Some Sample Problems from Linear Algebra that a student enrolling in this course should be able to solve.

1. Consider two distinct non-trivial proper subspaces, $M$ and $N$, of $\mathbb{R}^2$. Is it possible that $M + N = \mathbb{R}^2$, the trivial subspace (i.e. the subspace spanned by the 0-vector)? Is it possible that $M \cap N = \mathbb{R}^2$?

2. A polynomial $p(x)$ is said to be even if $p(x) = p(-x)$ and it is said to be odd if $p(x) = -p(x)$. What do the odd polynomials in $\mathbb{P}_5$ (the vector space of polynomials of degree not more than 5 over the reals)? Can a polynomial be both even and odd?

3. Show how every polynomial in $\mathbb{P}$ can be written as a sum of an even polynomial and an odd polynomial.

4. If $L, M,$ and $N$ are subspaces of a vector space $\mathbb{V}$, is it true that $L \cap (M + (L \cap N)) = (L \cap M) + (L \cap N)$?

5. A subset $E$ of a vector space $\mathbb{V}$ with the property that the only subspace of $\mathbb{V}$ that includes $E$ is $\mathbb{V}$ is called a total set. Since the only subspace of $\mathbb{V}$ that includes $E$ is $\mathbb{V}$, then the intersection of all subspaces that include $E$ is just $\mathbb{V}$ and hence $\bigvee E = \mathbb{V}$ (i.e. the span of $E$ is $\mathbb{V}$). List a total set for $\mathbb{P}_2$. List a total set for $\mathbb{P}$. List a total set for $\mathbb{O}$.

6. If $E$ is a total set for some vector space $\mathbb{V}$, and if $M$ is a subspace of $\mathbb{V}$, does it follow that some subset of $E$ is total for $M$?

7. A finite set of vectors is called linearly dependent if some non-trivial linear combination of them vanishes. Is the set consisting of the 0-vector linearly dependent? Is the set consisting of any non-trivial vector $x$ and the 0-vector, say $\{0, x\}$ linearly dependent? If $x$ and $y$ are arbitrary vectors, is the set $\{x, y, x + y\}$ linearly dependent? If $x, y, u,$ and $v$ are arbitrary vectors, is the set $\{x, y, x + y, u, v\}$ linearly dependent?

8. An (linearly) independent total set in a vector space is called a basis. Does every finite dimensional vector space have a finite basis?
9. If $M$ is a subspace of vector space $\mathbb{V}$, a **complement** of $M$ is defined as a subspace $N$ of $\mathbb{V}$ such that $M \cap N = \{0\}$ and $M + N = \mathbb{V}$. For subspaces this is equivalent to saying that $\mathbb{V}(M \cup N) = M + N$. Thus, a subspace *can* have more than one complement. It is possible for several subspaces to have a complement in common (a **simultaneous** complement). Under what conditions does a finite collection of subspaces of a finite-dimensional vector space have a simultaneous complement?

10. For which real numbers $x$ is it true that the vectors $x$ and 1 are linearly independent in the vector space $\mathbb{R}$ (the real numbers) over the field $\mathbb{Q}$ (the field of rational numbers)?

11. Under what conditions on the scalar $\alpha$ are the vectors $(1 + \alpha, 1 - \alpha)$ and $(1 - \alpha, 1 + \alpha)$ in $\mathbb{R}^2$ (over the field $\mathbb{Q}$ of rational numbers) linearly independent?

12. Is there a subset of $\mathbb{R}^3$ that is independent over $\mathbb{Q}$ but dependent over $\mathbb{R}$?

13. Under what conditions on the scalars $\alpha$ and $\beta$ are the vectors $(1, \alpha)$ and $(1, \beta)$ in $\mathbb{C}^2$ linearly independent?

14. Is there a set of three linearly independent vectors in $\mathbb{C}^2$?

15. Do there exist two bases in $\mathbb{C}^4$ such that the only vectors common to them are $(0,0,1,1)$ and $(1,1,0,0)$?

16. Do there exist two bases in $\mathbb{C}^4$ that have no vectors in common so that one of them contains the vectors $(1,0,0,0)$ and $(1,1,0,0)$ and the other contains the vectors $(1,1,1,0)$ and $(1,1,1,1)$?

17. Under what conditions on the scalar $x$ do the vectors $(0,1,x)$, $(x,0,1)$ and $(x,1,1+x)$ form a basis of $\mathbb{C}^3$?

18. Under what conditions on the scalar $x$ do the vectors $(1,1,1)$ and $(1,x,x^2)$ form a basis of $\mathbb{C}^3$?

19. A **maximal linearly independent subset** of $X$ is a subset $Y$ of $X$ that becomes linearly dependent every time that a vector of $X$ not already in $Y$ is adjoined to $Y$. If $X$ is the set consisting of the six vectors in $\mathbb{R}^4$ 
\{(1,1,0,0), (1,0,1,0), (1,0,0,1), (0,1,1,0), (0,1,0,1), (0,0,1,1)\},
Do there exist two different maximally independent subsets of $X$?

20. Every complex vector space $\mathbb{V}$ has an associated real vector space $\mathbb{V}^{real}$; The space $\mathbb{V}^{real}$ is obtained from $\mathbb{V}$ by refusing to multiply vectors in $\mathbb{V}$ by anything other than real scalars. If the dimension of the complex vector space $\mathbb{V}$ is $n$, what is the dimension of the real vector space $\mathbb{V}^{real}$?
21. Can a proper subspace of a finite-dimensional vector space have the same dimension as the whole space?

22. Can every finite independent set in a finite-dimensional vector space be extended to a basis?

23. Is every subspace of a finite-dimensional vector space finite-dimensional?

24. Is every minimal total set independent? Is every independent set a total minimal set?

25. Does every total set have a minimal total subset?

26. A set is called **infinitely total** if it is total and remains total when any finite number of its elements is discarded. Does every infinitely total set $E$ have an infinite subset $F$ such that the relative complement $E - F$ is total?

27. If $\mathbb{F}$ is a finite field with $q$ elements, how many bases are there in $\mathbb{F}^n$?