Modern Control Engineering

Fifth Edition

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2–7 LINEARIZATION OF NONLINEAR MATHEMATICAL MODELS

Nonlinear Systems. A system is nonlinear if the principle of superposition does not apply. Thus, for a nonlinear system the response to two inputs cannot be calculated by treating one input at a time and adding the results.

Although many physical relationships are often represented by linear equations, in most cases actual relationships are not quite linear. In fact, a careful study of physical systems reveals that even so-called "linear systems" are really linear only in limited operating ranges. In practice, many electromechanical systems, hydraulic systems, pneumatic systems, and so on, involve nonlinear relationships among the variables. For example, the output of a component may saturate for large input signals. There may be a dead space that affects small signals. (The dead space of a component is a small range of input variations to which the component is insensitive.) Square-law nonlinearity may occur in some components. For instance, dampers used in physical systems may be linear for low-velocity operations but may become nonlinear at high velocities, and the damping force may become proportional to the square of the operating velocity.

Linearization of Nonlinear Systems. In control engineering a normal operation of the system may be around an equilibrium point, and the signals may be considered small signals around the equilibrium. (It should be pointed out that there are many exceptions to such a case.) However, if the system operates around an equilibrium point and if the signals involved are small signals, then it is possible to approximate the nonlinear system by a linear system. Such a linear system is equivalent to the nonlinear system considered within a limited operating range. Such a linearized model (linear, time-invariant model) is very important in control engineering.

The linearization procedure to be presented in the following is based on the expansion of nonlinear function into a Taylor series about the operating point and the retention of only the linear term. Because we neglect higher-order terms of the Taylor series expansion, these neglected terms must be small enough; that is, the variables deviate only slightly from the operating condition. (Otherwise, the result will be inaccurate.)

Linear Approximation of Nonlinear Mathematical Models. To obtain a linear mathematical model for a nonlinear system, we assume that the variables deviate only slightly from some operating condition. Consider a system whose input is $x(t)$ and output is $y(t)$. The relationship between $y(t)$ and $x(t)$ is given by

$$y = f(x) \tag{2–42}$$

If the normal operating condition corresponds to $\bar{x}$, $\bar{y}$, then Equation (2–42) may be expanded into a Taylor series about this point as follows:

$$\begin{aligned} y &= f(x) \\ &= f(\bar{x}) + \frac{df}{dx}(x - \bar{x}) + \frac{1}{2!}\frac{d^2f}{dx^2}(x - \bar{x})^2 + \cdots \end{aligned} \tag{2–43}$$

where the derivatives $df/dx, d^2f/dx^2, \ldots$ are evaluated at $x = \bar{x}$. If the variation $x - \bar{x}$ is small, we may neglect the higher-order terms in $x - \bar{x}$. Then Equation (2–43) may be written as

$$y = \bar{y} + K(x - \bar{x}) \tag{2–44}$$

where

$$\bar{y} = f(\bar{x})$$

$$K = \left.\frac{df}{dx}\right|_{x=\bar{x}}$$

Equation (2–44) may be rewritten as

$$y - \bar{y} = K(x - \bar{x}) \tag{2–45}$$

which indicates that $y - \bar{y}$ is proportional to $x - \bar{x}$. Equation (2–45) gives a linear mathematical model for the nonlinear system given by Equation (2–42) near the operating point $x = \bar{x}, y = \bar{y}$.

Next, consider a nonlinear system whose output y is a function of two inputs x_1 and x_2, so that

$$y = f(x_1, x_2) \tag{2–46}$$

To obtain a linear approximation to this nonlinear system, we may expand Equation (2–46) into a Taylor series about the normal operating point $\bar{x}_1, \bar{x}_2$. Then Equation (2–46) becomes

$$\begin{aligned} y = f(\bar{x}_1, \bar{x}_2) &+ \left[\frac{\partial f}{\partial x_1}(x_1 - \bar{x}_1) + \frac{\partial f}{\partial x_2}(x_2 - \bar{x}_2)\right] \\ &+ \frac{1}{2!}\left[\frac{\partial^2 f}{\partial x_1^2}(x_1 - \bar{x}_1)^2 + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1 - \bar{x}_1)(x_2 - \bar{x}_2) \right. \\ &\left. + \frac{\partial^2 f}{\partial x_2^2}(x_2 - \bar{x}_2)^2\right] + \cdots \end{aligned}$$

where the partial derivatives are evaluated at $x_1 = \bar{x}_1, x_2 = \bar{x}_2$. Near the normal operating point, the higher-order terms may be neglected. The linear mathematical model of this nonlinear system in the neighborhood of the normal operating condition is then given by

$$y - \bar{y} = K_1(x_1 - \bar{x}_1) + K_2(x_2 - \bar{x}_2)$$

where

$$\bar{y} = f(\bar{x}_1, \bar{x}_2)$$

$$K_1 = \left.\frac{\partial f}{\partial x_1}\right|_{x_1=\bar{x}_1,\, x_2=\bar{x}_2}$$

$$K_2 = \left.\frac{\partial f}{\partial x_2}\right|_{x_1=\bar{x}_1,\, x_2=\bar{x}_2}$$

The linearization technique presented here is valid in the vicinity of the operating condition. If the operating conditions vary widely, however, such linearized equations are not adequate, and nonlinear equations must be dealt with. It is important to remember that a particular mathematical model used in analysis and design may accurately represent the dynamics of an actual system for certain operating conditions, but may not be accurate for other operating conditions.

EXAMPLE 2–5 Linearize the nonlinear equation

$$z = xy$$

in the region $5 \le x \le 7$, $10 \le y \le 12$. Find the error if the linearized equation is used to calculate the value of z when $x = 5$, $y = 10$.

Since the region considered is given by $5 \le x \le 7$, $10 \le y \le 12$, choose $\bar{x} = 6$, $\bar{y} = 11$. Then $\bar{z} = \bar{x}\bar{y} = 66$. Let us obtain a linearized equation for the nonlinear equation near a point $\bar{x} = 6$, $\bar{y} = 11$.

Expanding the nonlinear equation into a Taylor series about point $x = \bar{x}$, $y = \bar{y}$ and neglecting the higher-order terms, we have

$$z - \bar{z} = a(x - \bar{x}) + b(y - \bar{y})$$

where

$$a = \left.\frac{\partial(xy)}{\partial x}\right|_{x=\bar{x},\, y=\bar{y}} = \bar{y} = 11$$

$$b = \left.\frac{\partial(xy)}{\partial y}\right|_{x=\bar{x},\, y=\bar{y}} = \bar{x} = 6$$

Hence the linearized equation is

$$z - 66 = 11(x - 6) + 6(y - 11)$$

or

$$z = 11x + 6y - 66$$

When $x = 5$, $y = 10$, the value of z given by the linearized equation is

$$z = 11x + 6y - 66 = 55 + 60 - 66 = 49$$

The exact value of z is $z = xy = 50$. The error is thus $50 - 49 = 1$. In terms of percentage, the error is 2%.

Design of Feedback Control Systems

FOURTH EDITION

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D1.5 Draw diagrams similar to Figure 1.2 for the following systems:

(a) Control of human skin temperature by sweating

(b) Control of a nuclear reactor

(c) The learning process with feedback, assuming that available study time is a disturbance

1.3 Modeling

Start with a process.

Control engineers must be able to analyze and design systems of many kinds. For example, to design a speed control system for an automobile, it is necessary to understand how the vacuum pressure of an engine affects throttle setting (pneumatics), how temperature and pressure within a cylinder affect the power out as the gas–air mixture from the carburetor explodes (thermodynamics), how the car will respond to the power applied by the pistons in the cylinders (mechanics), and how electrical devices may be created to measure and store important variables like temperature and vacuum pressure (electrical circuits).

Create a model.

In each case it is necessary to create a mathematical model that behaves similarly to the actual system within some operating range. The result is the description of a plant for which a controller and measurement device may then be designed. For example, certain values of a spring-mass-damper may be able to simulate the motion of a car within some range of power applied while other values are needed for different powers applied.

The process of linearization may be used to construct a model that is valid for some range of operating conditions. For example, suppose a system output y (maybe speed) depends on some input x (perhaps power), as represented by Figure 1.3, in which

$$y = f(x) = x^2 \qquad [1.1]$$

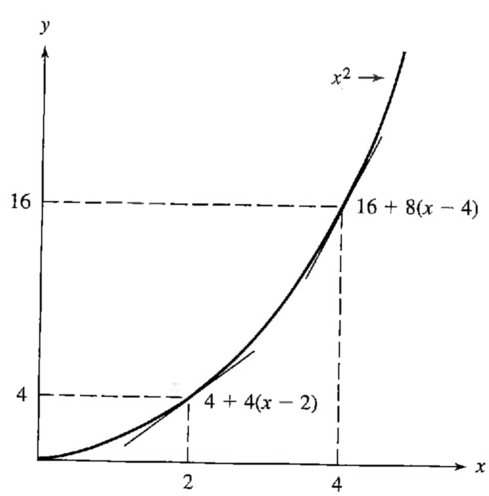

Figure 1.3 Two approximations for $y = x^2$.

Table 1.2 *Two Approximations to* $y = x^2$

x	x^2	$4 + 4(x - 2)$	$16 + 8(x - 4)$
2	4.00	4.00	0.00
2.1	4.41	4.40	0.80
2.2	4.84	4.80	1.60
3.0	9.00	8.00	8.00
4.1	16.81	12.40	16.80

Linearization.

Instead of this nonlinear equation, it may be more useful to create a linear model that operates near some value of x called x_0. A Taylor series approximation to $f(x)$ at the point x_0 is given by the following where $f^1(x_0)$ means that $f(x)$ is differentiated with respect to x and then evaluated when x equals x_0. The tilde symbol ($\sim$) implies an approximation

$$y \sim y_0 + f^1(x_0)(x - x_0) \quad \textbf{[1.2]}$$

$$y \sim x_0^2 + 2x_0(x - x_0) \quad \textbf{[1.3]}$$

If we choose x_0 to be 2, then the approximation of Equation (1.3) becomes

$$y \sim 4 + 4(x - 2) \quad \textbf{[1.4]}$$

Table 1.2 and Figure 1.3 show values of Equation (1.4) near $x_0 = 2$ and also results farther away from $x_0 = 2$.

Notice that Equation (1.4) is good approximation to x^2 for values of x near 2 but that the approximation becomes worse for x values which are higher than 2. For example, if x moves to the vicinity of 4, then Equation (1.3) becomes $16 + 8(x - 4)$, which yields an approximate value of 16.80 at $x = 4.1$, very close to the true value of 16.81 and much better than the value 12.40 that we would get using the other approximation. Even Ohm's famous law that $v = iR$ is good only for some range of voltage versus current. In Figure 1.4 there is a linear region where the slope of v versus i is constant (and Ohm's law applies) and other regions where the slope is not constant (and Ohm's law does not apply).

❑ DRILL PROBLEMS

D1.6 Approximate $y = \sqrt{x}$ for values of $x = 2.2, 2.4, 2.6, 2.8$, and 3.0 by linearizing $\sqrt{x}$ about $x_0 = 2$, using Equation (1.2). Compare approximate values with true values.

Ans. $y \approx 1.414 + 0.354(x - 2)$

x	Approximate	True
2.2	1.485	1.483
2.4	1.556	1.549
2.6	1.626	1.612
2.8	1.697	1.673
3.0	1.768	1.732

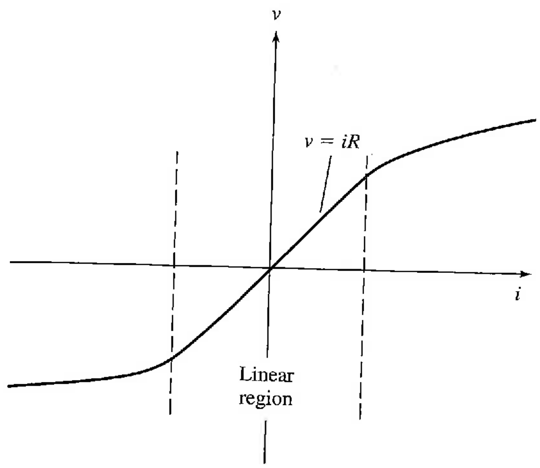

Figure 1.4 Linear region where $v = iR$.

D1.7 If x is in radians, use Equation (1.2) to approximate $y = \sin x$ for values of $x = 0.1, 0.2, 0.3$, and 0.4, by linearizing $\sin x$ about $x_0 = 0$. Compare approximate values with true values.

Ans. $y \approx x$

x (rad)	x (deg)	Approximate	True
0.1	5.7	0.100	0.100
0.2	11.5	0.200	0.199
0.3	17.2	0.300	0.296
0.4	22.9	0.400	0.389

1.4 System Dynamics

A controls engineer usually works from the Laplace-transformed description of a system. Each application has its own unique properties. Some systems are purely electrical while others may employ electrical, hydraulic, and mechanical subsystems, all tied together in a coordinated effort to maintain some desired performance. We shall examine methods for analyzing components of the following types:

Applications.

Electrical (mesh analysis, node analysis, state variables, operational amplifier applications)
Mechanical translational (free-body diagrams and state variables)
Mechanical rotational (free-body diagrams)
Electromechanical
Aerodynamic
Hydraulic
Thermodynamic

Data-Driven Science and Engineering

Machine Learning, Dynamical Systems, and Control

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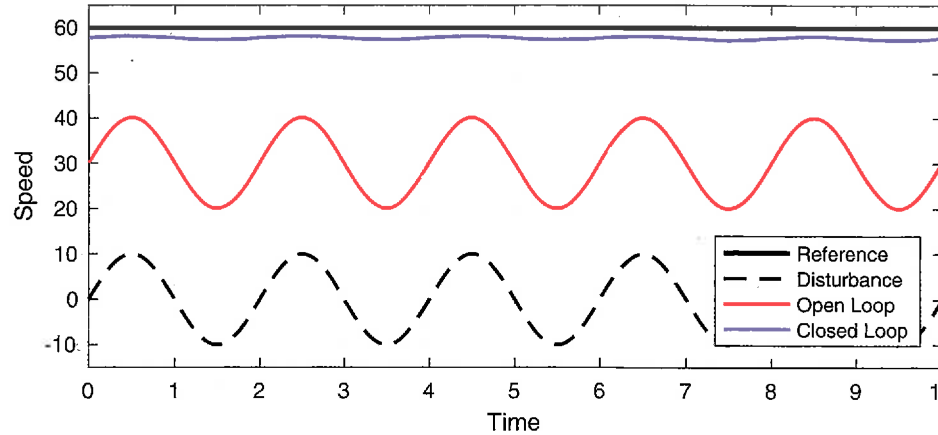

Figure 8.5 Open-loop vs. closed-loop cruise control.

Code 8.1 Compare open-loop and closed-loop cruise control.

```
clear all, close all, clc

t = 0:.01:10;              % time

wr = 60*ones(size(t));     % reference speed
d = 10*sin(pi*t);          % disturbance

aModel = 1;                % y = aModel*u
aTrue = .5;                % y = aTrue*u

uOL = wr/aModel;           % Open-loop u based on model
yOL = aTrue*uOL + d;       % Open-loop response

K = 50;                    % control gain, u=K(wr-y);
yCL = aTrue*K/(1+aTrue*K)*wr + d/(1+aTrue*K);
```

8.2 Linear Time-Invariant Systems

The most complete theory of control has been developed for linear systems [492, 165, 22]. Linear systems are generally obtained by linearizing a nonlinear system about a fixed point or a periodic orbit. However, instability may quickly take a trajectory far away from the fixed point. Fortunately, an effective stabilizing controller will keep the state of the system in a small neighborhood of the fixed point where the linear approximation is valid. For example, in the case of the inverted pendulum, feedback control may keep the pendulum stabilized in the vertical position where the dynamics behave linearly.

Linearization of Nonlinear Dynamics

Given a nonlinear input–output system

$$\frac{d}{dt}\mathbf{x} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \tag{8.7a}$$

$$\mathbf{y} = \mathbf{g}(\mathbf{x}, \mathbf{u}) \tag{8.7b}$$

it is possible to linearize the dynamics near a fixed point $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ where $\mathbf{f}(\bar{\mathbf{x}}, \bar{\mathbf{u}}) = \mathbf{0}$. For small $\Delta\mathbf{x} = \mathbf{x} - \bar{\mathbf{x}}$ and $\Delta\mathbf{u} = \mathbf{u} - \bar{\mathbf{u}}$ the dynamics $\mathbf{f}$ may be expanded in a Taylor series about the point $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$:

$$\mathbf{f}(\bar{\mathbf{x}} + \Delta\mathbf{x}, \bar{\mathbf{u}} + \Delta\mathbf{u}) = \mathbf{f}(\bar{\mathbf{x}}, \bar{\mathbf{u}}) + \underbrace{\left.\frac{\mathbf{df}}{\mathbf{dx}}\right|_{(\bar{\mathbf{x}}, \bar{\mathbf{u}})}}_{\mathbf{A}} \cdot \Delta\mathbf{x} + \underbrace{\left.\frac{\mathbf{df}}{\mathbf{du}}\right|_{(\bar{\mathbf{x}}, \bar{\mathbf{u}})}}_{\mathbf{B}} \cdot \Delta\mathbf{u} + \cdots. \tag{8.8}$$

Similarly, the output equation $\mathbf{g}$ may be expanded as:

$$\mathbf{g}(\bar{\mathbf{x}} + \Delta\mathbf{x}, \bar{\mathbf{u}} + \Delta\mathbf{u}) = \mathbf{g}(\bar{\mathbf{x}}, \bar{\mathbf{u}}) + \underbrace{\left.\frac{\mathbf{dg}}{\mathbf{dx}}\right|_{(\bar{\mathbf{x}}, \bar{\mathbf{u}})}}_{\mathbf{C}} \cdot \Delta\mathbf{x} + \underbrace{\left.\frac{\mathbf{dg}}{\mathbf{du}}\right|_{(\bar{\mathbf{x}}, \bar{\mathbf{u}})}}_{\mathbf{D}} \cdot \Delta\mathbf{u} + \cdots. \tag{8.9}$$

For small displacements around the fixed point, the higher order terms are negligibly small. Dropping the Δ and shifting to a coordinate system where $\bar{\mathbf{x}}$, $\bar{\mathbf{u}}$, and $\bar{\mathbf{y}}$ are at the origin, the linearized dynamics may be written as:

$$\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{8.10a}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}. \tag{8.10b}$$

Note that we have neglected the disturbance and noise inputs, $\mathbf{w}_d$ and $\mathbf{w}_n$, respectively; these will be added back in the discussion on Kalman filtering in Section 8.5.

Unforced Linear System

In the absence of control (i.e., $\mathbf{u} = \mathbf{0}$), and with measurements of the full state (i.e., $\mathbf{y} = \mathbf{x}$), the dynamical system in (8.10) becomes

$$\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x}. \tag{8.11}$$

The solution $\mathbf{x}(t)$ is given by

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0), \tag{8.12}$$

where the matrix exponential is defined by:

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2} + \frac{\mathbf{A}^3t^3}{3} + \cdots. \tag{8.13}$$

The solution in (8.12) is determined entirely by the eigenvalues and eigenvectors of the matrix $\mathbf{A}$. Consider the eigendecomposition of $\mathbf{A}$:

$$\mathbf{A}\mathbf{T} = \mathbf{T}\mathbf{\Lambda}. \tag{8.14}$$

In the simplest case, $\mathbf{\Lambda}$ is a diagonal matrix of distinct eigenvalues and $\mathbf{T}$ is a matrix whose columns are the corresponding linearly independent eigenvectors of $\mathbf{A}$. For repeated eigenvalues, $\mathbf{\Lambda}$ may be written in Jordan form, with entries above the diagonal for degenerate eigenvalues of multiplicity ≥ 2; the corresponding columns of $\mathbf{T}$ will be generalized eigenvectors.