Chapter 4

Design of discrete-time control systems via transform methods

procedure

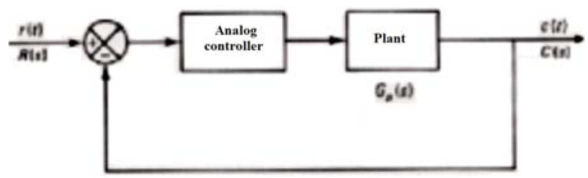


Figure 1: Continuous-time control system

The analog controller is to be replaced by a digital controller

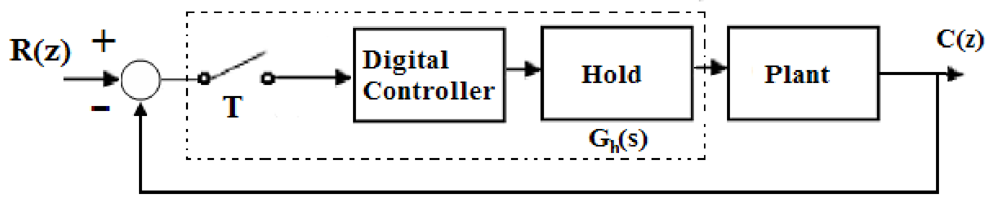


Figure 2: Digital control system

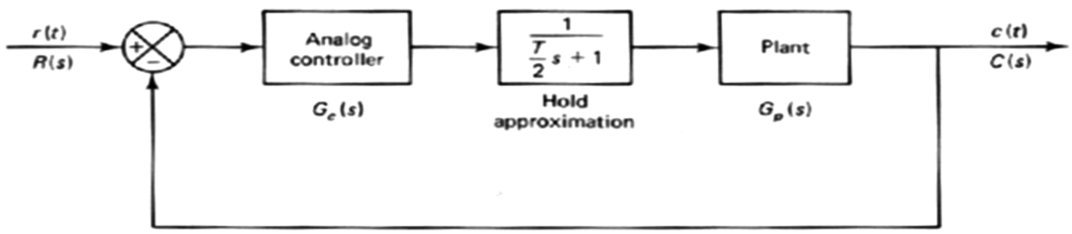


Figure 3: Continuous-time control system modified to allow for time lag of hold

ZOH: $\frac{1-e^{-Ts}}{s}$

Padé approximation $e^{-Ts} \approx \frac{1-\frac{Ts}{2}}{1+\frac{Ts}{2}}$

$$\Rightarrow \quad \frac{1-e^{-Ts}}{s} = \frac{1}{s}\left(1 - \frac{1-\frac{Ts}{2}}{1+\frac{Ts}{2}}\right) = \frac{1}{\frac{T}{2}s+1}$$

We will approximate $G_h(s)$ by

$$G_h(s) = \frac{1}{\frac{T}{2}s+1}$$

DC gain = 1
The DC gain will be determined in the final stage of the design.

The design procedure is

1. Design analog controller for the system of figure 3.
2. Discretize the controller using one of s to z transformations which will be presented next.
3. Perform computer simulation of system to check performance.
4. If performance is not adequate, use different s-to-z mapping.
5. Iterate steps (3) and (4) until adequate performance.

Transform Methods

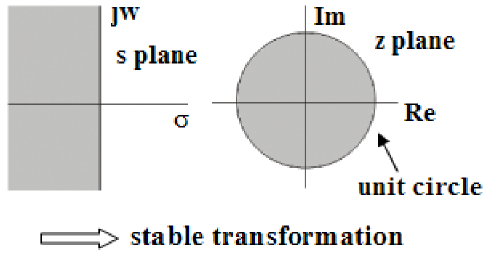
1. Backward difference
2. Forward difference
3. Bilinear transformation
4. Bilinear transformation with frequency prewarping
   ⋆ *Those first four methods are numerical integration methods.*
5. Impulse-invariance
6. Step-invariance
7. Matched pole-zero mapping

TABLE 4–1 EQUIVALENT DISCRETE-TIME FILTERS FOR A CONTINUOUS-TIME FILTER $G(s) = a/(s+a)$

Mapping method	Mapping equation	Equivalent discrete-time filter for $G(s) = \frac{a}{s+a}$
Backward difference method	$s = \frac{1-z^{-1}}{T}$	$G_D(z) = \frac{a}{\frac{1-z^{-1}}{T} + a}$
Forward difference method	$s = \frac{1-z^{-1}}{Tz^{-1}}$	This method is not recommended, because the discrete-time equivalent may become unstable.
Bilinear transformation method	$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$	$G_D(z) = \frac{a}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} + a}$
Bilinear transformation method with frequency prewarping	$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$ $\left(\omega_A = \frac{2}{T}\tan\frac{\omega_D T}{2}\right)$	$G_D(z) = \frac{\tan\frac{aT}{2}}{\frac{1-z^{-1}}{1+z^{-1}} + \tan\frac{aT}{2}}$
Impulse-invariance method	$G_D(z) = T\,\mathcal{Z}[G(s)]$	$G_D(z) = \frac{Ta}{1-e^{-aT}z^{-1}}$
Step-invariance method	$G_D(z) = \mathcal{Z}\left[\frac{1-e^{-Ts}}{s}G(s)\right]$	$G_D(z) = \frac{(1-e^{-aT})z^{-1}}{1-e^{-aT}z^{-1}}$
Matched pole-zero mapping method	A pole or zero at $s = -a$ is mapped to $z = e^{-aT}$. An infinite pole or zero is mapped to $z = -1$.	$G_D(z) = \frac{1-e^{-aT}}{2}\,\frac{1+z^{-1}}{1-e^{-aT}z^{-1}}$

There is no optimum method for a given system as this depends on the sampling frequency, the highest-frequency component in the system, etc.

s-plane to z-plane mapping



Note that the entire $j\omega$ axis maps into one complete revolution of the unit circle.
($z = e^{Ts}$ maps $j\omega$ axis into infinite number of revolutions of the unit circle)

Bilinear and $z = e^{Ts}$ transformations have considerable differences between them in their transient and frequency response characteristics.

A discrete controller can be obtained using bilinear transformation as

$$G_D(z) = G(s) \mid_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

Bilinear transformation with frequency prewarping

Discretizing the filter

$$G(s) = \frac{a}{s+a}$$

Define $G_D(z) = \frac{a}{s+a} \mid_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \frac{a}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} + a}$

frequency response:
continuous-time $G(j\omega)$
discrete-time $G_D(e^{j\omega T})$
Comparing frequency responses

$$\textit{substitute} \quad s = j\omega_A \quad \textit{and} \quad z = e^{j\omega_D T} \quad \textit{into}$$

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

$$\Rightarrow \quad \omega_A = \frac{2}{T}\tan\frac{\omega_D T}{2} \tag{1}$$

(1) shows the frequency distortion.

note: for $\omega_D T$ small, $\omega_A \cong \frac{2}{T}\,\frac{\omega_D T}{2} = \omega_D$

Now, $G(j\omega_A) = G_D(e^{j\omega_D T})$

The responses are equal when

$$\omega_A = \frac{2}{T} \tan \frac{\omega_D T}{2}$$

Procedure for prewarping

Consider low-pass filter:

$$G(s) = \frac{a}{s+a}$$

1. warp the frequency scale before transforming

$$\frac{\frac{2}{T}\tan\frac{aT}{2}}{s + \frac{2}{T}\tan\frac{aT}{2}}$$

2. transform

$$\begin{aligned} G_D(z) &= \frac{\frac{2}{T}\tan\frac{aT}{2}}{s + \frac{2}{T}\tan\frac{aT}{2}} \Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} \\ &= \frac{\tan\frac{aT}{2}}{\frac{1-z^{-1}}{1+z^{-1}} + \tan\frac{aT}{2}} \end{aligned}$$

Impulse-invariance method

We require

$$g_D(kT) = T\ g(t)\ |_{t=kT}$$

Now,

$$G_D(z) = \mathcal{Z}[g_D(kT)] = T\ \mathcal{Z}[g(t)] = T\ \mathcal{Z}[G(s)] = T\ G(z)$$

$$\textit{If} \quad G(s) = \frac{a}{s+a} \qquad \Rightarrow\ G_D(z) = T\ G(z) = \frac{Ta}{1 - e^{-aT} z^{-1}}$$

Step-invariance method

$$\underbrace{\mathcal{Z}^{-1}\left[G_D(z)\frac{1}{1-z^{-1}}\right]}_{\textit{step response of } G_D(z)} = \underbrace{\mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right]_{t=kT}}_{\textit{step response of } G(s) \textit{ at } t=kT}$$

$$\Rightarrow \qquad G_D(z)\frac{1}{1-z^{-1}} = \mathcal{Z}\left\{\mathcal{L}^{-1}\left[\frac{G(s)}{s}\right]\right\} = \mathcal{Z}\left[\frac{G(s)}{s}\right]$$

or

$$\begin{aligned} G_D(z) &= (1 - z^{-1})\mathcal{Z}\left[\frac{G(s)}{s}\right] \\ &= \mathcal{Z}\left[\frac{1 - e^{-Ts}}{s}\,G(s)\right] \end{aligned}$$

For $G(s) = \frac{a}{s+a}$

$$\begin{aligned} G_D(z) &= (1 - z^{-1})\ \mathcal{Z}\left[\frac{G(s)}{s}\right] \\ &= (1 - z^{-1})\ \mathcal{Z}\left[\frac{a}{s(s+a)}\right] \\ &= \frac{(1 - e^{-aT})z^{-1}}{1 - e^{-aT}z^{-1}} \end{aligned}$$

Matched pole-zero mapping method

Finite poles and zeros at $s = -b$ are replaced with $z = e^{-bT}$. For infinite poles and zeros in s, we replace with $z = -1$
Also, the gains should be matched.

Consider $G(s) = \frac{a}{s+a}$

$\Rightarrow \quad G_D(z) = K\ \frac{a(z+1)}{z-e^{-aT}}$

require $\quad G_D(1) = K\ \frac{2a}{1-e^{-aT}} = G(0) = 1$

$\Rightarrow \quad K = \frac{1-e^{-aT}}{2a}$

$\Rightarrow \quad G_D(z) = \frac{1-e^{-aT}}{2}\ \frac{(1+z^{-1})}{(1-e^{-aT}z^{-1})}$

Implementation

All of the methods above which produce stable filters except for the step-invariance method, give results of the following form

$$G_D(z) = \frac{Y(z)}{X(z)} = K\ \frac{1 + \alpha z^{-1}}{1 + \beta z^{-1}}, \qquad K,\ \alpha,\ \textit{and}\ \beta\ \textit{are constants}$$

The corresponding difference equation is

$$y(kT) = -\beta y((k-1)T) + Kx(kT) + \alpha K x((k-1)T)$$

These require $y[(k-1)T]$, $x[(k-1)T]$ and $x(kT)$

The step-invariance method gives

$$G_D(z) = \frac{Y(z)}{X(z)} = \frac{\alpha z^{-1}}{1 + \beta z^{-1}}$$

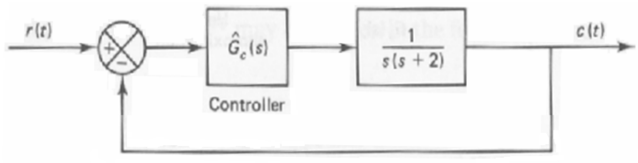
Difference equation

$$y(kT) = -\beta \; y((k-1)T) + \alpha \; x((k-1)T)$$

which requires only $y[(k-1)T]$ and x[(k-1)T]

So, if $x(kT)$ cannot be included to get $y(kT)$, then the step-invariance method must be used.

Design Example

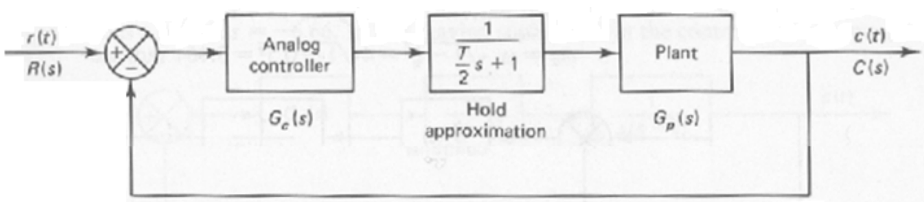


Specifications: damping ratio of the dominant closed-loop poles is 0.5 and settling time $=(\frac{4}{\zeta\omega_n}) = 2\ sec.$

$\Rightarrow$ unit step response: max. overshoot 16.3 %, $\omega_n = 4\ rad/sec.$

Wish to design a digital controller

First, design "analog" system taking into consideration the frequency effects of a ZOH



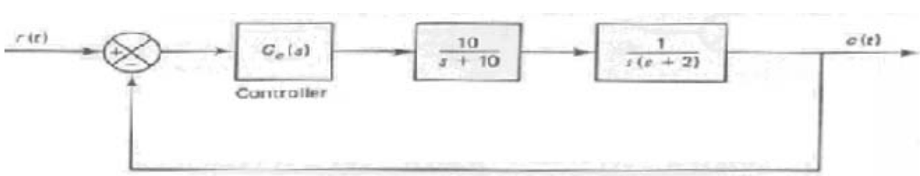
We need to decide an T, the sampling period,

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 4\sqrt{1-0.5^2} = 3.464\ rad/sec$$

$\nearrow$ $\Rightarrow$ damped oscillation of period $\frac{2\pi}{\omega_d} = 1.814\ sec$ will occur
We want at least 8 samples per period, so choose $T = 0.2\ sec$

$$\Rightarrow \quad G_h(s) = \frac{1}{\frac{T}{2}s+1} = \frac{1}{0.1s+1} = \frac{10}{s+10}$$

We now need to design a controller for the following system



let $G_c(s) = 20.25(\frac{s+2}{s+6.66})$

zero at $s = -2$ cancels pole of plant.

Closed-loop TF

$$\frac{C(s)}{R(s)} = \frac{202.5}{(s+2+j2\sqrt{3})(s+2-2j\sqrt{3})(s+12.66)}$$

Pole at $s = -12.66$ is far away, so we can neglect it and use the complex poles.
Note, complex poles have $\zeta = 0.5$ and $\omega_n = 4\ rad/sec$

Now, discretize the controller . *Use matched pole-zero mapping.*

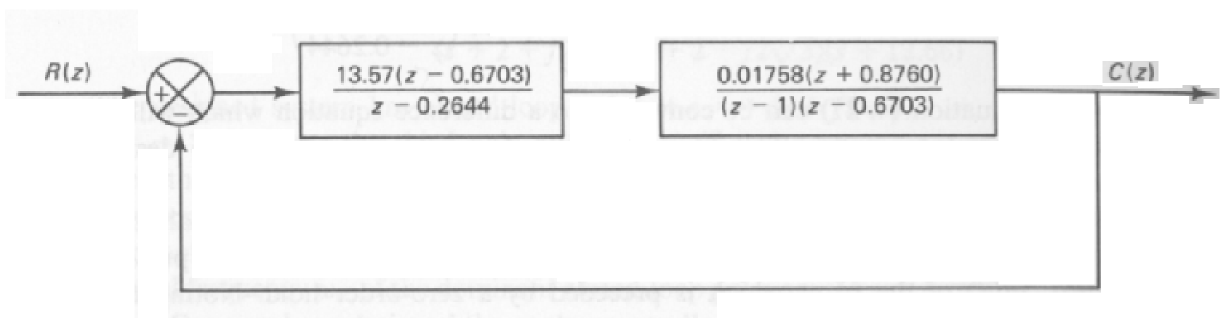
(Since the analog controller was designed to cancel the undesired plant pole at $s = -2$)
thus

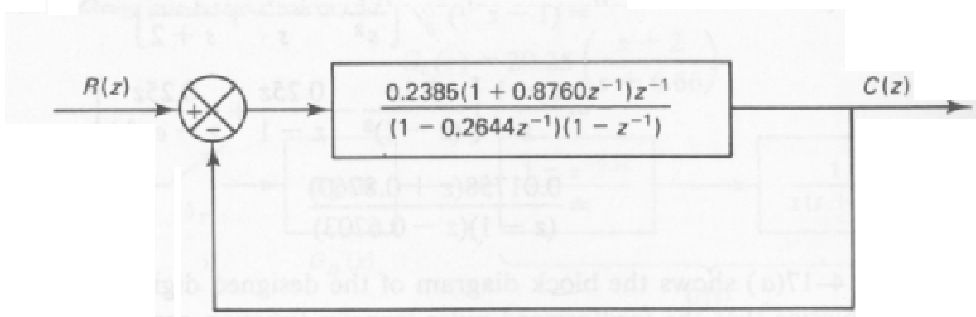
$$G_D(z) = 13.57 \left(\frac{z - 0.6703}{z - 0.2644} \right)$$

<u>**Check design**</u>

↙ pulse transfer function of plant

$$\begin{aligned} G(z) &= \mathcal{Z}\left[\frac{1 - e^{-0.2s}}{s} \frac{1}{s(s+2)} \right] \\ &= \frac{0.01759\ (z + 0.8760)}{(z-1)\ (z - 0.6703)} \end{aligned}$$





Closed-loop pulse transfer function

$$\frac{C(z)}{R(z)} = \frac{0.2385z^{-1} + 0.2089z^{-2}}{1 - 1.0259z^{-1} + 0.4733z^{-2}}$$

Can check the step response of this system to see if the specifications are satisfied.

4-6 Design based on the frequency response method

Advantage of the Bode diagram approach to design

1. Transient response specs. can be translated into the frequency response specs. of phase margin, gain margin, bandwidth, etc.

2. Design of a controller is undertaken straightforwardly and simply.

Bilinear transformation and the w plane

Given a pulse transfer function of a system G(z), the frequency response is given by $G(z)\mid_{z=e^{j\omega T}} = G(e^{j\omega T})$.

Since in the z plane, the frequency appears as $z = e^{j\omega T}$, if we treat frequency response in the z plane, the simplicity of logarithmic plots will be lost.
(Note that the z transformation maps the primary and complementary strips of the left half of the s plane into the unit circle in the z plane. Thus conventional frequency response methods, which deal with the entire left half plane do not apply to the z plane.)

We overcome this difficulty by transforming the pulse transfer function in the z plane into one in the w plane.

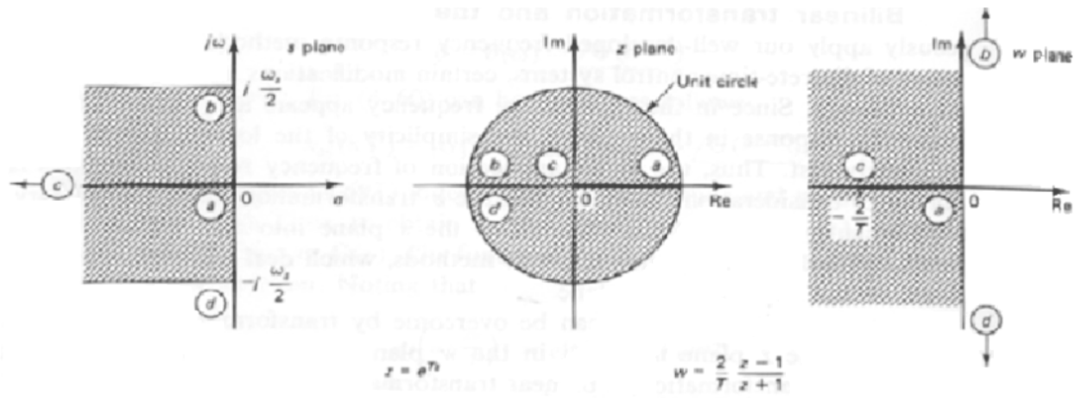
The w transformation is a bilinear transformation given by

$$z = \frac{1 + \frac{T}{2}\mathrm{w}}{1 - \frac{T}{2}\mathrm{w}}$$

T is the sampling period.
The inverse transformation is

$$\mathrm{w} = \frac{2}{T}\,\frac{z-1}{z+1}$$

Through the z transformation and the w transformation, the primary strip of the left half of the s plane is first mapped into the inside of the unit circle in the z plane and then mapped into the entire left half of the w plane.



The origin in the z plane maps into the point $w = -\frac{2}{T}$ in the w plane.

As s varies from $0 \rightarrow j\frac{\omega_s}{2}$ along $j\omega$ axis, z varies from 1 to -1 along the unit circle in the z plane, and w varies from 0 to ∞ along the imaginary axis in the w plane.

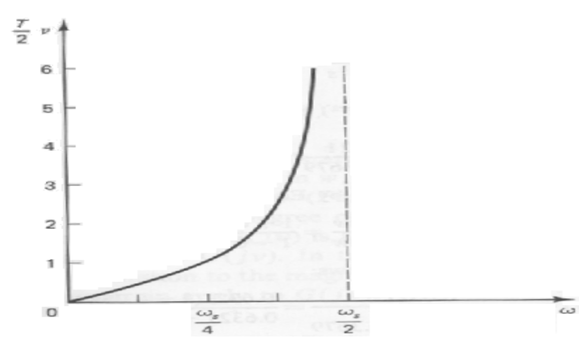
The difference between the s plane and w plane is that the frequency range $-\frac{1}{2}\omega_s \leq \omega \leq \frac{1}{2}\omega_s$ in the s plane maps to the range $-\infty < \nu < \infty$ in the w plane, where ν is the fictitious frequency on the w plane. Thus there is a compression of the frequency scale. $G(\mathrm{w})$ is treated as conventional transfer function. Replacing w by $j\nu$ we can draw Bode plots.

Although the w plane resembles the s plane geometrically, the frequency axis in the w plane is distorted. The fictitious frequency ν and the actual frequency ω are related as follows

$$
\begin{aligned}
\mathrm{w}\,|_{\mathrm{w}=j\nu} &= j\nu = \frac{2}{T}\,\frac{z-1}{z+1}\,|_{z=e^{j\omega T}} = \frac{2}{T}\,\frac{e^{j\omega T}-1}{e^{j\omega T}+1} \\
&= \frac{2}{T}\,\frac{e^{j\frac{1}{2}(\omega T)} - e^{-j\frac{\omega T}{2}}}{e^{j\frac{\omega T}{2}} + e^{-j\frac{\omega T}{2}}} = \frac{2}{T} j \tan\frac{\omega T}{2}
\end{aligned}
$$

or

$$\nu = \frac{2}{T}\ \tan\frac{\omega T}{2} \tag{2}$$



thus if the bandwidth is specified as ω_b, then the corresponding bandwidth in the w plane is

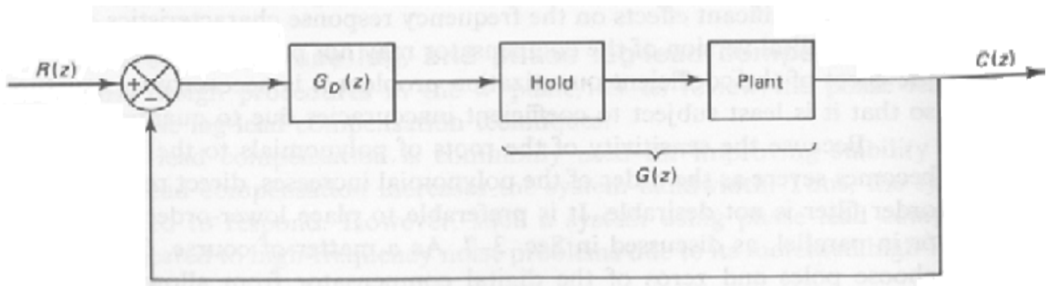
$$\frac{2}{T}\tan\frac{\omega_b T}{s}$$

Similarly, $G(j\nu_1)$ corresponds to $G(j\omega_1)$ where

$$\omega_1 = (\frac{2}{T})\tan^{-1}\frac{\nu_1 T}{2}$$

Note, for ωT small, $\nu \approx \omega$

Design procedure in the w plane



1. Obtain G(z), the z transform of the plant preceded by a hold. Then transform G(z) into a transfer function G(w)

$$G(\text{w}) = G(z)\,|_{z=\frac{1+\frac{T}{2}\text{w}}{1-\frac{T}{2}\text{w}}}$$

   Choose T about 10 times the bandwidth of the closed-loop system.

2. Substitute w $= j\nu$ into G(w) and plot the Bode diagram for $G(j\nu)$

3. Read from the plot the gain and phase margins and the low frequency gain (which will determine static accuracy).

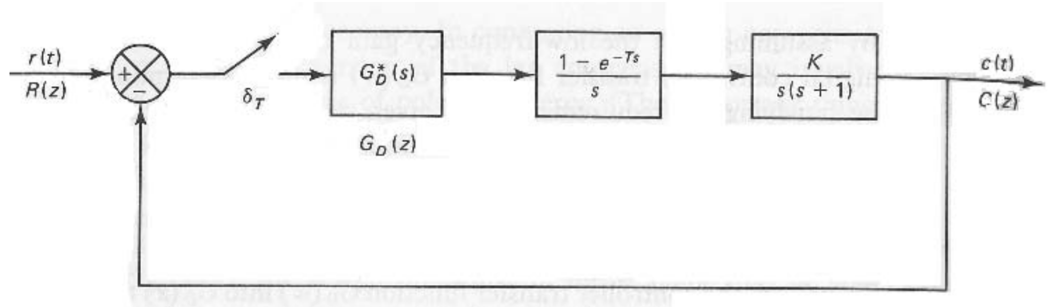
4. Design $G_D(\text{w})$ to achieve desired loop transfer function

5. Transform the $G_D(\text{w})$ into $G_D(z)$

$$G_D(z) = G_D(\text{w})\,|_{\text{w}=\frac{2}{T}\ \frac{z-1}{z+1}}$$

6. Realize $G_D(z)$ by a computational algorithm.

Example



Design a digital controller in the w plane such that the phase margin is 50^o, the gain margin is $\geq 10dB$ and static velocity constant K_v is 2 sec^{-1}. Assume $T = 0.2$

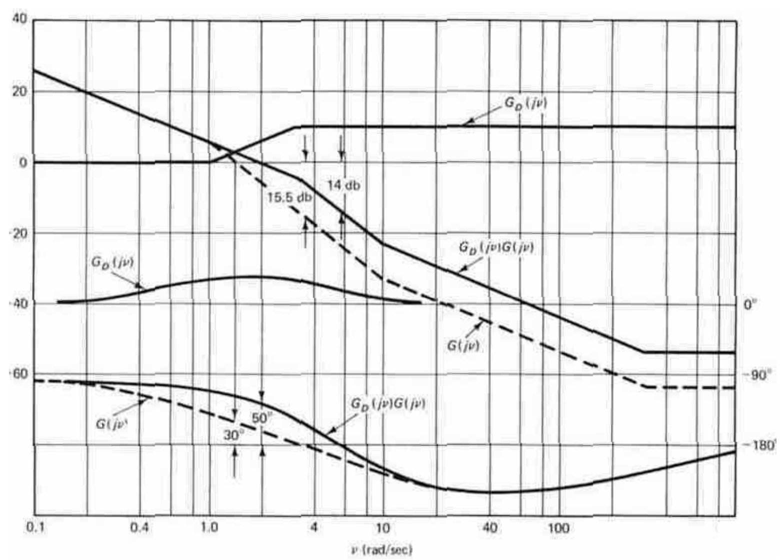
Solution

$$
\begin{aligned}
G(z) &= \mathcal{Z}\left[\frac{1-e^{-0.2s}}{s}\,\frac{K}{s(s+1)}\right] \\
&= 0.01873\left[\frac{K(z+0.9356)}{(z-1)(z-0.8187)}\right]
\end{aligned}
$$

$$
\begin{aligned}
G(\mathrm{w}) &= G(z)\,|_{z=\frac{1+0.1\mathrm{W}}{1-0.1\mathrm{W}}} \\
&= \frac{K(\frac{\mathrm{W}}{300.6}+1)(1-\frac{\mathrm{W}}{10})}{\mathrm{w}(1+\frac{\mathrm{W}}{0.997})}
\end{aligned}
$$

Poles at w = 0 and w = 0.997

LHP zero at w= 300.6 and RHP zero at w= 10



Try a lead compensator

$$G_D(\mathrm{w}) = \frac{1+\frac{\mathrm{w}}{\alpha}}{1+\frac{\mathrm{w}}{\beta}}$$

Need to adjust K, α *and* β to satisfy specifications. Adjust K to meet static accuracy specification.
Open-loop transfer function is

$$G_D(\mathrm{w})\ G(\mathrm{w}) = \frac{1+\frac{\mathrm{w}}{\alpha}}{1+\frac{\mathrm{w}}{\beta}}\ \frac{K(\frac{\mathrm{w}}{300.6}+1)(1-\frac{\mathrm{w}}{10})}{\mathrm{w}(\frac{\mathrm{w}}{0.997}+1)}$$

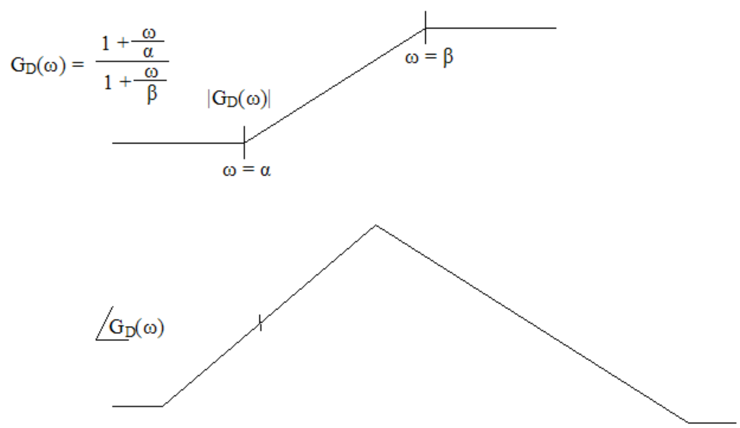
Require $K_v = 2$

where

$$K_v = \lim_{\mathrm{w}\to 0} \mathrm{w} G_D(\mathrm{w})\ G(\mathrm{w})$$

$\Rightarrow \quad K = 2$

With this value of K, we can read the gain and phase margins. We find 30^0 phase margin and 15.5 dB gain margin. To give a boost in the phase margin, we adjust the parameters of the lead network α *and* β



we decide on

$$G_D(\mathrm{w}) = \frac{1+\frac{\mathrm{w}}{0.997}}{1+\frac{\mathrm{w}}{3.27}}$$

$\Rightarrow \quad 50^o$ phase margin and 14 dB gain margin.

Now transform the controller to the z plane

$$G_D(z) = G_D(\mathrm{w})\ |_{\mathrm{w}=10\ \frac{z-1}{z+1}}$$

$$\Rightarrow \qquad G_D(z) = 2.718\, \frac{z - 0.8187}{z - 0.5071}$$

The open-loop pulse transfer function of the compensated system is

$$G_D(z)\ G(z) = 0.1018\ \frac{z + 0.9356}{(z-1)(z-0.5071)}$$

The closed-loop transfer function is

$$\frac{C(z)}{R(z)} = \frac{0.1018(z + 0.9356)}{(z - 0.7026 + j0.3296)(z - 0.7026 - j0.3296)}$$

closed-loop poles z=0.7026 ±j0.3296

$$\Rightarrow \qquad \zeta = 0.5$$

We find that $w_s = \frac{2\pi}{T} = 14.3\ w_d$
where w_d is the damped natural frequency of these poles.

$$w_d = w_n \sqrt{1 - \zeta^2}$$