Some Sample Problems from Linear Algebra that a student enrolling in this course should be able to solve.

1. (a) Consider the complex vector space $\mathbb{C}^3$ and the subsets $M$ of $\mathbb{C}^3$ consisting of those vectors $(\alpha, \beta, \gamma)$ for which:
   
   (1) $\alpha = 0$
   (2) $\beta = 0$
   (3) $\alpha + \beta = 1$
   (4) $\alpha + \beta = 0$
   (5) $\alpha + \beta \geq 0$
   (6) $\alpha$ is real

   In which of these spaces is $M$ a subspace of $\mathbb{C}^3$?

2. Consider the complex vector space, $\mathbb{P}$, of all polynomials with complex coefficients and the subsets $M$ of all those vectors (polynomials) $p$ for which:
   
   (1) $p$ has degree 3
   (2) $2p(0) = p(1)$
   (3) $p(t) \geq 0$, whenever $0 \leq t \leq 1$
   (4) $p(t) = p(1-t)$ for all $t$

   In which of these cases is $M$ a subspace of $\mathbb{P}$?

3. Under what conditions is the set-theoretic intersection of two subspaces a subspace?

4. Is the intersection of infinitely many subspaces a subspace?

5. Under what conditions is the set-theoretic union of two subspaces a subspace?

6. Is the union of more than two subspaces a subspace?

7. Can two disjoint subsets of $\mathbb{R}^2$, each containing two vectors have the same span?

8. What is the span in $\mathbb{R}^3$ of $\{(1,1,1), (0,1,1), (0,0,1)\}$?

9. State three properties of the span of a set of vectors.
10. Let $V(x, y)$ denote the span of the set of vectors $\{x, y\}$. Given three vectors $x, y, z$ and we know $x \in V(y, z)$, does this mean that $V(x, z) = V(y, z)$? Explain your answer.

11. Is there any finite set of vectors that span $\mathbb{P}$?

12. Explain why the set of subspaces of a vector space do not form a group with the operation of addition?

13. Given the vector space, $\mathbb{R}^2$. Let $M$ be the subspace of $\mathbb{R}^2$ consisting of all vectors of the form $(\alpha, 2\alpha)$ and let $N$ be the subspace consisting of all vectors of the form $(\alpha, 3\alpha)$. Which vectors can be represented in the form $(\alpha + \beta, 2\alpha + 3\beta)$ as $\alpha$ and $\beta$ are allowed to range over all real numbers? Let $M + N$ denote this addition.

14. Which of the following are true?
   (1) $M \subset M + N$
   (2) $N \subset M + N$
   (3) $M \cup N \subset M + N$
   (4) $V(M \cup N) \subset M + N$
   (5) $M + N \subset V(M \cup N)$
   (6) $V(M \cup N) = M + N$

Given the following definition of a vector space: A vector space over a field $\mathbb{F}$ (of elements called scalars) is an additive commutative group $\mathbb{V}$ (of elements called vectors) together with an operation that assigns to each scalar $\alpha$ and vector $\mathbf{x}$ a product (scalar multiplication) $\alpha \mathbf{x}$ that is again a vector, satisfying the following laws:

1) $(\alpha + \beta)x = \alpha x + \beta x$
2) $\alpha(x + y) = \alpha x + \alpha y$
3) $(\alpha \beta)x = \alpha (\beta x)$
4) $1x = x$

15. Which of the following are vector spaces?
   (1) Let $\mathbb{F}$ be $\mathbb{C}$, and let $\mathbb{V}$ also be $\mathbb{C}$ (the set of complex numbers). Define addition in $\mathbb{C}$ in the usual way, and let scalar multiplication (denoted by *) be defined as follows: $\alpha * x = \alpha^2 \cdot x$

   (2) Let $\mathbb{F}$ be a field, let $\mathbb{V}$ be $\mathbb{F}^2$ (i.e. $\mathbb{F} \times \mathbb{F}$ the set of all ordered pairs of elements from $\mathbb{F}$). Let addition in $\mathbb{V}$ be the usual one (coordinate-wise) and define a new scalar multiplication by $\alpha * (\beta, \gamma) = (\alpha \beta, 0)$ for all $\alpha, \beta, \gamma \in \mathbb{F}$.

   (3) Let $\mathbb{F}$ be the field of four elements. Let $\mathbb{V}$ be $\mathbb{F}^2$ with the usual addition and define scalar multiplication by $\alpha * (\beta, \gamma) = (\alpha \beta, \alpha \gamma)$ if $\gamma \neq 0$ and $\alpha * (\beta, 0) = (\alpha^2 \beta, 0)$.
Note: The field with elements is $\mathbb{P}[x]/x^2 + x + 1$, i.e. the field of all polynomials in indeterminate $x$ with coefficients in $\mathbb{Z}_2 = \{0, 1\}$ modulo the polynomial $x^2 + x + 1$. $\mathbb{F}$ has the four elements (polynomials) $\{0, 1, x, x + 1\}$. Addition and multiplication tables are included below.

<table>
<thead>
<tr>
<th>+</th>
<th>0</th>
<th>1</th>
<th>$x$</th>
<th>$x + 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$x$</td>
<td>$x + 1$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$x + 1$</td>
<td>$x$</td>
</tr>
<tr>
<td>$x$</td>
<td>$x$</td>
<td>$x + 1$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$x + 1$</td>
<td>$x + 1$</td>
<td>$x$</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\times$</th>
<th>0</th>
<th>1</th>
<th>$x$</th>
<th>$x + 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$x$</td>
<td>$x + 1$</td>
</tr>
<tr>
<td>$x$</td>
<td>0</td>
<td>$x$</td>
<td>$x + 1$</td>
<td>1</td>
</tr>
<tr>
<td>$x + 1$</td>
<td>0</td>
<td>$x + 1$</td>
<td>1</td>
<td>$x$</td>
</tr>
</tbody>
</table>

(4) Let $\mathbb{F}$ be $\mathbb{R}$ and let $\mathbb{V}$ be the set $\mathbb{R}_+$ of all positive real numbers. Define the “sum” denoted by $\alpha \boxplus \beta$ of any two positive real numbers $\alpha$ and $\beta$, and define the “scalar product” denoted by $\alpha \boxdot \beta$ of any positive real number $\alpha$ by an arbitrary (not necessarily positive) real number $\beta$ as follows: $\alpha \boxplus \beta = \alpha \beta$ and $\alpha \boxdot \beta = \beta^\alpha$.

(5) Let $\mathbb{F}$ be $\mathbb{C}$ and let $\mathbb{V}$ also be $\mathbb{C}$. Vector addition is defined as the ordinary addition of complex numbers, but the product of a scalar $\alpha$ in $\mathbb{C}$ and a vector $x$ in $\mathbb{C}$ is defined by forming the real part of $\alpha$ first, i.e. $\alpha \ast x = (Re \alpha)x$.

(6) Let $\mathbb{F}$ be $\mathbb{Q}$ (the field of rational numbers) and let $\mathbb{V}$ be the field $\mathbb{R}$ of real numbers. Addition in $\mathbb{V}$ is the usual addition in $\mathbb{R}$ and scalar multiplication is defined to be $\alpha \ast x = \alpha x$ for all $\alpha \in \mathbb{Q}$ and for all $x \in \mathbb{R}$.

16. Which of the following transformations are linear transformations?

(1) The vector space is $\mathbb{R}^2$. The transformation is $T(x, y) = (y, x)$
(2) The vector space is $\mathbb{R}^2$. The transformation is $T(x, y) = (x^2, y^2)$
(3) The vector space is $\mathbb{R}^2$. The transformation is $T(x, y) = (e^x, e^y)$
(4) The vector space is $\mathbb{P}$. The transformation is $T(p(x)) = \int_1^x p(t)dt$
(5) The vector space is $\mathbb{R}^2$. The transformation is $T(x, y) = (2x + 3y, 7x - 5y)$

17. Which of the following three transformations on $\mathbb{P}$ give linear transformations?
The equations are intended to hold for all arbitrary polynomials $p(x)$.

(1) $T(p(x)) = p(x^2)$
(2) $T(p(x)) = (p(x))^2$
(3) $T(p(x)) = x^2p(x)$