Chapter 10

Linear Quadratic Regulator Problem

Minimize the cost function $J$ given by

$$J = \frac{1}{2} \int_0^\infty (x'Qx + u'Ru) dt$$

$R > 0$ positive definite (symmetric with positive eigenvalues)
$Q \geq 0$ positive semi definite (symmetric with nonnegative eigenvalues)

subject to

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

LQR SOLUTION:

Find the positive-definite solution $P$ of the ARE (Algebraic Ricatti Equation)

$$A'P + PA + Q - PBR^{-1}B'P = 0$$

$$u = -Kx \quad \text{where} \quad K = R^{-1}B'P$$

The positive-definite solution of the ARE results in an asymptotically stable closed-loop system if:

1) the system is controllable
2) $R > 0$
3) $Q = C_q'C_q$ where $(C_q, A)$ is observable

These conditions are necessary and sufficient

We can define another output $z$ where

$$z = C_q'x \quad \rightarrow \text{controlled or regulated output}$$

Therefore $$x'Qx = x'C_q'C_qx = z'z$$

LQR design of double integrator

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C_q = [1 \ 0]$$

assume $$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad R = 1$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = C_q'C_q$$
(A, B) is controllable
(C, A) is observable

ARE:

\[
egin{bmatrix}
0 & 0 & p_1 & p_2 \\
p_1 & p_2 & p_3
\end{bmatrix} +
\begin{bmatrix}
p_1 & p_2 & 0 & 1 \\
p_2 & p_3 & 0 & 0
\end{bmatrix} -
\begin{bmatrix}
p_2 & 0 & 0 & 1 \\
p_3 & p_1 & 0 & 0
\end{bmatrix} -
\begin{bmatrix}
p_2 & p_3 & 0 & 0 \\
p_3 & p_1 & 0 & 0
\end{bmatrix}
= 0
\]

\begin{bmatrix}
p_2^2 & p_2p_3 \\
p_2p_3 & p_3^2
\end{bmatrix}

solving

\[
\begin{bmatrix}
0 & 0 \\
p_1 & p_2
\end{bmatrix} +
\begin{bmatrix}
0 & p_1 \\
0 & p_2
\end{bmatrix} +
\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix} -
\begin{bmatrix}
p_2^2 & p_2p_3 \\
p_2p_3 & p_3^2
\end{bmatrix} = 0
\]

\begin{align*}
1 - p_2^2 &= 0 & \Rightarrow & & p_2^2 &= 1 \\
p_1 - p_2p_3 &= 0 & \Rightarrow & & p_1 &= p_2p_3 \\
p_1 - p_2p_3 &= 0 & \Rightarrow & & p_1 &= p_2p_3 \\
2p_2 - p_3^2 &= 0 & \Rightarrow & & p_2 &= 1
\end{align*}

\begin{align*}
p_3 &= \sqrt{2} \\
p_1 &= \sqrt{2}
\end{align*}

\Rightarrow \quad P = \begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix}

K = R^{-1}B'P = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \sqrt{2} \end{bmatrix}

The closed loop system matrix becomes

\[
A - BK = \begin{bmatrix} 0 & 1 \\ -1 & -\sqrt{2} \end{bmatrix}
\]

Closed loop roots are:

\begin{align*}
\lambda^2 + \sqrt{2}\lambda + 1 &= 0 \\
\lambda &= \frac{\sqrt{2}}{2} (-1 \pm j)
\end{align*}

damping ratio is 0.707
The loop transfer function is:

\[ K\phi(s)B = K(sI - A)^{-1}B = \sqrt{2} \left[ s + \frac{\sqrt{2}}{2} \right] \frac{1}{s^2} \]

\[ \rightarrow 65^\circ \text{ phase margin} \]
\[ \rightarrow \text{ infinite gain margin} \]

\[ L(s) = \frac{\sqrt{2} \left[ s + \frac{\sqrt{2}}{2} \right]}{s^2} \]
\[ = \frac{\sqrt{2} \cdot \frac{\sqrt{2}}{2} \left[ 1 + \frac{\sqrt{2}}{2} s \right]}{s^2} \]
\[ = \frac{1 + \frac{s}{\sqrt{2}}}{s^2} \]
\[ = \frac{1 + \frac{s}{0.7071}}{s^2} \]

\[ |L(j\omega)|_{\omega=0.7071} \approx \frac{|j\omega|_{0.7071}}{(j\omega)^2} \]
\[ = \frac{1.4142}{\omega} \]

\[ |L(j\omega)|_{\omega=0.7071} \approx \frac{1.4142}{\omega} \]
\[ \left| L(j\omega_c) \right| = 1 = \frac{1.4142}{\omega_c} \]
\[ \omega_c = 1.4142 \]

Phase @ \( \omega_c = -180 + \tan^{-1}\left( \frac{\omega_c}{0.7071} \right) \)

Phase margin = \( \tan^{-1}\frac{1.4142}{0.7071} = \tan^{-1}(2) = 63.4^\circ \)

**USING MATLAB TO GET EXACT RESULTS**

Matlab

\[
\text{num} = \sqrt{2} \times \begin{bmatrix} 1 & \sqrt{2}/2 \end{bmatrix} \\
\text{den} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}
\]

**results:**

\[
\text{margin(num,den)}
\]

\[
\text{gm} = \infty \\
\text{pm} = 65.53 \quad @ \quad \omega = 1.554
\]

**Properties of LQR design**

From the ARE we can derive the relation

\[
\left| 1 + L(j\omega) \right|^2 = 1 + \frac{1}{\rho} \left| G_q(j\omega) \right|^2 \quad \text{(*)}
\]

\[ \rho \text{ is a scalar} \]

where \( L(s) = K\phi(s)B \)

- loop gain

\[ \phi(s) \equiv (sI - A)^{-1} \]

and

\[
Q = C_q^T C_q \\
G_q(s) = C_q \phi(s)B
\]

From (*) we see

\[
\left| 1 + L(j\omega) \right| \geq 1
\]

This implies that the Nyquist plot of the loop transfer function of an LQR design always stays outside of a unit circle centered at (-1,0).
In SISO case, LQR design has > 60° phase margin, infinite gain margin and a gain reduction tolerance of -6dB (i.e. the gain can be reduced by a factor of \( \frac{1}{2} \) before instability occurs).

Recall pole placement does not guarantee stability margins.

High-frequency roll-off rate

Closed loop transfer function \( T(j\omega) = -K(j\omega I - A + BK)^{-1} B \)

\[
\lim_{\omega \to \infty} T(j\omega) = \frac{1}{j\omega} KB = \frac{1}{j\omega} R^{-1} B' PB < 0
\]

\( \rightarrow \) -20dB/dec roll off rate at high frequencies
- not good for noise suppression

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Optimal Observers – Kalman Filter

State estimation – plant represented as

\[
\dot{x} = Ax + Bu + \omega \quad \leftarrow \text{process noise}
\]
\[
y = Cx + v \quad \leftarrow \text{measurement noise}
\]

The optimal filter is given by

\[
\hat{x} = \dot{x} + Bu + L(y - C\hat{x})
\]
where \( L = \Sigma C' R_0^{-1} \)

where \( \Sigma \) is the positive definite solution of

\[
A \Sigma + \Sigma A' + Q_0 - \Sigma C' R_0^{-1} C \Sigma = 0
\]

\( Q_0 \) and \( R_0 \) are noise covariance matrices, which represent the intensity of the process and sensor noise inputs.

Require \( Q_0 \geq 0, R_0 > 0 \) and system to be observable.

If we combine the Kalman-Bucy Filter (optimal estimator) with LQR design, we have LQG (Linear Quadratic Gaussian). Let’s do a LQG design for double integrator plant. We already have the LQR design.

For Kalman filter, assume

\[
Q_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad R_0 = 1
\]

Solving Ricatti equation with \( \Sigma = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \)

\[
a^2 = 2b + 1
\]

we find

\[
ab = c
\]

\[
b^2 = 1
\]

\[
\Rightarrow \Sigma = \begin{bmatrix} \sqrt{3} & 1 \\ 1 & \sqrt{3} \end{bmatrix}
\]

and \( L = \Sigma C' R_0^{-1} = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} \)

Transfer function of compensator is given by

\[
H(s) = K \left( sI - A + BK + LC \right)^{-1} L
\]

\[
= \frac{3.14 \left( s + 0.3 \right)}{(s + 1.57 + j1.4) (s + 1.57 - j1.4)}
\]

Comparison of LQR and LQG

- LQR has guaranteed stability margins
- LQG has no guaranteed stability margins
- High freq. roll off in LQG can be $> 20$ dB/dec exhibited by LQR $\Rightarrow$ greater noise filtering in LQG

- LQG is not robust $\Rightarrow$ uncertainty in plant may cause system to go unstable

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**Loop Transfer Recovery (LTR)**

**LQR**

$\Rightarrow$ $> 60^\circ$ phase margin

infinite gain margin

**LQG**

$\Rightarrow$ no guaranteed margins

The properties of LQR can be recovered asymptotically by using $Q_o$ and $R_0$ as tuning parameters

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**LQR**

loop gain, $L(s) = K\phi(s)B$

$= K(sI - A)^{-1}B$

---

**LQG**

$\Rightarrow$

$s\mathbf{\hat{X}} = (-BK + A - LC)\mathbf{\hat{X}} + LY(s)$

$\frac{\mathbf{\hat{X}}(s)}{Y(s)} = (sI - A + BK + LC)^{-1}L$

loop gain, $L_{LQG}(s) = K(sI - A + BK + LC)^{-1}LC\phi(s)B$
If the following two conditions hold then LQR loop properties can be recovered if:

1) $G(s)$ is minimum phase
2) $R_0 = 1$ and $Q_0 = q^2B'B$

Then it can be shown

$$\lim_{q \to \infty} L_{LQG}(s) = L(s)$$

The variable $y$ that is recovered may be different from the variable $z$ that is to be controlled.

where $y = Cx$ and $z = C_q x$

**Loop Shaping Steps**

1) Determine the controlled variable and set
   
   $Q = CC'$ and $Q = C_q'C_q$

2) Get a desired loop gain in LQR design. Use $R$ as tuning parameter.

3) Select scalar $q$ and solve the filter Ricatti equation

   $$A\Sigma + \Sigma A' + q^2BB' - \Sigma C'C\Sigma = 0$$
   $$L = \Sigma C''$$

4) Increase $q$ until the resulting loop transfer function is close to the LQR design.

**Do not make $q$ too high since**

1) large gains in $L$ are required
2) the undesirable -20dB/dec high freq. roll-off of LQR will be recovered

**Example**

Double integrator system

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad R = 1$$

$\Rightarrow$ Gave 65° phase margin for LQR design
Figure 10.19  Step response, Bode plots, and filter poles for LTR using $q = (1, 10, 100, 1000)$. (a) Closed-loop step response. (b) and (c) Open-loop magnitude and phase Bode plots. (d) Filter poles.
### Graphs

**Graph 1:**
- Title: TFL Mag
- Axes: dB vs. ω
- Curves for q = 1, 10, 100, 1000

**Graph 2:**
- Title: TFL Phase
- Axes: Phase vs. ω
- Curves for q = 1, 10, 100, 1000

### Table

<table>
<thead>
<tr>
<th>q</th>
<th>1</th>
<th>10</th>
<th>100</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>PM</td>
<td>32.6</td>
<td>41.9</td>
<td>55.0</td>
<td>61.7</td>
</tr>
<tr>
<td>GM</td>
<td>9.5</td>
<td>13.0</td>
<td>21.1</td>
<td>30.4</td>
</tr>
<tr>
<td>L</td>
<td>1.4</td>
<td>4.5</td>
<td>14.1</td>
<td>44.7</td>
</tr>
<tr>
<td>Filter poles</td>
<td>-0.7 + 0.7j</td>
<td>-2.2 + 2.2j</td>
<td>-7.0 + 7.0j</td>
<td>-22.3 + 22.3j</td>
</tr>
<tr>
<td>poles</td>
<td>-0.7 - 0.7j</td>
<td>-2.2 - 2.2j</td>
<td>-7.0 - 7.0j</td>
<td>-22.3 - 22.3j</td>
</tr>
</tbody>
</table>
Robustness

1) Robust stability – stable in the face of plant uncertainties
2) Robust performance – performance met even in the face of plant uncertainties

Two important properties of feedback –

1) sensitivity reduction
2) disturbance rejection

General feedback system

\[
Y(s) = \frac{G(s)H(s)}{1 + G(s)H(s)} R(s) + \frac{1}{1 + G(s)H(s)} D(s) - \frac{G(s)H(s)}{1 + G(s)H(s)} N(s)
\]

Tracking error \( e = r - y \)

\[
E(s) = \frac{1}{1 + G(s)H(s)} R(s) - \frac{1}{1 + G(s)H(s)} D(s) - \frac{1}{1 + G(s)H(s)} N(s)
\]

Actuator output (i.e. plant input) is given by

\[
U(s) = \frac{H(s)}{1 + G(s)H(s)} [R(s) - D(s) - N(s)] \quad \text{note: } U(s) = H^{-1}(s)E(s) \quad \Rightarrow E(s) = H^{-1}(s)U(s)
\]

Define the following terms

\[
J(s) = 1 + GH \quad \text{return difference}
\]

\[
S(s) = \frac{1}{1 + GH} \quad \text{sensitivity}
\]

\[
T(s) = \frac{GH}{1 + GH} \quad \text{complementary sensitivity}
\]

\[
\text{note: } S(s) + T(s) = 1
\]

Using these definitions

system output: \( Y(s) = S(s)D(s) + T(s) [R(s) - N(s)] \)

tracking error: \( E(s) = S(s) [R(s) - D(s) - N(s)] \)
plant input: \[ U(s) = H(s)S(s)[R(s) - D(s) - N(s)] \]

From these expressions we see that we need

1) **Disturbance rejection:** From \( Y(s) \) expression we see we require \( S \) small \( \rightarrow GH>>1 \) (since SD)

2) **Tracking:** \( S \) small

3) **Noise suppression:** From \( Y(s) \) we have \( T(s)N(s) \) \( \rightarrow \) require \( T \) small

4) **Actuator limits:** From \( U(s) \) expression want \( H(s)S(s) \) bounded

Tracking and Disturbance rejection require small \( S \)
Noise suppression requires small \( T \)

\[ S + T = 1 \]

however command inputs and disturbances are low frequency whereas measurement noise is high frequency signal

\[ \rightarrow \] keep \( S \) small in low frequency range and \( T \) small in high frequency range

Also

\[ H(s)S(s) = \frac{H(s)}{1 + G(s)H(s)} = \frac{T(s)}{G(s)} \]

\[ \rightarrow \] making \( T \) small we reduce control energy

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**Loop Gain Properties**

<table>
<thead>
<tr>
<th>Performance (R)</th>
<th>Low Frequency</th>
<th>Mid. Frequency</th>
<th>High Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Disturbance Rejection (D)</td>
<td>High Gain</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Noise Suppression (N)</td>
<td></td>
<td></td>
<td>Low Gain</td>
</tr>
</tbody>
</table>

**Uncertainty Modeling**

Two categories --

1) structured uncertainty
2) unstructured uncertainty

We will deal with unstructured uncertainty
Additive uncertainty: actual model $\tilde{G}(s)$

$$\tilde{G}(s) = \frac{G(s)}{\text{model}} + \frac{\Delta_a(s)}{\text{uncertainty or error}}$$

Multiplicative uncertainty: $\tilde{G}(s) = [1 + \Delta_m(s)]G(s)$

Robust Stability

We say a compensator robustly stabilizes a system if the closed-loop system remains stable for the true plant $\tilde{G}(s)$.

Robustness results can be derived using the small gain theorem.

Small Gain Theorem

The closed-loop system will remain stable if

$$|G(s)H(s)| < 1$$

no since $|G(s)H(s)| \leq |G(s)||H(s)|$
then closed-loop stability is guaranteed if

\[ |G(s)H(s)| < 1 \]

There is no possibility of encirclements of (−1,0) point by Nyquist plot.

Two equations that the small gain theorem can help us to answer

1) Given that the uncertainty is stable and bounded, will the closed-loop system be stable for the given uncertainty?
2) For a given system, what is the smallest uncertainty that will destabilize the system?

To answer these questions we first do some block diagram manipulation

\[
M(s) = \frac{-G(s)H(s)}{1 + G(s)H(s)}
\]

**Determine** \( M(s) \), **the transfer function seen by** \( \Delta_m \)
By small gain theorem, closed-loop system will be robustly stable if

$$|\Delta_m| < \frac{1}{GH(1+GH)^{-1}}$$

i.e. $$|\Delta_m| < \frac{1}{|T|}$$  

T – complementary sensitivity

If the uncertainty is bounded by $\gamma$ so that

$$|\Delta_m| < \gamma$$

then the closed-loop system will be stable if

$$|T| < \frac{1}{\gamma} \quad \text{or} \quad |\gamma T| < 1$$

This answers the first question

Second question: find the size of the smallest stable uncertainty that will destabilize the system

Because the uncertainty must be smaller that $1/T$, it must be smaller that the minimum of $1/T$. We must find the maximum of $T$.

Define $$M_r = \sup_{\omega} T(j\omega)$$

sup = supremum (least upper bound)

Then the smallest destabilizing uncertainty, we call this the multiplicative stability margin or MSM, is given by

$$MSM = \frac{1}{M_r}$$

For additive uncertainty

$$M(s) = \frac{-H(s)}{1 + G(s)H(s)}$$

closed-loop will be robustly stable if

$$|\Delta_a| < \frac{1}{H(1+GH)^{-1}}$$  

or  

$$|\Delta_a| < \frac{1}{|HS|}$$
if uncertainty is stable and bounded by

$$|\Delta_a| < \gamma$$

then we guarantee closed-loop stability if

$$|HS| \leq \frac{1}{\gamma} \quad \text{or} \quad |\phi HS| < 1$$

we can define additive stability margin (ASM) by

$$ASM = \frac{1}{\sup_{\omega} |H(j\omega)S(j\omega)|}$$

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**Example**

$$G(s) = \frac{5 - s}{(s + 5)(s^2 + 0.2s + 1)} \quad , \quad H(s) = \frac{5(s + 0.1)}{s} \frac{s + 0.2}{s + 5}$$

phase margin: 38°
gain margin: 2.8 (9dB)

Find MSM and ASM:

**MSM**

Find peak of T (complementary sensitivity function)

peak = 1.52 \(\rightarrow\) MSM = 0.65

\(\rightarrow\) the system will be robustly stable against unmodelled multiplicative uncertainties with transfer function magnitude < 0.65
Problem 10.9

a.) \[ \tilde{G} = (1 + \Delta_m)G \quad \Rightarrow \quad \Delta_m = \frac{\tilde{G}}{G} - 1 \]

\[ \Delta_m = \frac{2(s + 1)}{s^2(s^2 + s + 1)} - 1 \]

\[ = \frac{2(s + 1)}{s^2(s^2 + s + 1)} - \frac{s^2 + s + 1}{s^2(s^2 + s + 1)} \]

\[ = -\frac{s^2 + s + 1}{s^2 + s + 1} \]

b.)

b.)

\[ M(s) = \left( \frac{-G}{1 + GH} \right) = \frac{-20(s + 1)}{s^2(s + 10)} = \frac{-20(s + 1)}{s^2(s + 10) + 20(s + 1)} = \frac{20(s + 1)}{s^3 + 10s^2 + 20s + 20} \]

c.) SGT: \[ |\Delta_m||M| < 1 \]

\[ \Rightarrow |\Delta_m| < \left| \frac{1}{M} \right| = \frac{1}{GH} \frac{1}{1 + GH} \]

\[ \Rightarrow |\Delta_m| < \left| 1 + (GH)^{-1} \right| \]
Additive uncertainty

\[ \tilde{G} = G + \Delta_a \]

\[ \Rightarrow \Delta_a = \tilde{G} - G \]

\[ = \frac{2(s+1)}{s^2(s^2+s+1)} - \frac{1}{s^2} \]

\[ = \frac{1}{s^2} \left[ \frac{2s+2 - s^2 - s - 1}{s^2+s+1} \right] \]

\[ = \frac{-s^2 + s + 1}{s^2(s^2+s+1)} \]

\[ M(s) = \frac{-H}{1+GH} \]

SGT: \[ |\Delta_a||M| < 1 \]

\[ \Rightarrow |\Delta_a| < \frac{1}{|M|} \]

\[ \Rightarrow |\Delta_a| < |H^{-1} + G| \]
\[ |\Delta_a| < \left| \frac{s + 10}{20s + 20} + \frac{1}{s^2} \right| \]

\[< \frac{s^2 (s + 10) + 20s + 20}{s^2 (20s + 20)} \]

\[|\Delta_a| < \frac{s^3 + 10s^2 + 20s + 20}{s^2 (20s + 20)} \]
Basic Bode Magnitude Plots

\[ G(s) = \frac{A}{1 + \frac{s}{\omega_0}} \Rightarrow \]

\[ |G| \quad A \quad \frac{A}{s} \omega_0 \quad -20\,\text{dB/dec} \]

\[ G(s) = A(1 + \frac{s}{\omega_0}) \Rightarrow \]

\[ |G| \quad A \quad \frac{A}{s} \omega_0 \quad +20\,\text{dB/dec} \]

\[ G(s) = \frac{A}{1 + \frac{1}{Q} \left( \frac{s}{\omega_0} \right) + \left( \frac{s}{\omega_0} \right)^2} \Rightarrow \]

\[ |G| \quad A \quad \frac{A}{s} \omega_0 \quad Q \quad A \left( \frac{s}{\omega_0} \right)^2 \quad -40\,\text{dB/dec} \]

\[ G(s) = A \left[ 1 + \frac{1}{Q} \left( \frac{s}{\omega_0} \right) + \left( \frac{s}{\omega_0} \right)^2 \right] \Rightarrow \]

\[ |G| \quad Q \quad A \quad \frac{A}{s} \omega_0 \quad \frac{A}{Q} \quad \left( \frac{s}{\omega_0} \right)^2 \quad +40\,\text{dB/dec} \]

If \( Q < \frac{1}{2} \) then roots are real. Factor the expression and use the resulting product of two first order transfer functions to find magnitude response.
Example

\[
G(s) = \frac{A}{(1 + \frac{s}{\omega_0})[1 + \frac{1}{Q} (\frac{s}{\omega_1}) + (\frac{s}{\omega_1})^2]} \Rightarrow
\]

\[\omega_0 < \omega_1\]
Example

The $\Delta M$ structure of a system has been determined to be given by

$$\Delta = \frac{A_\Delta}{1 + \frac{1}{Q} \left( \frac{s}{\omega_2} \right) + \left( \frac{s}{\omega_3} \right)^2}$$

$$M = \frac{A_M \left( 1 + \frac{s}{\omega_2} \right)}{(1 + \frac{s}{\omega_3})(1 + \frac{s}{\omega_1})}$$

where $\omega_1 << \omega_2 << \omega_3$ and $\omega_s = \sqrt{\omega_2 \omega_3}$

Determine the conditions under which robust stability is assured.

Answer

By SGT we require $|\Delta M| < 1$ or $|\Delta| < \frac{1}{|M|}$

$$\frac{1}{M} = \frac{A_M^{-1} \left( 1 + \frac{s}{\omega_2} \right) \left( 1 + \frac{s}{\omega_3} \right)}{(1 + \frac{s}{\omega_2})}$$

From the above diagram we can see that we require

- $A_\Delta < A_M^{-1}$ and $QA_\Delta < A_M^{-1} \frac{\omega_2}{\omega_1}$

or

- $A_\Delta < A_M^{-1} \frac{\omega_2}{\overline{Q} \omega_1}$