Stable, Unstable and Marginally stable Systems

the total response of a system is the sum of the forced and natural responses, or

$$c(t) = c_{\text{forced}}(t) + c_{\text{natural}}(t)$$

definitions of stability, instability, and marginal stability:

A linear, time-invariant system is *stable* if the natural response approaches zero as time approaches infinity.

A linear, time-invariant system is *unstable* if the natural response grows without bound as time approaches infinity.

A linear, time-invariant system is *marginally stable* if the natural response neither decays nor grows but remains constant or oscillates as time approaches infinity.

alternate definition of *stability*, one that regards the total response and implies the first definition based upon the natural response, is

A system is stable if *every* bounded input yields a bounded output.

We call this statement the bounded-input, bounded-output (BIBO) definition of stability.

Let us summarize our definitions of stability for linear, time-invariant systems. Using the natural response:

1. A system is stable if the natural response approaches zero as time approaches infinity.
2. A system is unstable if the natural response approaches infinity as time approaches infinity.
3. A system is marginally stable if the natural response neither decays nor grows but remains constant or oscillates.

Using the total response (BIBO):

1. A system is stable if *every* bounded input yields a bounded output.
2. A system is unstable if *any* bounded input yields an unbounded output.

Tests for stability (given the pole locations):

Stable systems:

stable systems have closed-loop transfer functions with poles only in the left half-plane.

Unstable systems:

unstable systems have closed-loop transfer functions with at least one pole in the right half-plane and/or poles of multiplicity greater than 1 on the imaginary axis.

Marginally stable systems

marginally stable systems have closed-loop transfer functions with only imaginary axis poles of multiplicity 1 and poles in the left half-plane.

Coefficient Tests for Stability

First- and Second-Order Systems

The stability of first- and second-order systems can be determined by inspection of the coefficients of the characteristic polynomial. Both systems are stable, with all roots in the left half of the complex plane, if and only if all polynomial coefficients have the same algebraic sign.

For first-order polynomials, the proof is trivial. Now consider the following second-order polynomial with leading coefficient equal to one.

$$s^2 + a_1 s + a_0 = 0$$

Applying the quadratic formula,

$$s = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0}}{2}$$

$$= -\frac{a_1}{2} \pm \frac{\sqrt{a_1^2 - 4a_0}}{2}$$

Clearly $a_1 > 0$ is required. In addition, for real and distinct roots,

$$-a_1 + \sqrt{a_1^2 - 4a_0} < 0$$

$$\sqrt{a_1^2 - 4a_0} < a_1$$

$$a_1^2 - 4a_0 < a_1^2$$

$$-4a_0 < 0$$

$$\underline{\underline{a_0 > 0}} \leftarrow$$

Higher-Order Systems

For higher-order polynomials representing higher-order systems, the algebraic signs of the polynomial coefficients may or may not yield information as to stability. The following two conditions do result in conclusions about polynomial roots.

1. Differing algebraic signs – At least one RHP root.
2. Zero-valued coefficients – Imaginary axis roots or RHP roots or both.

Examples:

$$s^5 + 4s^4 - 3s^3 + s^2 + 7s + 10 = 0$$

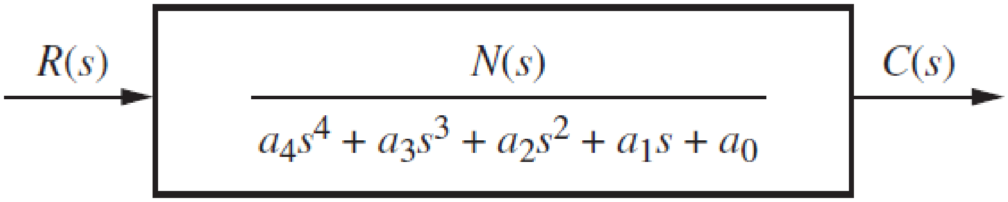
roots = 0.9098 + j 0.7826
0.9098 − j 0.7826
−1.3689
−0.6254 + j 0.7895
−0.6254 − j 0.7895

$$s^4 + 3s^3 + 2s + 6 = 0$$

roots = −3.0000
0.6300 + j 1.0911
0.6300 − j 1.0911
−1.2599

Routh-Hurwitz Criterion

(test for stability when given the characteristic polynomial)



Initial layout for Routh table

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2			
s^1			
s^0			

Completed Routh table

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Routh-Hurwitz criterion

$$
\begin{array}{l|lllll}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & & & \\
s^2 & k_1 & k_2 & & & \\
s^1 & l_1 & & & & \\
s^0 & m_1 & & & &
\end{array}
$$

The number of roots
in the open right half-plane
is equal to
the number of sign changes
in the first column of Routh array.