Dimension

• The dimension of a square matrix, $D$, or a column matrix, $v$, is the number of rows of the matrix and will be written $\dim(D)$ or $\dim(v)$ respectively.
Definitions

• The **adjoint** (or Hermitian conjugate), $D^\dagger$, of a matrix $D$, is the **transpose** of the **complex conjugate**.

• The **commutator** of two matrices, $D_1$ and $D_2$, of the same dimension is $[D_1, D_2] = D_1 D_2 - D_2 D_1$

• A **normal matrix** is a square matrix such that its commutator with its adjoint is zero – i.e. $[D, D^\dagger] = 0$

• Note: $DD^\dagger + D^\dagger D = 0$ implies $D = 0$
Important Classes of Normal Matrices

• A matrix D is **unitary** if $DD^\dagger = I$ (identity)
• A matrix D is **hermitian** if $D = D^\dagger$
• A matrix is an **orthogonal matrix** if $DD^\dagger = I$ or alternatively, $D^* = D$. In the case of real matrices, real orthogonal matrices are real unitary matrices. (Not true for complex matrices).
• A matrix D is a **rotation matrix** if D is an orthogonal matrix and $\det(D) > 0$
Definitions (cont.)

• A matrix $D$ is a **symmetric matrix** if $D^t = D$
• Note: For real matrices $D^* = D$
• A **permutation matrix** is a matrix whose entries are all either zero or one and which has exactly one non-zero entry in each row and exactly one non-zero entry in each column.
• A **pseudopermutation matrix** is a matrix obtained from a permutation matrix by replacing some of the ones by *minus* ones.
• A **diagonal matrix** is a **square** matrix for which all off-diagonal elements are zero.
Definitions (cont.)

• A **scalar matrix**, $D$, is a **diagonal** matrix with a **single** scalar value for each of the diagonal elements. The scalar is any element in the field of scalars from which the matrix’s elements are drawn.

• A **unimodular matrix** is a square matrix whose determinant is 1. $(\det(D) = 1)$

• An **integral matrix** is a matrix in which all elements are **integers**.
Definitions (cont.)

• An **idempotent matrix**, $D$, is a square matrix such that $D^2 = D$

• A **nilpotent matrix**, $D$, is a matrix such that $D^N = 0$ for some integer $N$

• A **monomial matrix** is a square matrix having only one non-zero entry in each row and only one non-zero entry in each column.

• A **real matrix**, $D$, is a matrix such that $D^* = D$

• A **skew symmetric matrix**, $D$, is a square matrix such that $D^t = -D$
Some Matrix Theory

- The **trace** (or spur) of a square $n$-dimensional matrix, $D$, is the **sum** of the **diagonal** elements of $D$. \[ \text{tr}(D) \equiv \sum_{i=1}^{n} D_{ii} \]

- If $D_1$ and $D_2$ are **square** matrices of the **same** dimension, then $\text{tr}(D_1D_2) = \text{tr}(D_2D_1)$

- If $D$ is a **unitary matrix** of dimension $d$ and $\text{tr}(D) = \pm d$ then $D = \pm I$ according to the sign of the trace.
Matrix Theory (cont.)

• Let D be a d-dimensional square matrix:
  - The **characteristic polynomial of D** is $|D - \lambda I|$ which is a $d$th degree polynomial in $\lambda$ with leading coefficient $(-1)^d$
  - The **characteristic equation** of D is $|D - \lambda I| = 0$
  - The **eigenvalues** of D are the $d$ roots of the characteristic equation of D.

• **Note**: The eigenvalues of **hermitian** matrices are **real**.

• **Note**: The eigenvalues of **unitary** matrices have **modulus one**.
Matrix Theory (cont.)

• If $D$ is a square $d$-dimensional matrix with eigenvalues $\lambda_1 \lambda_2 \ldots \lambda_d$, then $tr(D) = \sum_{i=1}^{d} \lambda_i$

• The **degeneracy of an eigenvalue** of a square matrix, $D$, is the **number** of times it occurs as a **root** of the characteristic equation of $D$.

• A **positive definite** square matrix is a **hermitian** matrix whose **eigenvalues** are all **positive**.

• **Theorem**: $\det(D) = \prod_{i=1}^{d} \lambda_i$

• **Corollary**: A skew symmetric matrix of odd dimension has determinant zero.

• **Corollary**: The dimension of a skew symmetric unitary matrix cannot be odd.
Matrix Theory (cont.)

• A square matrix is **nonsingular** (regular) if and only if it has **no eigenvalues equal to zero**.

• If **all** elements of a matrix are **positive**, then the matrix has **one nondegenerate positive eigenvalue** whose magnitude is **larger than** that of any other eigenvalue.