Chapter 5: Circuit Theorems

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5.1 Motivation

If you are given the following circuit, are there any other alternative(s) to determine the voltage across 2Ω resistor?

In Chapter 4, a circuit is analyzed without tampering with its original configuration.

*What are they? And how? Can we work it out by inspection?*

In Chapter 5, some theorems have been developed to simplify circuit analysis such as Thevenin’s and Norton’s theorems. The theorems are applicable to *linear circuits*.

*Discussion: Source Transformation, Linearity, & superposition.*
5.2 Source Transformation (1)

- Like series-parallel combination and wye-delta transformation, source transformation is another tool for simplifying circuits.
- An **equivalent circuit** is one whose $v$-$i$ characteristics are identical with the original circuit.
- A **source transformation** is the process of replacing a voltage source $v_s$ in series with a resistor $R$ by a current source $i_s$ in parallel with a resistor $R$, and vice versa.

  - **Transformation of independent sources**
    - The arrow of the current source is directed toward the positive terminal of the voltage source.
    - The source transformation is not possible when $R = 0$ for voltage source and $R = \infty$ for current source.

  - **Transformation of dependent sources**
A voltage source $v_s$ connected in series with a resistor $R_s$ and a current source $i_s$ is connected in parallel with a resistor $R_p$ are equivalent circuits provided that

$$R_p = R_s \quad \& \quad v_s = R_s \, i_s$$
Example: Find $v_o$ in the circuit using source transformation.

\[ i = \frac{2}{2 + 8} = 0.4 \text{ A} \]

\[ v_o = 8i = 8(0.4) = 3.2 \text{ V} \]
Example: Find $i_o$ in the circuit using source transformation.

Combining the 6-Ω and 3-Ω resistors in parallel gives 2Ω. Adding the 1-Ω and 4-Ω resistors in series gives $1 + 4 = 5Ω$. Transforming the left current source in parallel with the 2-Ω resistor gives the equivalent circuit.

*Refer to in-class illustration, textbook, answer $i_0 = 1.78$ A
5.3 Superposition Theorem (1)

- **Superposition** states that the voltage across (or current through) an element in a linear circuit is the algebraic sum of the voltage across (or currents through) that element due to **EACH independent source acting alone**.

- The principle of superposition helps us to analyze a linear circuit with more than one independent source by calculating the contribution of each independent source separately.

- **Steps to Apply Superposition Principle**:
  1. Turn off all indep. sources except one source. Find the output (v or i) due to that active source using techniques in Chapters 2 & 3.
  2. Repeat Step 1 for each of the other indep. sources.
  3. Find Total contribution by adding all contributions from indep. sources.

**Note**: In Step 1, this implies that we replace every **voltage source by 0 V** (or a **short circuit**), and every **current source by 0 A** (or an **open circuit**). Dependent sources are left intact because they are controlled by others.
5.3 Superposition Theorem (2)

**Example:** Use the superposition theorem to find $v$ in the circuit.

\[ v = v_1 + v_2 = 2 + 8 = 10 \text{ V} \]

\[
\begin{align*}
12i_1 - 6 &= 0 \\
\Rightarrow \quad i_1 &= 0.5 \text{ A} \\
v_1 &= 4i_1 = 2 \text{ V}
\end{align*}
\]

\[
\begin{align*}
i_3 &= \frac{8}{4 + 8(3)} = 2 \text{ A} \\
v_2 &= 4i_3 = 8 \text{ V}
\end{align*}
\]
5.3 Superposition Theorem (3)

Example: Use superposition to find $v_x$ in the circuit.

2A is discarded by open circuit

10V is discarded by open circuit

Dependant source keep unchanged
5.4 Thevenin’s Theorem (1)

It states that a linear two-terminal circuit (Fig. a) can be replaced by an equivalent circuit (Fig. b) consisting of a voltage source $V_{Th}$ in series with a resistor $R_{Th}$, where

- $V_{Th}$ is the open-circuit voltage at the terminals.
- $R_{Th}$ is the input or equivalent resistance at the terminals when the independent sources are turned off.

$V_{Th} = v_{oc}$

$R_{Th} = R_{in}$
To find $R_{Th}$:

**Case 1**: If the network has no dependent sources, we turn off all indep. Source. $R_{Th}$ is the input resistance of the network looking btw terminals $a$ & $b$.

**Case 2**: If the network has depend. Sources. Depend. sources are not to be turned off because they are controlled by circuit variables. (a) Apply $v_o$ at $a$ & $b$ and determine the resulting $i_o$. Then $R_{Th} = v_o/i_o$. Alternatively, (b) insert $i_o$ at $a$ & $b$ and determine $v_o$. Again $R_{Th} = v_o/i_o$. 

(a) Circuit with all independent sources set equal to zero

(b) Circuit with all independent sources set equal to zero

\[ R_{Th} = \frac{v_o}{i_o} \]
Example: Find the Thevenin equivalent circuit at the terminals $a$ & $b$.

\[-32 + 4i_1 + 12(i_1 - i_2) = 0, \quad i_2 = -2 \text{ A}\]

for $i_1$, we get $i_1 = 0.5 \text{ A}$. Thus,

\[V_{Th} = 12(i_1 - i_2) = 12(0.5 + 2.0) = 30 \text{ V}\]

\[R_{Th} = 4 \parallel 12 + 1 = \frac{4 \times 12}{16} + 1 = 4 \Omega\]

Indep. voltage source as a short circuit & the current source as an open circuit.
5.4 Thevenin’s Theorem (4)

**Example:** Find the Thevenin equivalent circuit with *dep. source*.

1. Indep. voltage source as a short circuit & the current source as an open circuit.
2. Set \( v_0 = 1 \) V to excite the circuit, and then to find \( i_0 \). Then \( R_{Th} = v_0 / i_0 \).

\[
\begin{align*}
R_{Th} &= \frac{1 \text{ V}}{1/6 \text{ A}} = 6 \Omega \\
v_0 &= 1 \text{ V} \\
-2v_x + 2(i_1 - i_2) &= 0 \quad \text{or} \quad v_x = i_1 - i_2 \\
-4i_2 &= v_x = i_1 - i_2; \quad \text{hence,} \quad i_1 = -3i_2 \\
4i_2 + 2(i_2 - i_1) + 6(i_2 - i_3) &= 0 \\
6(i_3 - i_2) + 2i_3 + 1 &= 0 \quad \therefore i_3 = -\frac{1}{6} \text{ A} \\
i_1 &= 5 \\
-2v_x + 2(i_3 - i_2) &= 0 \quad \Rightarrow \quad v_x = i_3 - i_2 \\
4(i_3 - i_1) + 2(i_3 - i_3) + 6i_2 &= 0 \\
4(i_1 - i_2) &= v_x. \quad \text{leads to} \quad i_2 = 10/3. \\
V_{Th} &= v_{oc} = 6i_2 = 20 \text{ V}
\end{align*}
\]
**Example:** Find the Thevenin equivalent circuit with *only dep. source.*

\[
\begin{align*}
V_{Th} & = 0 \\
\text{Using the Thevenin equivalent is quite easy since we have only one loop, as shown in Fig.}
\end{align*}
\]

\[
\begin{align*}
2(-v_o/2) + (v_o - 0)/4 + (v_o - 0)/2 + (-1) & = 0 \\
& = (-1 + \frac{1}{4} + \frac{1}{2})v_o - 1 \quad \text{or} \quad v_o = -4 \text{ V}
\end{align*}
\]

\[
v_o = 1 \times R_{Th}, \text{ then } R_{Th} = v_o/1 = -4 \Omega.
\]

\[
\begin{align*}
(4 + 2 - 8)i_1 + (-2 + 8)i_2 & = 0 \\
-2i_1 + 6i_2 & = 0 \quad \text{or} \quad i_1 = 3i_2 \\
-2i_1 + 11i_2 & = -10
\end{align*}
\]

Substituting the first equation into the second gives

\[
\begin{align*}
-6i_2 + 11i_2 & = -10 \quad \text{or} \quad i_2 = -10/5 = -2 \text{ A}
\end{align*}
\]
5.5 Norton’s Theorem (1)

It states that a linear two-terminal circuit (Fig. a) can be replaced by an equivalent circuit (Fig. b) consisting of a current source $I_N$ in parallel with a resistor $R_N$,

(a) \[\text{Linear two-terminal circuit}\]  \hspace{1cm} (b) \[\text{Current source } I_N, \hspace{1cm} R_N\]

where

- $I_N$ is the short-circuit current through the terminals.
- $R_N$ is the input or equivalent resistance at the terminals when the indepen. sources are turned off.

\[ R_N = R_{Th} \hspace{1cm} R_{Th} = R_{in} \]
5.5 Norton’s Theorem (2)

The Thevenin and Norton equivalent circuits are related by a source transformation.

\[ V_{TH} = V_{oc} \]
\[ I_N = I_{sc} \]
\[ R_{TH} = \frac{V_{oc}}{I_{sc}} = R_N \]

**Example:** Find the Norton equivalent circuit at the terminals \(a\) & \(b\).

1. **Thevenin Circuit:**
   - \(V_{TH} = 4\) V
   - \(I_N = I_{sc} = 1\) A
   - \(R_{TH} = R_N = 4\) Ω

2. **Norton Circuit:**
   - \(V_{oc}/I_{sc} = 4\) Ω
   - \(V_{TH} = V_{oc} = 4\) V

\[ V_{TH} = V_{oc} = 4\text{V} \]

\[ I_N = I_{sc} = 1\text{A} \]

\[ R_{TH} = R_N = 4\text{Ω} \]
5.5 Norton’s Theorem (3)

Example: Find the Norton equivalent circuit with dep. source.

\[
I_N = i_{sc}
\]

\[
R_N
\]
5.6 Maximum Power Transfer (1)

- There are applications where it is desirable to maximize the power delivered to a load. Also, power utility systems are designed to transport the power to the load with the greatest efficiency by reducing the losses on the power lines.

- If the entire circuit is replaced by its **Thevenin equivalent** except for the load, the power delivered to the load is:

\[
    P = i^2 R_L = \left( \frac{V_{Th}}{R_{Th} + R_L} \right)^2 R_L
\]

- **Maximum power** is transferred to the load resistance equals the Thevenin resistance as seen from the load.

\[
    R_L = R_{Th} \quad \Rightarrow \quad P_{\text{max}} = \frac{V_{Th}^2}{4R_L}
\]

The power transfer profile with different $R_L$.
Example: Determine the value of $R_L$ that will draw the maximum power. Calculate the maximum power.

\[
R_L = R_{Th} \Rightarrow P_{\text{max}} = \frac{V_{Th}^2}{4R_L}
\]

Fig. (a) => To determine $R_{Th}$

Fig. (b) => To determine $V_{Th}$

*Refer to in-class illustration, textbook, $R_L = 4.22W$, $P_m = 2.901W$
5.6 Maximum Power Transfer (3)

- Practical voltage

- Practical current

- To measure $v_s$ and $R_s$:
5.7 Summary

Table 5.11-1  Source Transformations

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<thead>
<tr>
<th>THÉVENIN CIRCUIT</th>
<th>NORTON CIRCUIT</th>
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<tbody>
<tr>
<td><img src="image1" alt="Thévenin Circuit" /></td>
<td><img src="image2" alt="Norton Circuit" /></td>
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Table 5.11-2  Thévenin and Norton Equivalent Circuits

<table>
<thead>
<tr>
<th>ORIGINAL CIRCUIT</th>
<th>THÉVENIN CIRCUIT</th>
<th>NORTON EQUIVALENT CIRCUIT</th>
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</thead>
<tbody>
<tr>
<td><img src="image3" alt="Original Circuit" /></td>
<td><img src="image4" alt="Thévenin Circuit" /></td>
<td><img src="image5" alt="Norton Circuit" /></td>
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