ECE317 : Feedback and Control

Lecture :
Routh-Hurwitz stability criterion
Examples

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Course roadmap

Modeling
- Laplace transform
- Transfer function
- Block Diagram
- Linearization
- Models for systems
  - electrical
  - mechanical
  - example system

Analysis
- Stability
  - Pole locations
  - Routh-Hurwitz
- Time response
  - Transient
    - Steady state (error)
- Frequency response
  - Bode plot

Design
- Design specs
- Frequency domain
- Bode plot
- Compensation
- Design examples

Matlab & PECS simulations & laboratories
Definitions of stability (review)

• **BIBO** (Bounded-Input-Bounded-Output) **stability**
  *Any bounded input generates a bounded output.*

• **Asymptotic stability**
  *Any ICs generates y(t) converging to zero.*
Stability summary (review)

Let $s_i$ be poles of $G(s)$. Then, $G(s)$ is ...

- (BIBO, asymptotically) stable if $\Re(s_i) < 0$ for all $i$.
- marginally stable if
  - $\Re(s_i) \leq 0$ for all $i$, and
  - simple pole for $\Re(s_i) = 0$
- unstable if it is neither stable nor marginally stable.
Routh-Hurwitz criterion (review)

• This is for LTI systems with a *polynomial* denominator (without sin, cos, exponential etc.)

• It determines if all the roots of a polynomial
  • lie in the open LHP (left half-plane),
  • or equivalently, have negative real parts.

• It also determines the number of roots of a polynomial in the open RHP (right half-plane).

• It does **NOT** explicitly compute the roots.

• No proof is provided in any control textbook.
## Routh array (review)

From the given polynomial:

\[ Q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 \]

<table>
<thead>
<tr>
<th>( s^n )</th>
<th>( a_n )</th>
<th>( a_{n-2} )</th>
<th>( a_{n-4} )</th>
<th>( a_{n-6} )</th>
<th>( \cdots )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s^{n-1} )</td>
<td>( a_{n-1} )</td>
<td>( a_{n-3} )</td>
<td>( a_{n-5} )</td>
<td>( a_{n-7} )</td>
<td>( \cdots )</td>
</tr>
<tr>
<td>( s^{n-2} )</td>
<td>( b_1 )</td>
<td>( b_2 )</td>
<td>( b_3 )</td>
<td>( b_4 )</td>
<td>( \cdots )</td>
</tr>
<tr>
<td>( s^{n-3} )</td>
<td>( c_1 )</td>
<td>( c_2 )</td>
<td>( c_3 )</td>
<td>( c_4 )</td>
<td>( \cdots )</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( s^2 )</td>
<td>( k_1 )</td>
<td>( k_2 )</td>
<td></td>
<td></td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( s^1 )</td>
<td>( l_1 )</td>
<td></td>
<td></td>
<td></td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( s^0 )</td>
<td>( m_1 )</td>
<td></td>
<td></td>
<td></td>
<td>( \vdots )</td>
</tr>
</tbody>
</table>
Routh array
(How to compute the third row)

<table>
<thead>
<tr>
<th>$s^n$</th>
<th>$a_n$</th>
<th>$a_{n-2}$</th>
<th>$a_{n-4}$</th>
<th>$a_{n-6}$</th>
<th>$\cdots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s^{n-1}$</td>
<td>$a_{n-1}$</td>
<td>$a_{n-3}$</td>
<td>$a_{n-5}$</td>
<td>$a_{n-7}$</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>$s^{n-2}$</td>
<td>$b_1$</td>
<td>$b_2$</td>
<td>$b_3$</td>
<td>$b_4$</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>$s^{n-3}$</td>
<td>$c_1$</td>
<td>$c_2$</td>
<td>$c_3$</td>
<td>$c_4$</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$s^2$</td>
<td>$k_1$</td>
<td>$k_2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s^1$</td>
<td>$l_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s^0$</td>
<td>$m_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
\begin{align*}
  b_1 & = \frac{a_{n-2}a_{n-1} - a_n a_{n-3}}{a_{n-1}} \\
  b_2 & = \frac{a_{n-4}a_{n-1} - a_n a_{n-5}}{a_{n-1}} \\
  & \vdots
\end{align*}
\]
Routh array
(How to compute the fourth row)

<table>
<thead>
<tr>
<th>$s^n$</th>
<th>$a_n$</th>
<th>$a_{n-2}$</th>
<th>$a_{n-4}$</th>
<th>$a_{n-6}$</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s^{n-1}$</td>
<td>$a_{n-1}$</td>
<td>$a_{n-3}$</td>
<td>$a_{n-5}$</td>
<td>$a_{n-7}$</td>
<td>...</td>
</tr>
<tr>
<td>$s^{n-2}$</td>
<td>$b_1$</td>
<td>$b_2$</td>
<td>$b_3$</td>
<td>$b_4$</td>
<td>...</td>
</tr>
<tr>
<td>$s^{n-3}$</td>
<td>$c_1$</td>
<td>$c_2$</td>
<td>$c_3$</td>
<td>$c_4$</td>
<td>...</td>
</tr>
</tbody>
</table>

For $s^{n-3}$ row:

\[
c_1 = \frac{a_{n-3}b_1 - a_{n-1}b_2}{b_1} \]
\[
c_2 = \frac{a_{n-5}b_1 - a_{n-1}b_3}{b_1} \]

...
Routh-Hurwitz criterion

The number of roots in the open right half-plane is equal to the number of sign changes in the first column of Routh array.
Example 1

\[ Q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10 \]

Routh array

| \( s^5 \) | 1   | 2   | 11  |
| \( s^4 \) | 2   | 4   | 10  |
| \( s^3 \) | 0 \( \epsilon \) | 6   |
| \( s^2 \) | \( \frac{4\epsilon - 12}{\epsilon} \) | 10  |
| \( s^1 \) | \( \approx 6 \) |
| \( s^0 \) | 10  |

If 0 appears in the first column of a nonzero row in Routh array, replace it with a small positive number. In this case, \( Q \) has some roots in RHP.

Two sign changes in the first column \( \Rightarrow \) Two roots in RHP

\[ \epsilon \rightarrow \frac{4\epsilon - 12}{\epsilon} \rightarrow 6 \]
Example 2

$$Q(s) = s^4 + s^3 + 3s^2 + 2s + 2$$

Routh array

<table>
<thead>
<tr>
<th>$s^4$</th>
<th>1</th>
<th>3</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s^3$</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$s^2$</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$s^1$</td>
<td>0</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$s^0$</td>
<td>2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

If zero row appears in Routh array, $Q$ has roots either on the imaginary axis or in RHP.

No sign changes in the first column $\rightarrow$ No roots in RHP

But some roots are on imag. axis.

Take derivative of an auxiliary polynomial (which is a factor of $Q(s)$) $s^2 + 2$
Example 3

\[ Q(s) = s^3 + s^2 + s + 1 \quad (= (s + 1)(s^2 + 1)) \]

Routh array

\[
\begin{array}{c|cc}
  & s^3 & 1 \\
  & s^2 & 1 \\
  s^1 & 0 & 2 \\
  s^0 & 1 & \\
\end{array}
\]

Derivative of auxiliary poly.

\[(s^2 + 1)' = 2s\]

(Auxiliary poly. is a factor of \(Q(s)\).)

No sign changes in the first column

No root in OPEN(!) RHP
Example 4

\[ Q(s) = s^5 + s^4 + 2s^3 + 2s^2 + s + 1 = (s+1)(s^2+1)^2 \]

Routh array

\[
\begin{array}{c|ccc}
 s^5 & 1 & 2 & 1 \\
 s^4 & 1 & 2 & 1 \\
 s^3 & 0 & 4 & 4 \\
 s^2 & 1 & 1 & \text{no root in OPEN(!) RHP} \\
 s^1 & 0 & 2 & \text{no sign changes in the first column} \\
 s^0 & 1 & & \\
\end{array}
\]

Derivative of auxiliary poly.

\[(s^4 + 2s^2 + 1)' = 4s^3 + 4s\]

\[(s^2 + 1)' = 2s\]
Example 5

\[ Q(s) = s^4 - 1 = (s + 1)(s - 1)(s^2 + 1) \]

Routh array

\[
\begin{array}{ccc|c}
s^4 & 1 & 0 & -1 \\
s^3 & 4 & 0 & 0 \\
s^2 & 0 & \varepsilon & -1 \\
s^1 & 4/\varepsilon & 4 & 0 \\
s^0 & -1 & 0 & 0 \\
\end{array}
\]

Derivative of auxiliary poly.

\[ (s^4 - 1)' = 4s^3 \]

One sign changes in the first column \[ \rightarrow \] One root in OPEN(!) RHP
Notes on Routh-Hurwitz criterion

• Advantages
  • No need to explicitly compute roots of the polynomial.
    • High order $Q(s)$ can be handled by hand calculations.
  • Polynomials including undetermined parameters (plant and/or controller parameters in feedback systems) can be dealt with.
    • Root computation does not work in such cases!

• Disadvantage
  • Exponential functions (delay) cannot be dealt with.
    • Example: $Q(s) = e^{-s} + s^2 + s + 1$
Example 6

\[ Q(s) = s^3 + 3Ks^2 + (K + 2)s + 4 \]

Find the range of K s.t. \( Q(s) \) has all roots in the left half plane. (Here, K is a design parameter.)

Routh array

\[
\begin{array}{c|cc}
\text{s}^3 & 1 & K + 2 \\
\text{s}^2 & 3K & 4 \\
\text{s}^1 & \frac{3K(K + 2) - 4}{3K} & \text{No sign changes in the first column} \\
\text{s}^0 & 4 & \begin{cases} 3K > 0 \\
3K(K + 2) - 4 > 0 \\
K > -1 + \frac{\sqrt{21}}{3} \end{cases}
\end{array}
\]
Example 7

- Design $K(s)$ that stabilizes the closed-loop system for the following cases.
  - $K(s) = K$ (constant, P controller)
  - $K(s) = K_P + K_I/s$ (PI (Proportional-Integral) controller)
Example 7: \( K(s) = K \)

- **Characteristic equation**

\[
1 + K \frac{2}{s^3 + 4s^2 + 5s + 2} = 0
\]

\[
\Rightarrow s^3 + 4s^2 + 5s + 2 + 2K = 0
\]

- **Routh array**

| \( s^3 \) | 1 | 5 |
| \( s^2 \) | 4 | \( 2 + 2K \) |
| \( s^1 \) | \( \frac{18 - 2K}{4} \) |
| \( s^0 \) | \( 2 + 2K \) |

\[
-1 < K < 9
\]
Example 7: $K(s)=K_P+K_I/s$

- Characteristic equation

$$1 + \left(K_P + \frac{K_I}{s}\right) \frac{2}{s^3 + 4s^2 + 5s + 2} = 0$$

$$s^4 + 4s^3 + 5s^2 + (2 + 2K_P)s + 2K_I = 0$$

- Routh array

<table>
<thead>
<tr>
<th>$s^4$</th>
<th>$s^3$</th>
<th>$s^2$</th>
<th>$s^1$</th>
<th>$s^0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>$\frac{18-2K_P}{4}$</td>
<td>(*)</td>
<td>$2K_I$</td>
</tr>
<tr>
<td>5</td>
<td>2 + $2K_P$</td>
<td>$2K_I$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2$K_I$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2K_I$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$K_P < 9$

$K_I > 0$
Example 7: Range of \((K_P, K_I)\)

- From Routh array,
  
  \[ K_P < 9 \]
  
  \[ K_I > 0 \]
  
  \[(*) \iff (1 + K_P)(9 - K_P) - 8K_I > 0\]

CL system: stable

CL system: unstable
Example 8

- Determine the range of $K$ that stabilize the closed-loop system.
Example 8 (cont’d)

\[ \frac{1}{(s + 2)(s + 3)} \]
Example 8 (cont’d)

• Characteristic equation

\[ 1 + K \frac{1}{(s+2)(s+3)} \cdot \frac{1}{s} = 0 \]

\[ 1 + K \cdot \frac{1}{s(s + 2)(s + 3) + s} = 0 \]

\[ s(s + 2)(s + 3) + s + K = 0 \]

\[ s^3 + 5s^2 + 7s + K = 0 \]
Example 8 (cont’d)

• Routh array

\[
\begin{array}{c|cc}
  s^3 & 1 & 7 \\
  s^2 & 5 & K \\
  s^1 & \frac{35-K}{5} & \\
  s^0 & K & \\
\end{array}
\]

\[0 < K < 35\]

• If \(K=35\), the closed-loop system is marginally stable. Output signal will oscillate with frequency corresponding to

\[
\frac{1}{5s^2 + 35} = \frac{1}{5} \cdot \frac{1}{s^2 + 7} = \frac{1}{5} \cdot \frac{1}{s^2 + (\sqrt{7})^2}
\]
Summary

• Examples for Routh-Hurwitz criterion
  • Cases when zeros appear in Routh array
  • P controller gain range for stability
  • PI controller gain range for stability

• Next
  • Frequency response