| \cdot |z_2| \cdots |z_n| \)
- Every nonzero complex number has two square roots
- The complex field \( \mathbb{C} \) has no zero divisors
- For \( z \in \mathbb{C} \) \( z \neq 0 \)
  - \( z \) is real \( \iff \arg z = n\pi \) \( (n \in \mathbb{Z}) \)
  - \( z \) is pure imaginary \( \iff \arg z = \pm \frac{\pi}{2} + n\pi \) \( (n \in \mathbb{Z}) \)
  - \( |1| = 1 \quad \arg(-1) = (2n + 1)\pi \) \( (n \in \mathbb{Z}) \)
  - \( |i| = 1 \quad \arg(i) = \frac{\pi}{2} + 2n\pi \) \( (n \in \mathbb{Z}) \)
  - \( |1 - i| = \sqrt{1^2 + (-1)^2} = \sqrt{2} \quad \arg(1 - i) = -\frac{\pi}{4} + 2n\pi \) \( (n \in \mathbb{Z}) \)
  - \( \arg(z_1z_2) = \arg z_1 + \arg z_2 \)
  - \( \arg(z_1 \cdots z_2) = \arg z_1 + \arg z_2 + \cdots + \arg z_n \)
**Defn:** The polar form of a complex number is $z = r \cdot e^{i\theta}$ where $r \in \mathbb{R}$ and $\theta \in [0,2\pi)$.

**Note:**
- $e^{i\theta} = \cos \theta + i \sin \theta$
- $e^{-i\theta} = \cos \theta - i \sin \theta$
- $\cos^2 \theta + \sin^2 \theta = 1$
- $\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$
- $\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

**Problems:**

1. Perform the following operations and reduce each of the following numbers to the form $x + yi$ ($x, y \in \mathbb{R}$):
   a. $(1 - i)(2 - i)(3 - i)$
   b. $(\sqrt{3} + i)^6$
   c. $\frac{4 + 3i}{3 - 4i}$
   d. $\frac{5 - z}{5 + z}$, where $z = 4 + 3i$

2. Find the real numbers $x, y, u, v$ satisfying
   $z = x + i$     $w = 3 + iy$     $z + w = u - i$     $zw = 14 + iv$

3. Show that any complex number $z$ with $|z| = 1$, but $z \neq -1$ can be expressed as $z = \frac{1 + it}{1 - it}$ with an appropriate choice of the real parameter $t$.

4. Let $z = a + ib$ ($a, b \in \mathbb{R}$). Find conditions on $a$ and $b$ such that
   a) $z^4$ is real
   b) $z^4$ is purely imaginary

5. Find the absolute values of
   a. $3 + 2i$
   b. $-1 + i\sqrt{3}$
   c. $-i(1 + i)(2 - 3i)(4 + 3i)$
   d. $\frac{(3-i)(-1+2i)}{2-3i}$
6. Solve:
   a. $z^3 - i = 0$
   b. $z^4 + 1 = 0$
   c. $z^5 + 32 = 0$
   d. $z^6 - 1 = 0$

7. Given $z \in \mathbb{C}$, show that there exist $\alpha, \beta \in \mathbb{C}$ with $|\alpha| = |\beta| = 1$ such that $z = \alpha + \beta$ if and only if $|z| \leq 2$.

8. Given $z \in \mathbb{C}$, show that there exist $\alpha, \beta, \gamma \in \mathbb{C}$ with $|\alpha| = |\beta| = |\gamma| = 1$ such that $z = \alpha + \beta + \gamma$ if and only if $|z| \leq 3$.

9. If $z_1 + z_2 + z_3 = 0$ and $|z_1| = |z_2| = |z_3| = 1$, show that $z_1, z_2, z_3$ are the vertices of an equilateral triangle inscribed in the unit circle.

10. If $z_1 + z_2 + z_3 + z_4 = 0$ and $|z_1| = |z_2| = |z_3| = |z_4| = 1$, what can be said about the quadrangle with vertices $z_1, z_2, z_3, z_4$.

11. For any complex number $z \neq 0$, show that $z, -z, \frac{1}{z}, -\frac{1}{z}, 0$ are collinear.

12. Find the polar representations for
   a. $1 + i$
   b. $4 - 3i$
   c. $1 + \omega$
   d. $\frac{1}{\omega}$
      where $\omega^2 + \omega + 1 = 0$