Portland State University

ECE241

Introduction to Electrical Engineering
Prefixes

Prefixes modify some basic unit. Engineering notation refers to those prefixes where the exponent is a multiple of three. For example, the most commonly used prefixes for resistance (Ohm), inductance (Henry), and capacitance (Farad) are:

<table>
<thead>
<tr>
<th>Prefix</th>
<th>Symbol</th>
<th>Power of Ten</th>
<th>Ohm</th>
<th>Henry</th>
<th>Farad</th>
</tr>
</thead>
<tbody>
<tr>
<td>pico</td>
<td>p</td>
<td>$10^{-12}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>nano</td>
<td>n</td>
<td>$10^{-9}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>micro</td>
<td>µ</td>
<td>$10^{-6}$</td>
<td></td>
<td>µH</td>
<td>µF</td>
</tr>
<tr>
<td>milli</td>
<td>m</td>
<td>$10^{-3}$</td>
<td></td>
<td>mH</td>
<td></td>
</tr>
<tr>
<td>kilo</td>
<td>k</td>
<td>$10^{-3}$</td>
<td></td>
<td>kΩ</td>
<td>H</td>
</tr>
<tr>
<td>mega</td>
<td>M</td>
<td>$10^6$</td>
<td></td>
<td>MΩ</td>
<td></td>
</tr>
</tbody>
</table>

Voltage and Current

Voltage is created by the separation of charge, and current is created by the movement of charge. By definition,

\[
\frac{dv}{dq} \quad \frac{di}{dt}
\]

where

\[
v = \text{voltage in volts} \quad i = \text{current in amperes}
\]

\[
w = \text{energy in joules} \quad q = \text{charge in coulombs}
\]

\[
g = \text{charge in coulombs} \quad t = \text{time in seconds}
\]

These definitions of voltage and current describe magnitude but not polarity. Voltage requires a reference polarity (plus and minus) and current requires a reference direction for positive charge movement.
Basic Circuit Elements

Basic circuit elements are described in terms of the circuit variables voltage and/or current. The assignment of the reference polarity for voltage and the reference direction for current is completely arbitrary. However, all equations written must agree with the chosen references. In addition, the passive sign convention must be observed:

- Positive sign with voltage/current expression

- Negative sign with voltage/current expression

Power and Energy

Power is defined as the time rate of expending or absorbing energy. By definition,

\[ P = \frac{dw}{dt} \quad P = vi \]

Where

- \( p \) = power in watts
- \( w \) = energy in joules
- \( t \) = time in seconds
- \( v \) = voltage in volts
- \( i \) = current in amperes

The passive sign convention applied to the power equation is as follows:

- Positive sign: \( p = +vi \)
- Negative sign: \( p = -vi \)
An **ideal voltage source** is a circuit element that produces $v_s$ volts regardless of the current in the device. The symbols are

![Diagram of an ideal voltage source](image)

which denotes that terminal 1 is $v_s$ volts higher than that of terminal 2. However, the quantity $v_s$ can be either a positive or negative number. If $v_s$ is negative, the polarity of the voltage can be reversed to achieve an equivalent voltage source where $v_s$ is positive.

![Diagram of reversed voltage source](image)

Another symbol that is sometimes used when $v_s$ is a constant is that of the **ideal battery**.

![Diagram of an ideal battery](image)
The graph of voltage versus time for this device is

\[ v_s, \text{volts} \]

\[ t, \text{time} \]

An ideal current source is a circuit element that produces \( i_s \) amperes regardless of the voltage across the device. The symbols are

- \( i_s \) \text{ \( \uparrow \)}
  \hspace{1cm} \text{independent current source}

- \( i_s \) \text{ \( \downarrow \)}
  \hspace{1cm} \text{dependent current source}

\[ i_s = \begin{cases} \alpha v_x \\ \beta i_x \end{cases} \]

which denotes that \( i_s \) amperes flows from terminal 2 to terminal 1.

If \( i_s \) is negative, the direction of the current can be reversed to achieve an equivalent current source where \( i_s \) is positive.

\[ -3A \uparrow = 3A \downarrow \]
Consider the following circuit.

The graph of $v$ versus $i$ is

The proportionality constant for the straight line is called resistance $R$. The unit of resistance is the ohm. Therefore,

$$v = iR$$

and this equation is called Ohm's law.

The passive sign convention applied to Ohm's law is as follows.

$$v = +iR$$

$$v = -iR$$
Example:

Use Ohm's law to find the missing quantities.

\[ v_1 = +i_1R_1 = +\left(-\frac{1}{3}\right)(2) = -\frac{2}{3}V \]

\[ i_2 = \frac{v_2}{R_2} = +\frac{8}{3} = \frac{8}{3}A \]

\[ v_3 = -i_3R_3 = -\left(\frac{11}{9}\right)(1) = -\frac{11}{9}V \]

\[ i_4 = -\frac{v_4}{R_4} = -\frac{7}{3} = \frac{7}{3}A \]

\[ R_5 = -\frac{v_5}{i_5} = -\frac{14}{9} = 2\Omega \]
KVL: The algebraic sum of all voltages around a loop is zero.

\[ \sum_{L} V = v_1 - v_2 - v_3 + v_4 - v_5 + v_6 = 0 \]

\[ v_1 + v_4 + v_6 = v_2 + v_3 + v_5 \]

**Example:**

Consider a loop from the previous example.

\[ \sum_{L} V = \left( -\frac{2}{3} \right) - \frac{11}{9} + v_6 - 2 = 0 \]

\[ v_6 = \frac{6}{9} + \frac{11}{9} + \frac{18}{9} \]

\[ = \frac{35}{9} V \]
Kirkhoff's Current Law

KCL: The algebraic sum of all currents around a node is zero.

\[ \sum_{n} i = i_1 - i_2 + i_3 + i_4 - i_5 - i_6 = 0 \]

\[ i_1 + i_3 + i_4 = i_2 + i_5 + i_6 \]

Example:

\[ \sum_{n} i = (-\frac{1}{3}) - \frac{8}{9} + 2 + i_5 = 0 \]

\[ i_5 = \frac{3}{9} + \frac{8}{9} - \frac{18}{9} \]

\[ = -\frac{7}{9} A \]
Circuit elements connected in series carry the same current. When the circuit elements are resistors, the circuit becomes

\[ V_s \uparrow i_s R_1 \quad R_2 \quad R_3 \quad R_4 \]

\[ R_7 \quad R_6 \quad R_5 \]

This circuit can be simplified to

\[ V_s \uparrow i_s R_{eq} \]

where

\[ R_{eq} = \sum_{i=1}^{k} R_i = R_1 + R_2 + \ldots + R_k \]

Circuit elements connected in parallel have the same voltage. For resistors,

\[ V_s \uparrow i_s R_1 \quad \frac{1}{R_2} \quad \frac{1}{R_3} \quad \frac{1}{R_4} \]

This circuit can also be simplified to

\[ V_s \uparrow i_s R_{eq} \]
where
\[ \frac{1}{R_{eq}} = \sum_{i=1}^{K} \frac{1}{R_i} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_K} \]

Also,
\[ G_{eq} = \sum_{i=1}^{K} G_i = G_1 + G_2 + \cdots + G_K \]

**Example:**

Consider the following circuit.

Find (a) the voltage \( V \), (b) the power delivered by the current source, and (c) the power dissipated in the 10\( \Omega \) resistor.

(a) Simplifying

1.  
   \[ 5A \uparrow V \]
   3\( \Omega \)
   6\( \Omega \)
   7.2\( \Omega \)
   10\( \Omega \)

2.  
   \[ 5A \uparrow V \]
   3\( \Omega \)
   6\( \Omega \)
   12.8\( \Omega \)
   7.2\( \Omega \)

3.  
   \[ 5A \uparrow V \]
   3\( \Omega \)
   20\( \Omega \)

4.  
   \[ 5A \uparrow V \]
   12\( \Omega \)
Using Ohm's law,

\[ V = i R = (5)(12) = 60\text{V} \]

\[ (b) \]

\[ P = -V_i = - (60 \times 5) = -300\text{W} \]

(c) Using Ohm's law on circuit (3)

\[ i_1 = \frac{V}{20} = \frac{60}{20} = 3\text{A} \]

And on (2),

\[ V_1 = i_1 (7.2) = (3)(7.2) = 21.6\text{V} \]

Using KVL on (1),

\[ 21.6 + V_2 - 60 = 0 \]

\[ V_2 = 38.4\text{V} \]

Also,

\[ i_2 = \frac{V_2}{16} = \frac{38.4}{16} = 2.4\text{A} \]

Therefore,

\[ P = (i_1)^2 10 = (2.4)^2 (10) = 57.6\text{W} \]
The Voltage Divider Circuit

The voltage divider circuit produces a fractional part of an applied voltage.

\[ V_o = \frac{R_2}{R_1 + R_2} V_s \]

which permits the direct determination of \( V_o \) without determining \( i_s \).

The Current Divider Circuit

The current divider circuit produces a fractional part of an incoming current.

\[ i_2 = \frac{R_1}{R_1 + R_2} i_s \]

which permits the direct determination of \( i_2 \) (or \( i_1 \)) without determining \( V_s \).
Using the previous example,

\[ i_1 = \frac{30}{20+30} = 3A \]

\[ i_2 = \frac{64}{16+64} = 2.4A \]

And,

\[ P = (i_2)^2 R = (2.4)^2 (10) = 57.6 \text{W} \]
Network Terminology

**Planar Circuit:** A circuit that can be drawn on a plane with no crossing branches

**Node:** Point or portion of a circuit where 2 or more elements are joined

**Essential Node:** Point or portion of a circuit where 3 or more elements are joined

**Branch:** Path that connects 2 nodes

**Essential Branch:** Path that connects 2 essential nodes w/o passing through an essential node

**Loop:** Path with last node same as starting node that does not cross itself

**Mesh:** Loop that does not enclose any other loops

Note: this isn't in the text.
Consider the following planar circuit:

The following information is obtained:

1) nodes = 6
2) essential nodes = 4
3) branches = 11
4) essential branches = 9
5) essential branches with unknown currents = 7
6) meshes = 6
Nodal Analysis

Consider the following circuit with a reference node.

![Circuit Diagram]

Branch currents are now expressed in terms of node voltages and branch conductances (or resistances). For example,

![Current Expression]

Using KVL,

\[ V_1 - 10 - \nu = 0 \]

\[ \nu = V_1 - 10 \]

Therefore,

\[ i = \frac{\nu}{R} = \frac{V_1 - 10}{1} \]

The nodal equations for the above circuit are

\[
\begin{align*}
\frac{V_1 - 10}{1} + \frac{V_1}{5} + \frac{V_1 - V_2}{2} &= 0 \\
\frac{V_2 - V_1}{2} + \frac{V_2}{10} - 2 &= 0
\end{align*}
\]
Example:

Consider the following circuit.

Using KCL at node 1,

\[-1 + 1v_1 + 3(v_1 - v_2) = 0\]
\[-4v_1 - 3v_2 = 1\]

At node 2,

\[3(v_2 - v_1) + 2v_2 + 2 = 0\]
\[-3v_1 + 5v_2 = -2\]

In matrix terms,

\[
\begin{bmatrix}
4 & -3 \\
-3 & 5
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix} =
\begin{bmatrix}
1 \\
-2
\end{bmatrix}
\]

\[
\begin{bmatrix}
G
\end{bmatrix}
\begin{bmatrix}
v
\end{bmatrix} =
\begin{bmatrix}
i
\end{bmatrix}
\]

The solution using Cramer's rule is

\[
v_1 = \frac{1(-5) - (-3)6}{4 - 9} = \frac{-5 - 18}{-5} = \frac{-23}{-5} = \frac{23}{5} \quad \text{and} \quad v_1 = \frac{1}{11} \quad \text{V}
\]

\[
v_2 = \frac{4(-2) - (-3)(-1)}{4 - 9} = \frac{-8 + 3}{-5} = \frac{-5}{-5} = 1 \quad \text{V}
\]
If the inverse of matrix $G$ is known,

$$
\begin{align*}
[G][v] &= [i] \\
[I][v] &= [G][i] \\
[v] &= [G]^{-1}[i]
\end{align*}
$$

$$
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}
= \begin{bmatrix}
\frac{5}{11} & \frac{3}{11} \\
\frac{3}{11} & \frac{4}{11}
\end{bmatrix}
\begin{bmatrix}
1 \\
-2
\end{bmatrix}
= \begin{bmatrix}
-\frac{1}{11} \\
\frac{5}{11}
\end{bmatrix}
$$

---

**Example:**

Because the circuit contains only conductances (or resistances) and independent current sources, the $G$ matrix can be written by inspection.

$$
[G] = \begin{bmatrix}
G_{11} & G_{12} & G_{13} \\
G_{21} & G_{22} & G_{23} \\
G_{31} & G_{32} & G_{33}
\end{bmatrix} = \begin{bmatrix}
3 & -1 & -2 \\
-1 & 5 & 0 \\
-2 & 0 & 5
\end{bmatrix}
$$

$$
[i] = \begin{bmatrix}
i_1 \\
i_2 \\
i_3
\end{bmatrix} = \begin{bmatrix}
-2 \\
3 \\
-3
\end{bmatrix}
$$
The solution using Cramer's rule is

\[
V_1 = \begin{vmatrix}
-2 & -1 & -2 \\
3 & 5 & 0 \\
-3 & 0 & 5
\end{vmatrix} = \frac{-56 - 30 + 15}{75 - 20 - 5} = \frac{-65}{50} = -1.30 \text{V}
\]

\[
V_2 = \begin{vmatrix}
3 & -2 & -2 \\
-1 & 3 & 0 \\
-2 & -3 & 5
\end{vmatrix} = \frac{-45 + 12 - 10}{50} = \frac{17}{50} = 0.34 \text{V}
\]

\[
V_3 = \begin{vmatrix}
3 & -1 & -2 \\
-1 & 5 & 3 \\
-2 & 0 & -3
\end{vmatrix} = \frac{-45 + 6 - 20 + 3}{50} = \frac{-56}{50} = -1.12 \text{V}
\]
Using KCL at node 1,
\[
\frac{v_1}{4} + \frac{v_1 - v_3}{1} + 1 = 0
\]
\[v_1 + 4v_1 - 4v_3 + 4 = 0\]
\[5v_1 - 4v_3 = -4\]

At node 2,
\[
\begin{cases}
-1 + 3i + \frac{v_2 - v_3}{2} = 0 \\
\quad i = \frac{v_2 - v_3}{2}
\end{cases}
\]
\[-1 + 3\left(\frac{v_2 - v_3}{2}\right) + \frac{v_2 - v_3}{2} = 0\]
\[-2 + 3v_2 - 3v_3 + v_2 - v_3 = 0\]
\[4v_2 - 4v_3 = 2\]

At node 3,
\[
\frac{v_3 - v_2}{2} + \frac{v_3 - v_1}{1} + \frac{v_3}{3} = 0
\]
\[3v_3 - 3v_2 + 6v_3 - 6v_1 + 2v_3 = 0\]
\[-6v_1 - 3v_2 + 11v_3 = 0\]
the matrix equation is

\[
\begin{bmatrix}
5 & 0 & -4 \\
0 & 4 & -4 \\
-6 & -3 & 11
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}
= 
\begin{bmatrix}
-4 \\
2 \\
0
\end{bmatrix}
\]

therefore,

\[
v_1 = \frac{\begin{bmatrix}
-4 & 0 & -4 \\
2 & 4 & -4 \\
0 & -3 & 11
\end{bmatrix}}{64} = \frac{-176 + 24 + 48}{220 - 96 - 60} = \frac{-104}{64} = -\frac{52}{32} \text{ V}
\]

\[
v_2 = \frac{\begin{bmatrix}
5 & -4 & -4 \\
0 & 2 & -4 \\
-6 & 0 & 11
\end{bmatrix}}{64} = \frac{110 - 96 - 48}{64} = \frac{-34}{64} = -\frac{17}{32} \text{ V}
\]

\[
v_3 = \frac{\begin{bmatrix}
5 & 0 & -4 \\
0 & 4 & 2 \\
-6 & -3 & 0
\end{bmatrix}}{64} = \frac{-96 + 30}{64} = \frac{-66}{64} = -\frac{33}{32} \text{ V}
\]

Also,

\[
i = \frac{1}{2} (v_2 - v_3) = \frac{1}{2} \left[ -\frac{17}{32} - \left( -\frac{33}{32} \right) \right] = \frac{1}{2} \left( \frac{16}{32} \right) = \frac{1}{4} \text{ A}
\]

\[
v = \frac{2}{1+2} \cdot v_3 = \frac{2}{3} \left( -\frac{33}{32} \right) = -\frac{11}{16} \text{ V}
\]
Example:

At node 1,

\[-4.8 + \frac{v_1}{7.5} + \frac{v_1 - v_2}{2.5} = 0\]

\[-36 + v_1 + 3v_1 - 3v_2 = 0\]

\[4v_1 - 3v_2 = 36\]

At the supernode,

\[\frac{v_2 - v_1}{2.5} + \frac{v_2}{10} + \frac{v_3}{2.5} + \frac{v_3 - 12}{1} = 0\]

\[4v_1 - 4v_1 + v_1 + 4v_3 + 10v_3 - 120 = 0\]

\[-4v_1 + 5v_2 + 14v_3 = 120\]

The third equation is

\[v_3 - v_2 = i = \frac{v_1}{7.5}\]

\[15v_3 - 15v_2 = 2v_1\]

\[2v_1 + 15v_2 - 15v_3 = 0\]
The matrix equation is

\[
\begin{bmatrix}
4 & -3 & 0 \\
-4 & 5 & 14 \\
2 & 15 & -15
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}
= 
\begin{bmatrix}
36 \\
120 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}
= 
\begin{bmatrix}
2.7299E-01 & 4.3103E-02 & 4.0230E-02 \\
3.0651E-02 & 5.7471E-02 & 5.3640E-02 \\
6.7050E-02 & 6.3218E-02 & -1.6628E-03
\end{bmatrix}
\begin{bmatrix}
36 \\
120 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}
= 
\begin{bmatrix}
15 \\
8 \\
10
\end{bmatrix}
\]
Mesh Analysis

Consider the following circuit with branch currents $i_1$, $i_2$, and $i_3$.

\[ \begin{align*}
1 & \quad 1 \Omega \\
2 & \quad 2 \Omega \\
1V & \quad i_1 \\
3V & \quad i_2 \\
2V &
\end{align*} \]

Using KVL around the two meshes,

\[ \begin{align*}
i_1 + 3i_3 &= 1 \\
2i_2 - 3i_3 &= -2
\end{align*} \]

But KCL yields

\[ i_1 = i_2 + i_3 \]

Eliminating $i_3$ and gathering terms,

\[ \begin{align*}
4i_1 - 3i_2 &= 1 \\
-3i_1 + 5i_2 &= -2
\end{align*} \]

These equations are the mesh equations for the above circuit. Redrawing,

\[ \begin{align*}
1 & \quad 1 \Omega \\
2 & \quad 2 \Omega \\
1V & \quad i_1 \\
3V & \quad i_2 \\
2V &
\end{align*} \]

In matrix terms,

\[ \begin{bmatrix}
4 & -3 \\
-3 & 5
\end{bmatrix}
\begin{bmatrix}
i_1 \\
i_2
\end{bmatrix} =
\begin{bmatrix}
1 \\
-2
\end{bmatrix} \]

\[ [R] [i] = [v] \]
Example:

\[ R = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} = \begin{bmatrix} 6 & -2 & -3 \\ -2 & 5 & -1 \\ -3 & -1 & 5 \end{bmatrix} \]

\[ v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \]

The solution using Cramer's rule is

\[ i_1 = \frac{\begin{vmatrix} 2 & -2 & -3 \\ 1 & 5 & -1 \\ 0 & -1 & 5 \end{vmatrix}}{6 - 2 - 3} = \frac{50 + 3 \cdot -2 + 10}{150 - 6 - 45 - 6 - 20} = \frac{61}{67} A \]

\[ i_2 = \frac{\begin{vmatrix} 6 & 2 & -3 \\ -2 & 1 & -1 \\ -3 & 0 & 5 \end{vmatrix}}{67} = \frac{30 + 6 \cdot -9 + 20}{67} = \frac{47}{67} A \]

\[ i_3 = \frac{\begin{vmatrix} 6 & -2 & 2 \\ -2 & 5 & 1 \\ -3 & -1 & 0 \end{vmatrix}}{67} = \frac{6 + 4 + 30 + 6}{67} = \frac{46}{67} A \]
Mesh Analysis With Dependent Sources

Example:

\[ \begin{align*}
V_1 & \quad + \quad - \\
\quad & \quad + \\
\quad & \quad + \\
\quad & \quad 14.7k \quad i_e \\
\quad & \quad 15k \quad v_2 \\
\quad & \quad 27 \quad i_e \\
\quad & \quad 150 \quad i_c \\
\quad & \quad 1.5k \quad v_1 \\
\end{align*} \]

Find: \( \frac{v_2}{v_1} \) and \( \frac{v_1}{i_e} \)

Around loop \( i_e \),

\[ 27i_e + 150(i_e - i_c) - v_1 = 0 \]

\[ 177i_e = 150i_c + v_1 \]

\[ i_e = \frac{150i_c + v_1}{177} \]

Around loop \( i_c \),

\[ -14,700i_e + 16,500i_c + 150(i_c - i_e) = 0 \]

\[ -14,850i_e + 16,650i_c = 0 \]

\[ -14,850 \left( \frac{150i_c + v_1}{177} \right) + 16,650i_c = 0 \]

\[ -14,850(150i_c) - 14,850v_1 + 16,650(177)i_c = 0 \]

\[ 719,550i_c = 14,850v_1 \]

\[ i_c = 2.06 \times 10^{-2}v_1 \]
Now,
\[ V_2 = 1500 \cdot i_c \]
\[ = 1500 \left( 2.06 \times 10^{-2} V_i \right) \]
\[ = 30.90 \cdot V_i \]
\[ \therefore \frac{V_2}{V_i} = 30.90 \]

Also,
\[ i_e = \frac{150}{177} \cdot i_c + \frac{1}{177} \cdot V_i \]
\[ = \frac{150}{177} \left( 2.06 \times 10^{-2} \right) V_i + \frac{1}{177} \cdot V_i \]
\[ = 2.31 \times 10^{-2} \cdot V_i \]
\[ \therefore R_{in} = \frac{V_i}{i_e} = 43.28 \Omega \]
Example:

Around loop 1,
\[ i_1 = -2V = -2(-3i_3) = 6i_3 \]
\[ i_1 - 6i_3 = 0 \]

Around the supermesh loop,
\[ 3i_3 + 1i_2 + 4(i_2 - i_1) + 2(i_3 - i_1) = 0 \]
\[ -6i_1 + 5i_2 + 5i_3 = 0 \]

The third equation is
\[ i_3 - i_2 = 3 \]

The matrix equation is
\[
\begin{bmatrix}
1 & 0 & -6 \\
-6 & 5 & 5 \\
0 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
i_1 \\
i_2 \\
i_3
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
3
\end{bmatrix}
\]

Therefore,
\[
i_3 = \frac{15}{5-36+5} = \frac{15}{-26} A
\]
\[
i_1 = 6i_3 = 6\left(-\frac{15}{26}\right) = -\frac{90}{26} A
\]
\[
i_2 = i_3 - 3 = -\frac{15}{26} - \frac{78}{26} = -\frac{93}{26} A
\]
Nonideal sources are modeled from ideal sources and resistors.

**Practical Voltage Source**

**Practical Current Source**

Practical sources are equivalent if they produce the same effect in an arbitrary load.

Setting $i_L = i'_L$,

$$\frac{V_S}{R_S + R_L} = \frac{R_p i_s}{R_p + R_L}$$

This equality is satisfied when

$V_S = i_S R_S$

$R_S = R_p$

Therefore,

$$i_L = \frac{V_S}{R_S + R_L} = \frac{i_S R_S}{R_S + R_L} = i'_L$$
Source Transformations

Replacing a source by an equivalent source is called a source transformation. The two kinds of transformations are:

\[
\begin{align*}
\text{Voltage Source: } & \quad V_s \quad \Rightarrow \quad \frac{V_s}{R_s} \\
\text{Current Source: } & \quad I_s \quad \Rightarrow \quad I_s R_s
\end{align*}
\]

Example:

\[
\begin{align*}
\text{Input Circuit: } & \quad 28V \quad || \quad 1\Omega, 4\Omega, 3\Omega \\
& \quad \Rightarrow \quad \frac{28}{5}A
\end{align*}
\]

\[
\begin{align*}
\text{Input Circuit: } & \quad \frac{112}{9}V, \frac{10}{9}A, 1\Omega, 3\Omega \\
& \quad \Rightarrow \quad \frac{112}{9}V
\end{align*}
\]

Now,

\[
i = \frac{V_s}{R_{eq}} = \frac{\frac{112}{9}}{\frac{20}{9} + 1 + 3} = \frac{\frac{112}{9}}{\frac{60}{9}} = 2A
\]
Consider the following practical voltage source connected to a load resistance.

Using Ohm's law,

\[ i_L = \frac{V_S}{R_S + R_L} \]

the instantaneous power absorbed by the load is

\[ P = i_L^2 R_L = \left( \frac{V_S}{R_S + R_L} \right)^2 R_L = \frac{V_S^2 R_L}{(R_S + R_L)^2} \]

Differentiating,

\[ \frac{dP}{dR_L} = \frac{(R_S - R_L) V_S^2}{(R_S + R_L)^3} = 0 \]

Therefore, maximum power occurs when

\[ R_S = R_L \]

and is

\[ P_{max} = \frac{V_S^2}{4R_S} \]
The Superposition Principle

The voltage or current response of a linear circuit containing \( n \) independent sources is the sum of the response of each of these \( n \) sources considered one at a time with the remaining \( n-1 \) independent sources set to zero.

Example:

Consider the following circuit with \( n = 2 \) independent sources.

Determine \( i \) and \( v \) using the principle of superposition.

The response to the 2V source is

At node \( v_a \),

\[
\frac{v_a - 2}{1} + \frac{v_a}{3} + \frac{v_a - (-3v_a)}{3} = 0
\]

\[
3v_a - 6 + v_a + v_a + 3v_a = 0
\]

\[
8v_a = 6
\]

\[
v_a = \frac{3}{4} \text{V}
\]
Using Ohm’s law,

\[ i_a = \frac{V_a - (-3V_a)}{3} = \frac{4}{3}V_a = \frac{4}{3}\left(\frac{30}{59}\right) = 1A \]

The response to the \( \frac{30}{59} \) A source is

At node \( V_b \),

\[ \frac{V_b}{1} + \frac{V_b}{3} + \frac{V_b - V_1}{2} = 0 \]

\[ 6V_b + 2V_b + 3V_b - 3V_1 = 0 \]

\[ 11V_b - 3V_1 = 0 \]

At node \( V_1 \),

\[ \frac{V_1 - V_b}{2} + \frac{V_1 - (-3V_b)}{1} = \frac{30}{59} \]

\[ V_1 - V_b + 2V_1 + 6V_b = \frac{60}{59} \]

\[ 5V_b + 3V_1 = \frac{60}{59} \]

Solving,

\[
\begin{vmatrix}
0 & -3 \\
60 & 3 \\
11 & -3 \\
5 & 3
\end{vmatrix}
= \frac{180}{59} = \frac{180}{(48)(59)} = \frac{15}{286} \text{ V}
\]
\[ V_1 = \begin{bmatrix} 11 & 0 \\ 5 & 60 \\ 59 \end{bmatrix} \begin{bmatrix} \frac{60}{59} \\ \frac{48}{59} \end{bmatrix} = \frac{660}{48} = \frac{660}{(48)(59)} = \frac{55}{236} \text{ V} \]

Using Ohm's law,
\[ i_b = \frac{V_1 + (-3V_b)}{1} = \frac{55}{236} + 3 \left( \frac{15}{236} \right) = \frac{100}{236} \text{ A} \]

Summing the responses,
\[ i = i_a + i_b = 1 + \frac{100}{236} = \frac{236}{236} + \frac{100}{236} = \frac{336}{236} = \frac{84}{59} \text{ A} \]
\[ V = V_a + V_b = \frac{3}{4} + \frac{15}{236} = \frac{177}{236} + \frac{15}{236} = \frac{192}{236} = \frac{48}{59} \text{ V} \]

As a check,

Loop 1,
\[ 1i_1 + 3(i_2 - i_1) = 2 \\
-4i_1 - 3i_2 = 2 \]

Supermesh loop,
\[ 3(i_2 - i_1) + 2i_2 + i = 3V = 9(i_1 - i_2) \\
-12i_1 + 14i_2 + i = 0 \]

By inspection,
\[ i_2 - i = -\frac{30}{59} \]
And,
\[
\begin{vmatrix}
4 & -3 & 2 \\
-12 & 14 & 0 \\
0 & 1 & -30 \\
\end{vmatrix}
\frac{30}{59} = \frac{-56(\frac{30}{59}) - 24 + 36(\frac{30}{59})}{-56 + 36 - 4}
\]
\[
= \frac{-1680 - 1416 + 1080}{59} = \frac{2016}{(24)(59)} = \frac{84}{59} \text{ A}
\]

Node \( V \),
\[
\frac{V - 2}{1} + \frac{V}{3} + \frac{V - V_1}{2} = 0
\]
\( 6V - 12 + 2V + 3V - 3V_1 = 0 \)
\( 11V - 3V_1 = 12 \)

Node \( V_1 \),
\[
\frac{V_1 - V}{2} + \frac{V_1 - (-3V)}{1} = \frac{30}{59}
\]
\( 59V_1 - 59V + 118V_1 + 354V = 60 \)
\( 295V + 177V_1 = 60 \)

And,
\[
V = \begin{vmatrix}
12 & -3 \\
60 & 177 \\
11 & -3 \\
295 & 177
\end{vmatrix}
\frac{2124 + 180}{1947 + 885} = \frac{2304}{2832} = \frac{48}{59} \text{ V}
The Inductor

An ideal inductor is an energy storage device that is represented by the following symbol:

The relationship between the voltage and current is

\[ v = L \frac{di}{dt} \]

where \( L \) is the inductance in henrys.

The energy stored in the inductor at time \( t \) is found by integrating the instantaneous absorbed power in the inductor. By definition,

\[ \frac{dW_L}{dt} = P \]

\[ W_L = \int_{-\infty}^{t} p \, d\tau = \int_{-\infty}^{t} i \, v \, d\tau \]

\[ = \int_{-\infty}^{t} i \left( L \frac{di}{d\tau} \right) \, d\tau = L \int_{i(-\infty)}^{i(t)} i \, di \]

\[ = L \frac{i^2}{2} \bigg|_{i(-\infty)}^{i(t)} = \frac{1}{2} L \left[ i^2(t) - i^2(-\infty) \right] \]

Assuming \( i^2(-\infty) = 0 \),

\[ W_L = \frac{1}{2} L i^2 \]
Example:

Consider the following circuit and i waveform.

\[\begin{align*}
V_s & \quad + \quad \left\{ \begin{array}{ll}
2 + i \\
V_R \\
\end{array} \right. \\
2H & \quad \quad v_L
\end{align*}\]

\[i = \begin{cases}
0 & \text{for } -\infty < t < 0 \\
-t & \text{for } 0 \leq t < 1 \\
2t - 3 & \text{for } 1 \leq t < 2 \\
1 & \text{for } 2 \leq t < \infty
\end{cases}\]

The current waveform can be expressed as

\[i = \begin{cases}
0 & \text{for } -\infty < t < 0 \\
-t & \text{for } 0 \leq t < 1 \\
2t - 3 & \text{for } 1 \leq t < 2 \\
1 & \text{for } 2 \leq t < \infty
\end{cases}\]

\[\begin{align*}
v_L &= L \frac{di}{dt} = 2 \frac{di}{dt} = \begin{cases}
0 & \text{for } -\infty < t < 0 \\
-2 & \text{for } 0 \leq t < 1 \\
4 & \text{for } 1 \leq t < 2 \\
0 & \text{for } 2 \leq t < \infty
\end{cases}
\end{align*}\]
\[ w_L = \frac{1}{2} L i^2 = \frac{1}{2} (2) i^2 = \begin{cases} 0 & \text{for } -\infty < t < 0 \\ t^2 & \text{for } 0 \leq t < 1 \\ (2t-3)^2 & \text{for } 1 \leq t < 2 \\ 1 & \text{for } 2 \leq t < \infty \end{cases} \]

\[ p_R = R i^2 = 2 i^2 = \begin{cases} 0 & \text{for } -\infty < t < 0 \\ 2t^2 & \text{for } 0 \leq t < 1 \\ 2(2t-3)^2 & \text{for } 1 \leq t < 2 \\ 2 & \text{for } 2 \leq t < \infty \end{cases} \]

\[ v_R = R i = 2 i = \begin{cases} 0 & \text{for } -\infty < t < 0 \\ -2t & \text{for } 0 \leq t < 1 \\ 4t-6 & \text{for } 1 \leq t < 2 \\ 2 & \text{for } 2 \leq t < \infty \end{cases} \]
The integral relationship between the voltage and current can be derived from the differential relationship.

\[ V = L \frac{di}{dt} \]

\[ di = \frac{1}{L} V dt \]

\[ i = \frac{1}{L} \int_{-\infty}^{t} V d\tau \]

If \( i(t) \) is known at \( t = t_o \),

\[ i(t) = \frac{1}{L} \int_{-\infty}^{t_o} V d\tau + \frac{1}{L} \int_{t_o}^{t} V d\tau \]

\[ i(t) = i(t_o) + \frac{1}{L} \int_{t_o}^{t} V d\tau \]
The Capacitor

An ideal capacitor is an energy storage device that is represented by the following symbol.

\[ i \xrightarrow{C} v \]

The relationship between the voltage and current is

\[ i = C \frac{dv}{dt} \]

where \( C \) is the capacitance in farads.

The energy stored in the capacitor at time \( t \) is found by integrating the instantaneous absorbed power in the capacitor. By definition,

\[
\frac{d\omega_c}{dt} = P
\]

\[ \therefore \]

\[ \omega_c = \int_{-\infty}^{t} P \, dt = \int_{-\infty}^{t} v \, idt \]

\[ = \int_{-\infty}^{t} v \left( C \frac{dv}{dt} \right) \, dt = C \int_{-\infty}^{t} v \, dv \]

\[ = C \frac{v^2}{2} \bigg|_{v(-\infty)}^{v(t)} = \frac{1}{2} C \left[ v^2(t) - v^2(-\infty) \right] \]

Assuming \( v^2(-\infty) = 0 \),

\[ \omega_c = \frac{1}{2} C v^2 \]
Example:

Consider the following circuit and \( v \) waveform.

Determine \( i_c, w_c, P_R, i_R \), and \( i_s \).

The voltage waveform can be expressed as

\[
v = \begin{cases} 
0 & \text{for } -\infty < t < 0 \\
 t & \text{for } 0 \leq t < 1 \\
1 & \text{for } 1 \leq t < \infty
\end{cases}
\]

\[
i_c = C \frac{dv}{dt} = 2 \frac{dv}{dt} = \begin{cases} 
0 & \text{for } -\infty < t < 0 \\
2 & \text{for } 0 \leq t < 1 \\
0 & \text{for } 1 \leq t < \infty
\end{cases}
\]
\[ w_c = \frac{1}{2} C v^2 = \frac{1}{2} (2) v^2 = \begin{cases} 0 & \text{for } -\infty < t < 0 \\ t^2 & \text{for } 0 \leq t < 1 \\ 1 & \text{for } 1 \leq t < \infty \end{cases} \]

\[ p_r = \frac{v^2}{R} = 2v^2 = \begin{cases} 0 & \text{for } -\infty < t < 0 \\ 2t^2 & \text{for } 0 \leq t < 1 \\ 2 & \text{for } 1 \leq t < \infty \end{cases} \]

\[ i_r = \frac{v}{R} = 2v = \begin{cases} 0 & \text{for } -\infty < t < 0 \\ 2t & \text{for } 0 \leq t < 1 \\ 2 & \text{for } 1 \leq t < \infty \end{cases} \]
\[ i_s = i_R + i_C = \begin{cases} 
0 & \text{for } -\infty < t < 0 \\
2t + 2 & \text{for } 0 \leq t < 1 \\
2 & \text{for } 1 \leq t < \infty 
\end{cases} \]

The integral relationship between the voltage and current can be derived from the differential relationship.

\[ i = C \frac{dv}{dt} \]

\[ dv = \frac{1}{C} i dt \]

\[ v = \frac{1}{C} \int_{-\infty}^{t} i d\tau \]

If \( v(t) \) is known at \( t = t_0 \),

\[ v(t) = \frac{1}{C} \int_{-\infty}^{t_0} i d\tau + \frac{1}{C} \int_{t_0}^{t} i d\tau \]

\[ v(t) = v(t_0) + \frac{1}{C} \int_{t_0}^{t} i d\tau \]
Series-Parallel Combination of Inductance and Capacitance

Inductors in series:

\[ i \quad \begin{array}{c} + \quad L_1 \quad + \quad L_2 \quad + \quad L_3 \quad \Rightarrow \quad i \end{array} \]

Using KVL,

\[ \nu = \nu_1 + \nu_2 + \nu_3 = L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + L_3 \frac{di}{dt} = (L_1 + L_2 + L_3) \frac{di}{dt} = \frac{di}{dt} \]

In general,

\[ L_{eq} = \sum_{i=1}^{K} L_i = L_1 + L_2 + \cdots + L_K \]

Inductors in parallel:

Using KCL,

\[ i = i_1 + i_2 + i_3 = \frac{1}{L_1} \int_{t_0}^{t} \nu \, dt + i_1(t_0) + \frac{1}{L_2} \int_{t_0}^{t} \nu \, dt + i_2(t_0) + \frac{1}{L_3} \int_{t_0}^{t} \nu \, dt + i_3(t_0) \]

\[ = \left( \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} \right) \int_{t_0}^{t} \nu \, dt + i_1(t_0) + i_2(t_0) + i_3(t_0) = \frac{1}{L_{eq}} \int_{t_0}^{t} \nu \, dt + i(t_0) \]
In general,

\[
\frac{1}{L_{eq}} = \sum_{i=1}^{K} \frac{1}{L_i} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_K}
\]

\[
i(t_0) = \sum_{i=1}^{K} i_K(t_0) = i_1(t_0) + i_2(t_0) + \cdots + i_K(t_0)
\]

---

**Capacitors in series:**

\[\begin{align*}
\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} & \Rightarrow \frac{1}{C_{eq}} \\
\end{align*}\]

Applying KVL and generalizing,

\[
\frac{1}{C_{eq}} = \sum_{i=1}^{K} \frac{1}{C_i} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_K}
\]

\[
V(t_0) = \sum_{i=1}^{K} V_i(t_0) = V_1(t_0) + V_2(t_0) + \cdots + V_K(t_0)
\]

---

**Capacitors in parallel:**

\[\begin{align*}
\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} & \Rightarrow \frac{1}{C_{eq}} \\
\end{align*}\]
Applying KCL and generalizing,

\[ C_{eq} = \sum_{i=1}^{K} C_i = C_1 + C_2 + \cdots + C_K \]
The Natural Response of an RL Circuit

Consider the following RL circuit where $i_L$ at time $t=0$ is known.

By KVL,

\[ V_L - V_R = 0 \]
\[ L \frac{di_L}{dt} - (-Ri_L) = 0 \]
\[ \frac{di_L}{dt} + \frac{R}{L} i_L = 0 \]

This is a homogeneous first-order linear differential equation.

Solving,

\[ \frac{di_L}{dt} + \frac{R}{L} i_L = 0 \]
\[ \frac{di_L}{i_L} = -\frac{R}{L} dt \]
\[ \int \frac{di_L}{i_L} = -\frac{R}{L} \int dt \]
\[ \ln i_L(t) = -\frac{Rt}{L} + K_1 \]
\[ i_L(t) = e^{-\frac{Rt}{L} + K_1} = e^{-\frac{Rt}{L}} e^{K_1} \]
\[ i_L(t) = Ke^{-\frac{Rt}{L}} \]

At $t=0$,

\[ i_L(0) = Ke^0 = K = I_0 \]
therefore,

\[ i_L(t) = I_0 e^{-\frac{-Rt}{L}} , \quad t \geq 0 \]

The resistor voltage \( v_R \) can be determined using Ohm's law.

\[ v_R(t) = -R i_L(t) = -R I_0 e^{-\frac{-Rt}{L}} , \quad t \geq 0 \]

The graph of \( i_L(t) \) or \( v_R(t) \) is the zero-input or natural response of the circuit.

Both these equations have the same form,

\[ f(t) = f(0) e^{-\frac{t}{\tau}} \]

where \( \tau = \frac{L}{R} \) is called the time constant.

The inductor voltage \( v_L \) is

\[ v_L(t) = L \frac{di_L(t)}{dt} = L \frac{d}{dt} \left( I_0 e^{-\frac{-Rt}{L}} \right) = L \left( -\frac{R}{L} \right) I_0 e^{-\frac{-Rt}{L}} \]

\[ = -R I_0 e^{-\frac{-Rt}{L}} , \quad t \geq 0 \]

which is the same as \( v_R(t) \).
The initial energy stored in the inductor at \( t = 0 \) is
\[
W_L(0) = \frac{1}{2} LI_0^2
\]
The power absorbed by the resistor is
\[
P_R(t) = RI_0^2 e^{-\frac{2Rt}{L}} = RI_0^2 e^{-\frac{2R}{L}}
\]
The total energy absorbed by the resistor is
\[
W_R(t) = \int_0^\infty P_R(t) \, dt = \int_0^\infty RI_0^2 e^{-\frac{2R}{L}} \, dt
\]
\[
= -\frac{L}{2R} RI_0^2 e^{-\frac{2R}{L}} \bigg|_0^\infty
= -\frac{L}{2} I_0^2 (0 - 1)
\]
\[
= \frac{1}{2} LI_0^2 = W_L(0)
\]

Now consider the RL circuit where \( i_L \) at time \( t = t_0 \) is known. Starting with
\[
i_L(t) = Ke^{-\frac{R}{L}t}
\]
and evaluating \( K \) at \( t = t_0 \),
\[
i_L(t_0) = Ke^{-\frac{R}{L}t_0}
\]
\[
K = i_L(t_0)e^{\frac{R}{L}t_0}
\]
Substituting,
\[
i_L(t) = i_L(t_0)e^{-\frac{R}{L}t} e^{\frac{R}{L}t_0} = i_L(t_0)e^{-\frac{R}{L}(t-t_0)}
\]
\[
i_L(t) = i_L(t_0)e^{-\frac{R}{L}(t-t_0)}, \quad t \geq t_0
\]
This is a more general expression, and \( t_0 = 0 \) is merely a special case.
Example:

Find $i(t)$ and $v(t)$ for all time and sketch these functions.

For $t < 0$,

- $v(t) = 0\, \text{V}$
- $i(t) = \frac{q}{3} = 3\, \text{A} \Rightarrow I_0 = 3\, \text{A}$

For $0 \leq t < 2$,

- $i(t) = I_0 e^{-\frac{Rt}{L}} = 3e^{-2t}$
- $v(t) = L \frac{di(t)}{dt} = 6 \frac{d}{dt} \left(3e^{-2t}\right) = -36e^{-2t}$
- $i(2) = 3e^{-2(2)} = 0.055\, \text{A}$
- $v(2) = -36e^{-2(2)} = -0.166\, \text{V}$

For $t \geq 2$,

- $R_{eq} = \frac{(12)(4)}{12+4} = 3\, \Omega$
- $i(t) = i(t_0) e^{-\frac{R(t-t_0)}{L}} = i(2) e^{-\frac{(t-2)}{2}} = 0.055 e^{-\frac{(t-2)}{2}}$
- $v(t) = L \frac{di(t)}{dt} = 6 \frac{d}{dt} \left[0.055 e^{-\frac{(t-2)}{2}}\right] = -0.165 e^{-\frac{(t-2)}{2}}$
For all \( t \),

\[
i(t) = \begin{cases} 
3 & \text{for } t < 0 \\
3e^{-2t} & \text{for } 0 \leq t < 2 \\
0.055e^{-\frac{(t-2)}{2}} & \text{for } t \geq 2
\end{cases}
\]

\[
v(t) = \begin{cases} 
0 & \text{for } t < 0 \\
-36e^{-2t} & \text{for } 0 \leq t < 2 \\
-0.165e^{-\frac{(t-2)}{2}} & \text{for } t \geq 2
\end{cases}
\]

---

Sketching,
Consider the following RC circuit where \( V_c \) at time \( t = 0 \) is known.

\[
\begin{align*}
\text{By KCL,} & \\
i_c + i_R &= 0 \\
C \frac{dv_c}{dt} + \frac{V_c}{R} &= 0 \\
\frac{dv_c}{dt} + \frac{1}{RC} V_c &= 0
\end{align*}
\]

Solving,

\[
V_c(t) = V_0 e^{-\frac{t}{RC}}, \quad t \geq 0
\]

The time constant \( \tau \) is now \( RC \).

The resistor current \( i_R \) is

\[
i_R(t) = \frac{V_R(t)}{R} = \frac{V_c(t)}{R} = \frac{V_0}{R} e^{-\frac{t}{RC}}, \quad t \geq 0
\]
Also,
\[ i_c(t) = -i_R(t) = -\frac{V_0}{R} e^{-\frac{t}{RC}}, \quad t \geq 0 \]

or
\[ i_c(t) = C \frac{dv_c(t)}{dt} = C \frac{d}{dt} \left( V_0 e^{-\frac{t}{RC}} \right) = C\left(-\frac{1}{RC}\right)V_0 e^{-\frac{t}{RC}} \]
\[ = -\frac{V_0}{R} e^{-\frac{t}{RC}}, \quad t \geq 0 \]

The initial energy stored in the capacitor at \( t = 0 \) is
\[ w_c(0) = \frac{1}{2} C V_0^2 \]
The power absorbed by the resistor is
\[ P_R(t) = R I_R^2(t) = R \left( \frac{V_0}{R} e^{-\frac{t}{RC}} \right)^2 = \frac{V_0^2}{R^2} e^{-\frac{2t}{RC}} \]
The total energy absorbed by the resistor is
\[ w_R(t) = \int_{0}^{\infty} P_R(t) \, dt = \int_{0}^{\infty} \frac{V_0^2}{R^2} e^{-\frac{2t}{RC}} \, dt \]
\[ = -\frac{RC}{2} \left. \frac{V_0^2}{R} e^{-\frac{2t}{RC}} \right|_{0}^{\infty} = -\frac{C}{2} V_0^2 (0 - 1) \]
\[ = \frac{1}{2} C V_0^2 = w_c(0) \]

If \( v_c \) is known at \( t = t_0 \),
\[ v_c(t) = v_c(t_0) e^{-\frac{(t-t_0)}{RC}}, \quad t \geq t_0 \]
Example:

Find \( v(t) \) and \( i(t) \) for all time and sketch these functions. Also, determine the time when \( v_c(t) = 0.01 \, v_c(0) \).

For \( t < 0 \),

\[
i(t) = 0 \text{ A} \\
v(t) = \frac{4}{4+1}(10) = 8 \text{ V} \Rightarrow V_o = 8 \text{ V}
\]

For \( 0 \leq t < 1 \),

\[
i(t) = 4 \frac{d v(t)}{dt} = \frac{4}{4+1} \frac{d}{dt} (8e^{-t}) = -2e^{-t} \\
v(t) = V_0 e^{\frac{-t}{RC}} = 8e^{\frac{-t}{RC}}
\]

For \( t \geq 1 \),

\[
R_{eq} = \frac{(4)(6)}{4+6} = 2.40 \Omega \\
v(t) = v(t_o) e^{\frac{-(t-t_o)}{RC}} = v(1) e^{\frac{-5(t-1)}{3}} = 2.94 e^{\frac{-5(t-1)}{3}} \\
i(t) = C \frac{d v(t)}{dt} = \frac{1}{4} \frac{d}{dt} \left[ 2.94 e^{\frac{-5(t-1)}{3}} \right] = -1.23 e^{\frac{-5(t-1)}{3}}
\]
For all $t$,

$$v(t) = \begin{cases} 
8 & \text{for } t < 0 \\
8e^{-t} & \text{for } 0 \leq t < 1 \\
-5 \frac{(t-1)}{3} & \text{for } t \geq 1 
\end{cases}$$

$$i(t) = \begin{cases} 
0 & \text{for } t < 0 \\
-2e^{-t} & \text{for } 0 \leq t < 1 \\
-1.23e^{-\frac{5(t-1)}{3}} & \text{for } t \geq 1 
\end{cases}$$

Sketching,

$$v(t)$$

$$i(t)$$

For $t = 0$,

$$W_c(0) = \frac{1}{2} CV_b^2 = \frac{1}{2} (\frac{1}{4})(8)^2 = 8J$$
Therefore,

\[ w_c(t) = \frac{1}{2} C v^2(t) = 0.01 w_c(0) = 0.01 \times 8 = 0.08 \, J \]

\[ v^2(t) = \frac{2 \times 0.08}{C} = 0.64 \]

\[ v(t) = 0.80 \, V \]

Solving for \( t \),

\[ v(t) = \frac{-5(t-1)}{3} = 0.80 \]

\[ e^{\frac{-5(t-1)}{3}} = \frac{0.80}{2.94} = 0.27 \]

\[ -\frac{5(t-1)}{3} = \ln 0.27 = -1.31 \]

\[ t - 1 = \frac{(3)(1.31)}{5} = 0.79 \]

\[ t = 1.79 \, \text{sec} \]

---
The Step Response of an RL Circuit

Consider the following RL circuit where $i_L(0) = 0$.

![RL Circuit Diagram]

Applying KVL at $t = 0$,

$$L \frac{di_L}{dt} + Ri_L = V_s$$

$$\frac{di_L}{dt} + \frac{R}{L} i_L = \frac{V_s}{L}$$

The general solution of the homogeneous equation is

$$i_L\text{ (general)} = K_1 e^{-\frac{Rt}{L}}$$

For the particular solution, $i_L = K_2$ will be tried.

$$\frac{d}{dt} (K_2) + \frac{RK_2}{L} = \frac{V_s}{L}$$

$$\frac{RK_2}{L} = \frac{V_s}{L}$$

$$K_2 = \frac{V_s}{R}$$

Therefore,

$$i_L\text{ (particular)} = \frac{V_s}{R}$$
the complete solution is

\[ i_L(t) = i_L(\text{particular}) + i_L(\text{general}) \]
\[ = \frac{V_S}{R} + K_1 e^{-\frac{Rt}{L}} \]

Knowing \( i_L(0) = 0 \) permits the evaluation of \( K_1 \).

\[ 0 = \frac{V_S}{R} + K_1 e^0 \]
\[ \therefore K_1 = -\frac{V_S}{R} \]

Finally,

\[ i_L(t) = \frac{V_S}{R} - \frac{V_S}{R} e^{-\frac{Rt}{L}}, \quad t \geq 0 \]

As a check,

\[ \frac{di_L}{dt} + \frac{R}{L} i_L = \frac{V_S}{L} \]

\[ \frac{d}{dt} \left( \frac{V_S}{R} - \frac{V_S}{R} e^{-\frac{Rt}{L}} \right) + \frac{R}{L} \left( \frac{V_S}{R} - \frac{V_S}{R} e^{-\frac{Rt}{L}} \right) = \frac{V_S}{L} \]

\[ \frac{V_S}{L} e^{-\frac{Rt}{L}} + \frac{V_S}{L} - \frac{V_S}{L} e^{-\frac{Rt}{L}} = \frac{V_S}{L} \]

\[ \frac{V_S}{L} = \frac{V_S}{L} \]

the inductor voltage is

\[ V_L(t) = V_S - R i_L(t) = V_S - R \left( \frac{V_S}{R} - \frac{V_S}{R} e^{-\frac{Rt}{L}} \right) \]

\[ = V_S - V_S + \frac{V_S}{L} e^{\frac{-Rt}{L}} \]

\[ = V_S e^{\frac{-Rt}{L}}, \quad t \geq 0 \]
The graph of this particular zero-state response is a step response.

The graph of the voltage is

The following parallel RC circuit

is the dual of the series RL circuit. Therefore,

\[ V_c(t) = I_s R - I_s R e^{-\frac{t}{RC}}, \quad t \geq 0 \]

\[ i_c(t) = I_s e^{-\frac{t}{RC}}, \quad t \geq 0 \]
The step response of an RC circuit

Consider the following RC circuit where \( V_c(0) = V_o \).

![RC Circuit Diagram]

Applying KCL at \( t = 0 \),

\[
\begin{align*}
    i_c &= i_R \\
    C \frac{dV_c}{dt} &= \frac{V_s - V_c}{R} \\
    C \frac{dV_c}{dt} + \frac{V_c}{R} &= \frac{V_s}{R} \\
    \frac{dV_c}{dt} + \frac{1}{RC} V_c &= \frac{V_s}{RC}
\end{align*}
\]

The complete solution is

\[
V_c(t) = V_c(\text{particular}) + V_c(\text{general})
\]

\[
= V_s + Ke^{\frac{-t}{RC}}
\]

Knowing \( V_c(0) = V_o \),

\[
V_o = V_s + Ke^0
\]

\[
K = V_o - V_s
\]

Therefore,

\[
V_c(t) = V_s + (V_o - V_s)e^{\frac{-t}{RC}}, \quad t \geq 0
\]
If \( V_0 = 0 \),

\[
v_c(t) = V_s - V_v e^{-\frac{t}{RC}}, \quad t \geq 0
\]

The capacitor current is

\[
i_c(t) = \frac{V_s}{R} - \frac{V_c}{R}
\]

\[
= \frac{V_s}{R} - \frac{V_s}{R} - \frac{V_0 - V_s}{R} e^{-\frac{t}{RC}}
\]

\[
= \frac{V_s - V_0}{R} e^{-\frac{t}{RC}}, \quad t \geq 0
\]

Again, if \( V_0 = 0 \),

\[
i_c(t) = \frac{V_s}{R} e^{-\frac{t}{RC}}, \quad t \geq 0
\]

The graph of the voltage is

![Graph of the voltage](image)
The graph of the current is

The following parallel RL circuit

is the dual of the series RC circuit. Therefore,

\[ i_L(t) = I_s + (I_0 - I_s) e^{-\frac{Rt}{L}}, \quad t \geq 0 \]

\[ v_L(t) = (I_s R - I_0 R) e^{-\frac{Rt}{L}}, \quad t \geq 0 \]

For \( I_0 = 0 \),

\[ i_L(t) = I_s - I_s e^{-\frac{Rt}{L}}, \quad t \geq 0 \]

\[ v_L(t) = I_s R e^{-\frac{Rt}{L}}, \quad t \geq 0 \]
Forced and Natural Response

Consider the following general first-order linear differential equation with constant coefficients, where \( f(t) \) results from the input and \( x(t) \) represents the response.

\[
\frac{dx(t)}{dt} + ax(t) = f(t)
\]

To obtain a general solution, multiply both sides of the equation by \( e^{at} \).

\[
e^{at} \frac{dx(t)}{dt} + a e^{at} x(t) = e^{at} f(t)
\]

The left side of the equation is now an exact derivative.

\[
\frac{d}{dt} \left[ e^{at} x(t) \right] = e^{at} f(t)
\]

\[
\int d \left[ e^{at} x(t) \right] = \int e^{at} f(t) \; dt
\]

\[
e^{at} x(t) = \int e^{at} f(t) \; dt + K
\]

\[
x(t) = e^{-at} \int e^{at} f(t) \; dt + Ke^{-at}
\]

In summary, the general solution of the differential equation

\[
\frac{dx(t)}{dt} + ax(t) = f(t)
\]

is called the complete response.

\[
x(t) = e^{-at} \int e^{at} f(t) \; dt + Ke^{-at}
\]

- Complete response
- Forced response
- Natural response
- Steady-state response
- Transient response
If the forcing function \( f(t) \) is a constant, say \( b \), then the differential equation is

\[
\frac{dx(t)}{dt} + ax(t) = b
\]

and the complete response is

\[
x(t) = x_f(t) + x_h(t)
\]

\[
= \frac{b}{a} + Ke^{-at}
\]

where the constant \( K \) is determined from an initial (boundary) condition. For example, when \( t = 0 \),

\[
x(0) = \frac{b}{a} + Ke^0
\]

\[
K = x(0) - \frac{b}{a}
\]

Substituting,

\[
x(t) = \frac{b}{a} + \left[ x(0) - \frac{b}{a} \right] e^{-at}
\]

In general,

\[
x(t) = x_f(t) + \left[ x(0) - x_f(t) \right] e^{-\frac{t}{\tau}}
\]
Example:

Find \( i(t) \) for all time.

\[
\begin{align*}
\text{For } t < 0, & \quad i(t) = \frac{40}{8} = 5 \text{A} \Rightarrow i(0) = 5 \text{A} \\
\text{Superpositioning the } 40 \text{V source for } t < 0, & \quad i_{f,a} = \frac{6.4}{6.4 + 5.6} (5) = 2.166 \text{A} \\
\text{Superpositioning the } 10 \text{A source for } t < 0, & \quad i_{f,b} = \frac{5.6}{5.6 + 6.4} (-10) = -4.66 \text{A} \\
\text{Adding,} & \quad i_f(t) = i_{f,a} + i_{f,b} = 2.166 + (-4.66) = -2 \text{A}
\end{align*}
\]
The Thevenized resistance across the inductor terminals for \( t \geq 0 \) is:

\[
R = 6.4 \Omega + 5.6 \Omega = 12 \Omega
\]

The time constant is:

\[
\tau = \frac{L}{R} = \frac{24 \times 10^{-3}}{12} = 2 \text{ ms}
\]

Using the general equation,

\[
i(t) = i_f(t) + \left[ i(0) - i_f(t) \right] e^{-\frac{t}{\tau}}
\]

\[
= -2 + \left[ 5 - (-2) \right] e^{-\frac{t}{2 \times 10^{-3}}}
\]

Finally,

\[
i(t) = -2 + 7e^{-500t}, \quad t \geq 0
\]

For all \( t \),

\[
i(t) = \begin{cases} 
5 & \text{for } t < 0 \\
-2 + 7e^{-500t} & \text{for } t \geq 0
\end{cases}
\]
The Sinusoidal Source

Consider the graph of $\cos x$ where $x$ is an angle in radians.

By letting $x = wt$, the angle becomes a function of time, where $w$ is termed the radian or angular frequency in radians per second.

The time to complete one cycle of the sinusoid is termed the period of the sinusoid and is denoted by $T$. Therefore,

$$T = t = \frac{x}{\omega} = \frac{2\pi}{\omega} \text{ seconds/cycle}$$

Also,

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \text{ cycles/second or Hertz}$$

The relationships between real and radian frequency are

$$f = \frac{\omega}{2\pi} \text{ Hertz} \quad \omega = 2\pi f \text{ radians per second}$$

A more general sinusoid is $\cos (wt + \phi)$, where $\phi$ is termed the phase angle of the sinusoid. When $\phi$ is a negative quantity, the sinusoid is translated (shifted) to the right by the amount $\phi$. If $\phi$ is a positive quantity, the shift is to the left.
The graph of \( \cos(\omega t + \phi) \) is

\[
\begin{align*}
\text{for negative } \phi. \text{ Also, } \cos(\omega t - \frac{\pi}{2}) &= \sin \omega t \text{ and } \sin(\omega t + \frac{\pi}{2}) &= \cos \omega t.
\end{align*}
\]

**Example:**

Find the forced response \( v(t) \) of the following circuit.

Using KVL,

\[
\begin{align*}
L \frac{di}{dt} + Ri &= v_s \\
6 \frac{di}{dt} + 5i &= 12 \cos 2t + 7 \sin 2t \\
\frac{di}{dt} + \frac{5}{6}i &= 2 \cos 2t + \frac{7}{6} \sin 2t
\end{align*}
\]

The solution form for \( i(t) \) must be

\[
i(t) = A_1 \cos 2t + A_2 \sin 2t
\]

Therefore,

\[
\begin{align*}
(-2A_1 \sin 2t + 2A_2 \cos 2t) + \frac{5}{6} (A_1 \cos 2t + A_2 \sin 2t) &= 2 \cos 2t + \frac{7}{6} \sin 2t \\
\left(\frac{5}{6} A_1 + 2A_2\right) \cos 2t + \left(\frac{5}{6} A_2 - 2A_1\right) \sin 2t &= 2 \cos 2t + \frac{7}{6} \sin 2t
\end{align*}
\]
and,
\[
\begin{aligned}
\frac{5}{6} A_1 + 2 A_2 &= 2 \\
-2 A_1 + \frac{5}{6} A_2 &= \frac{7}{6}
\end{aligned}
\] \Rightarrow
\[
\begin{aligned}
2 A_1 + \frac{24}{5} A_2 &= \frac{24}{5} \\
-2 A_1 + \frac{5}{6} A_2 &= \frac{7}{6}
\end{aligned}
\]

\[
\frac{169}{30} A_2 = \frac{179}{30} \\
2 A_1 = \frac{5}{6} A_2 - \frac{7}{6} = \frac{5}{6} \left( \frac{179}{169} \right) - \frac{7}{6} \left( \frac{169}{169} \right) = -\frac{48}{169}
\]

\[
A_2 = \frac{179}{169} \\
A_1 = -\frac{24}{169}
\]

Finally,
\[
i(t) = A_1 \cos 2t + A_2 \sin 2t
\]
\[
= -\frac{24}{169} \cos 2t + \frac{179}{169} \sin 2t
\]

and
\[
v(t) = L \frac{di(t)}{dt}
\]
\[
= 6 \left( \frac{48}{169} \sin 2t + \frac{358}{169} \cos 2t \right)
\]
\[
= 2.148 \cos 2t + 2.88 \sin 2t
\]
\[
= 12.71 \cos 2t + 1.70 \sin 2t
\]
\[
= 12.82 \cos \left( 2t - 7.62^\circ \right)
\]

Rewriting \( v_s(t) \),
\[
v_s(t) = 12 \cos 2t + 7 \sin 2t
\]
\[
= 13.89 \cos \left( 2t - 30.26^\circ \right)
\]

and \( v(t) \) leads \( v_s(t) \) by \( 22.64^\circ \).
\[ v(t) = 12.82 \cos (2t - 7.62^\circ) \text{ V} \]

\[ v_s(t) = 13.89 \cos (2t - 30.26^\circ) \text{ V} \]
In the previous example problem, a sinusoid that was represented in quadratic form as 12.71 \cos 2t + 1.70 \sin 2t was converted to the standard form 12.82 \cos (2t - 7.62^\circ). To understand this conversion, consider the following trigonometric identity:

\[
A \cos (\omega t - \theta) = A \cos \omega t \cos \theta + A \sin \omega t \sin \theta
\]

Then,

\[
A \cos (\omega t - \theta) = (A \cos \theta) \cos \omega t + (A \sin \theta) \sin \omega t
\]

= \chi \cos \omega t + y \sin \omega t

Now,

\[
\frac{A \sin \theta}{A \cos \theta} = \tan \theta = \frac{y}{\chi}
\]

\[
\theta = \tan^{-1} \frac{y}{\chi}
\]

And,

\[
A^2 \cos^2 \theta + A^2 \sin^2 \theta = \chi^2 + y^2
\]

\[
A^2 (\cos^2 \theta + \sin^2 \theta) = \chi^2 + y^2
\]

\[
A = \sqrt{\chi^2 + y^2}
\]
The Phasor Transform

Phasor transforms permit sinusoidal circuits to be analyzed in the frequency domain in a manner analogous to resistive circuits by using the phasor versions of KCL, KVL, nodal analysis, mesh analysis, etc.

As an example, consider the earlier problem.

\[ V_s(t) = 12 \cos 2t + 7 \sin 2t \]
\[ = 13.89 \cos (2t - 30.26^\circ) \]
\[ = \text{Re} \left( 13.89 e^{i(2t - 30.26^\circ)} \right) \]
\[ = A e^{i(\omega t + \phi)} \]

The forced response to this complex sinusoid has the form \( B e^{i(\omega t + \phi)} \).

Therefore,

\[ 6 \frac{di}{dt} + 5i = 13.89 e^{i(2t - 30.26^\circ)} \]

\[ \frac{di}{dt} + \frac{5}{6}i = 2.32 e^{i(2t - 30.26^\circ)} \]

\[ \frac{d}{dt} \left[ B e^{i(2t + \phi)} \right] + \frac{5}{6} \left[ B e^{i(2t + \phi)} \right] = 2.32 e^{i(2t - 30.26^\circ)} \]

\[ j2Be^{i2t} + \frac{5}{6} Be^{i2t} i\phi = 2.32 e^{i2t} \]

Dividing by the common factor \( e^{i2t} \),

\[ (j2 + \frac{5}{6}) Be^{i\phi} = 2.32 e^{-j30.26^\circ} \]

\[ (5 + j12) Be^{i\phi} = 13.89 e^{-j30.26^\circ} \]

\[ Be^{i\phi} = \frac{13.89 e^{-j30.26^\circ}}{5 + j12} \]
Multiplying the numerator and denominator by the conjugate \( 5 - j12 \),

\[
Be^{j\phi} = \frac{13.89 (5 - j12)}{169} e^{-j30.26^\circ}
\]

\[
= (0.41 - j0.99) e^{-j30.26^\circ}
\]

\[
= 1.07 e^{-j67.56^\circ} e^{-j30.26^\circ}
\]

\[
= 1.07 e^{-j97.76^\circ}
\]

Thus,

\[
i(t) = Be^{j(\omega t + \phi)} = 1.07 e^{j(2t - 97.76^\circ)}
\]

and

\[
v(t) = L \frac{di(t)}{dt}
\]

\[
= 6 \frac{d}{dt} \left[ 1.07 e^{j(2t - 97.76^\circ)} \right]
\]

\[
= 6 \frac{d}{dt} \left[ 1.07 e^{j2t} e^{-j97.76^\circ} \right]
\]

\[
= j6 (1.07) 2e^{j2t} e^{-j97.76^\circ}
\]

\[
= j12.84 e^{j(2t - 97.76^\circ)}
\]

\[
= 12.84 e^{j90^\circ} e^{j(2t - 97.76^\circ)}
\]

\[
= 12.84 e^{j(2t - 7.76^\circ)}
\]

Finally,

\[
v(t) = \text{Re} \left[ 12.84 e^{j(2t - 7.76^\circ)} \right]
\]

\[
= 12.84 \cos (2t - 7.76^\circ) V
\]
A simpler notation results when the \( e^{j\omega t} \) factor is suppressed.

\[
Ae^{j(\omega t + \phi)} = Ae^{j\phi} e^{j\omega t} \Rightarrow Ae^{j\phi} = A/\theta
\]

For a resistor,

\[
i \rightarrow R \hspace{1cm} i \rightarrow \hspace{1cm} \frac{R}{\theta}
\]

and

\[
v = Ri \quad \text{(time domain)}
\]

\[
Ve^{j(\omega t + \phi)} = RIe^{j(\omega t + \phi)}
\]

\[
Ve^{j\omega t} e^{j\phi} = RIE^{j\omega t} e^{j\phi}
\]

\[
Ve^{j\phi} = RIE^{j\phi} \quad \text{(frequency domain)}
\]

\[
\frac{v}{\phi} = RI / \theta
\]

\[
\begin{array}{c}
\hat{v} \hspace{1cm} RI \\
\end{array}
\]

(phasor equation)

For an inductor,

\[
i \rightarrow L \hspace{1cm} i \rightarrow \hspace{1cm} \frac{L}{\theta}
\]

\[
v = \frac{L}{\theta} \frac{di}{dt}
\]

\[
Ve^{j(\omega t + \phi)} = \frac{L}{\theta} \frac{d}{dt} \left[ Ie^{j(\omega t + \phi)} \right]
\]

\[
Ve^{j\omega t} e^{j\phi} = j\omega LIe^{j\omega t} e^{j\phi}
\]

\[
Ve^{j\phi} = j\omega LIe^{j\phi}
\]

\[
\frac{v}{\phi} = j\omega LI / \theta
\]

\[
\begin{array}{c}
\hat{v} \hspace{1cm} j\omega LI \\
\end{array}
\]
And a capacitor,

\[ i = C \frac{dv}{dt} \]

\[ I e^{i(\omega t + \phi)} = C \frac{d}{dt} [ V e^{i(\omega t + \phi)} ] \]

\[ I e^{j\omega t} e^{i\phi} = j \omega C V e^{j\omega t} e^{i\phi} \]

\[ I / (j\omega) = j \omega C V / \phi \]

\[ \bar{I} = j \omega C \bar{V} \]

or

\[ \bar{V} = \frac{1}{j\omega C} \bar{I} \]

In general, each phasor equation is of the form

\[ \bar{V} = Z \bar{I} \]

where \( Z \) is the impedance of the element. Specifically,

\[ Z_R = R \]

\[ Z_L = j\omega L \]

\[ Z_C = \frac{1}{j\omega C} \]
Both KCL and KVL hold for current and voltage phasors respectively.

For example, in the time domain

\[ i_1(t) + i_2(t) + i_3(t) + \ldots + i_n(t) = 0 \]

For sinusoids,

\[ I_1 \cos(\omega t + \theta_1) + I_2 \cos(\omega t + \theta_2) + \ldots + I_n \cos(\omega t + \theta_n) = 0 \]

\[ \text{Re} \left[ I_1 e^{j(\omega t + \theta_1)} \right] + \text{Re} \left[ I_2 e^{j(\omega t + \theta_2)} \right] + \ldots + \text{Re} \left[ I_n e^{j(\omega t + \theta_n)} \right] = 0 \]

Dividing by \( e^{j\omega t} \),

\[ I_1 e^{j\theta_1} + I_2 e^{j\theta_2} + I_3 e^{j\theta_3} + \ldots + I_n e^{j\theta_n} = 0 \]

\[ I_1 \angle \theta_1 + I_2 \angle \theta_2 + I_3 \angle \theta_3 + \ldots + I_n \angle \theta_n = 0 \]

and in the frequency domain,

\[ \bar{I}_1 + \bar{I}_2 + \bar{I}_3 + \ldots + \bar{I}_n = 0 \]

Similarly, when

\[ v_1(t) + v_2(t) + v_3(t) + \ldots + v_n(t) = 0 \]

it follows that

\[ \bar{V}_1 + \bar{V}_2 + \bar{V}_3 + \ldots + \bar{V}_n = 0 \]
Now using the phasor transform on the earlier problem,

\[ V_s(t) = 13.89 \cos(2t - 30.26^\circ) \]

By voltage division,

\[ \overline{V} = \frac{j12}{5 + j12} \overline{V_s} = \frac{12/90^\circ}{13/67.38^\circ} \] \[ 13.89/ -30.26^\circ \]

\[ = 12.82/ -7.64^\circ \text{ V} \]

In the time domain,

\[ v(t) = 12.82 \cos(2t - 7.64^\circ) \text{ V} \]
Consider the following circuit with arbitrary impedance $Z$.

The current will also be a sinusoid, say $i = I_m \cos(\omega t + \theta_i)$, and the instantaneous power is

$$P = vi$$

$$= [V_m \cos(\omega t + \theta_V)] [I_m \cos(\omega t + \theta_i)]$$

By definition, the average power is

$$P = \frac{1}{T} \int_{t_0}^{t_0+T} p \, dt$$

Integration yields

$$P = \frac{1}{2} V_m I_m \cos(\theta_V - \theta_i)$$

If $Z = R$ then $\theta_V = \theta_i$, and

$$P_R = \frac{1}{2} I_m^2 R$$

If $Z = \pm jX$ then $\theta_V - \theta_i = \pm 90^\circ$, and

$$P_x = 0$$
Example:

Find the average power dissipated by each resistor.

\[ 4 \cos (4t-60^\circ) \]

In the frequency domain,

\[ 4 \frac{L}{60^\circ} \]
\[ \omega = 4 \]

By inspection,

\[ \bar{I}_1 = \frac{\frac{4}{60^\circ} j^4}{0 (4+j4)} \]
\[ = \frac{\frac{1}{60^\circ} j}{(2-j) j} \]
\[ = \frac{1+j}{4+j} \]

\[ = \frac{(1.414/45^\circ)(1/60^\circ)}{4.12/14.04^\circ} \]
\[ = 0.34/29.04^\circ \text{ A} \]

\[ \bar{I}_2 = \frac{(8-j4) 4/60^\circ}{j4 \quad 0} \]
\[ \frac{(8-j4) j4}{j4 \quad (4+j4)} \]

\[ = \frac{(2-j) 1/60^\circ}{j \quad 0} \]
\[ = \frac{-j}{4+j} \]

\[ = \frac{(1/90^\circ)(1/60^\circ)}{4.12/14.04^\circ} \]
\[ = 0.24/164.04^\circ \text{ A} \]
Therefore,

\[ P_{8a} = \frac{1}{2} I_m^2(\theta) = 4 |I_1|^2 = 4 (0.34)^2 = 0.46 \text{ W} \]

\[ P_{4n} = \frac{1}{2} I_m^2(4) = 2 |I_2|^2 = 2 (0.24)^2 = 0.12 \text{ W} \]
Effective Values

Consider the following circuit with a sinusoidal source:

\[ I_m \cos(\omega t + \theta_i) \]

\[ V_m \cos(\omega t + \theta_v) \]

\[ R \]

The average power is

\[ P_R = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = \frac{1}{2} V_m I_m = \frac{1}{2} I_m^2 R = \frac{1}{2} \frac{V_m^2}{R} \]

For a battery,

\[ V = V_{eff} \]

\[ R \]

and the average power is

\[ P_R = V_{eff} I_{eff} = I_{eff}^2 R = \frac{V_{eff}^2}{R} \]

If both sources are to produce the same power dissipation in \( R \), there must be some effective value of \( I_m \) and \( V_m \) for the sinusoidal source that is equivalent to the dc source (battery).

\[ I_{eff}^2 R = \frac{1}{2} I_m^2 R \]

\[ \frac{V_{eff}^2}{R} = \frac{1}{2} \frac{V_m^2}{R} \]

\[ I_{eff} = \frac{I_m}{\sqrt{2}} \]

\[ V_{eff} = \frac{V_m}{\sqrt{2}} \]
Therefore,

\[ P_R = \frac{1}{2} V_m I_m = \frac{1}{2} (V_{\text{eff}}^2 I_{\text{eff}}^2) = V_{\text{eff}} I_{\text{eff}} \]

and,

\[ P_R = V_{\text{eff}} I_{\text{eff}} = I_{\text{eff}}^2 R = \frac{V_{\text{eff}}^2}{R} \]

For an arbitrary impedance \( Z \),

\[ P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = V_{\text{eff}} I_{\text{eff}} \cos(\theta_v - \theta_i) \]

For a non-sinusoidal voltage across a resistor or current through a resistor, the average power absorbed is

\[ P = \frac{1}{T} \int_{t_0}^{t_0+T} p \, dt = \frac{1}{T} \int_{t_0}^{t_0+T} R i^2 \, dt = R I_{\text{eff}}^2 \]

Solving for \( I_{\text{eff}} \),

\[ I_{\text{eff}} = \sqrt{\frac{1}{T} \int_{t_0}^{t_0+T} i^2 \, dt} \]

and the effective value is also known as the root-mean-square or \textit{rms} value.
Also,
\[ V_{\text{eff}} = \sqrt{\frac{1}{T} \int_0^{T/2} v^2 \, dt} \]

**Example:**
Determine the effective value of the following half-wave rectified sine.

\[ V(t) \]

\[ \begin{align*}
V_{\text{eff}} &= \sqrt{\frac{1}{T} \int_0^{T/2} v^2 \, dt} = \sqrt{\frac{1}{T} \int_0^{T/2} \left[100 \sin \left(\frac{3\pi}{T} t\right)\right]^2 \, dt} \\
&= \sqrt{\frac{1}{T} \int_0^{T/2} \left[1 - \cos \left(\frac{3\pi}{T} t\right)\right] \, dt} \\
&= \sqrt{100 \int_0^{T/2} \left[1 - \cos \left(\frac{3\pi}{T} t\right)\right] \, dt} \\
&= \sqrt{100 \left[\left(\frac{1}{4}\right) - 0 - \frac{1}{4\pi} \left(\sin 2\pi - \sin 0\right)\right]} = \frac{100}{2} = 50 \text{V}
\end{align*} \]

The average or dc value is
\[ V_{\text{dc}} = \frac{1}{T} \int_0^{T/2} v \, dt = \frac{1}{T} \int_0^{T/2} 100 \sin \left(\frac{3\pi}{T} t\right) \, dt = -\frac{100}{\pi} \left(\frac{T}{2\pi}\right) \cos \left(\frac{3\pi}{T} t\right) \bigg|_0^{T/2} \\
= -\frac{50}{\pi} \left(\cos \pi - \cos 0\right) = -\frac{50}{\pi} (-1 - 1) = \frac{100}{\pi} = 31.83 \text{V} \]
Power Factor

Once again, consider the following circuit with arbitrary impedance $Z$.

$$\frac{V_m}{\theta_v} \rightarrow \frac{I_m}{\theta_i} \rightarrow \frac{V_m}{\theta_v} = \frac{V_m}{I_m} \cdot \frac{I_m}{\theta}$$

The average power absorbed by the load impedance is

$$P = \frac{1}{2} V_m I_m \cos \theta = V_{eff} I_{eff} \cos \theta = (\text{apparent power})(\cos \theta)$$

where

$$\cos \theta = \frac{P}{V_{eff} I_{eff}} = \frac{\text{average power}}{\text{apparent power}} = \text{power factor (pf)}$$

The angle $\theta = \theta_v - \theta_i$ is called the \underline{power factor angle}.

---

If the load impedance is inductive, $x > 0$ and

$$\Theta = \angle \theta = \tan^{-1} \frac{x}{R} > 0$$

The current lags the voltage resulting in a \underline{lagging power factor}. If the load impedance is \underline{capacitive}, $x < 0$ and

$$\Theta = \angle \theta = \tan^{-1} \frac{x}{R} < 0$$

Now the current leads the voltage producing a \underline{leading power factor}.

In each case, the current is referenced to the voltage to determine whether the power factor is lagging or leading.
Example:

A 1,000 W electric motor is designed to run from 220 V (rms) @ 60 Hz and has a lagging power factor of 0.8. If the motor is connected to the source through a 1Ω resistor, determine (a) the "line loss" and (b) the percent reduction in power loss with a 50 μF capacitor in parallel with the motor.

\[ I_{\text{motor}} = \ \begin{array}{c}
\frac{\overline{V}_s}{\overline{I}_s} = \frac{V_{\text{eff}}}{I_{\text{eff}}} \end{array} \]
\[ \omega = 377 \]

\[ V_s = 220/0^\circ \text{ (rms)} \]

\[ I_{\text{motor}} = 1\Omega \]

\[ \overline{I}_{\text{motor}} = \frac{220}{123.23} = 5.57/35.99^\circ \text{ (rms)} A \]

\[ P_L = I^2 R = (5.57)^2 1 = 31.02 \text{ W} \]

The equivalent circuit of the motor is

\[ Z_{\text{motor}} = \frac{V_{\text{eff}}}{I_{\text{eff}}} = \frac{\overline{V}_{\text{motor}}}{\overline{I}_{\text{motor}}} = \frac{V_{\text{eff}}}{\overline{I}_{\text{motor}}} \]

\[ = 30.98 + j23.23 \Omega \]

This represents an inductive load.

(a)

\[ V_s = 220/0^\circ \text{ (rms)} \]

\[ I_{\text{motor}} = 1\Omega \]

\[ Z_{\text{motor}} = 30.98 + j23.23 \]

Solving for \( \overline{I}_{\text{motor}} \),

\[ \overline{I}_{\text{motor}} = \frac{\overline{V}_s}{Z_{\text{motor}} + 1} = \frac{220/0^\circ}{31.98 + j23.23} = \frac{220/0^\circ}{39.53/35.99^\circ} = 5.57/35.99^\circ \text{ (rms)} A \]
with the 50 μF capacitor connected,

(b)

\[ V_s = 220/0^\circ \text{ (rms) } \]
\[ \omega = 317 \]

\[ I_m \]
\[ 30.98 \Omega \]
\[ 123.33 \Omega \]
\[ -j53.05 \Omega \]

The new equivalent load is

\[ Z_{eq} = \frac{(38.72/36.87^\circ)(53.05/-90^\circ)}{30.98 - j29.82} = \frac{2054.10/-53.13^\circ}{43.00} \]

\[ = 47.77/-9.22^\circ = 47.15 - j7.65 \Omega \]

which is capacitive. The new current drawn from the source is

\[ I_s = \frac{V_s}{Z_{eq} + 1} = \frac{220/0^\circ}{48.15 - j7.65} = \frac{220/0^\circ}{48.75/-9.03^\circ} = 4.51/9.03^\circ \text{ (rms) A} \]

The power factor for the new load (the motor in parallel with the capacitor) is

\[ \text{pf} = \cos(-9.22^\circ) = 0.99 \text{ leading} \]

The new line loss is

\[ P_L = I_{eff}^2 R = (4.51)^2 1 = 20.34 \text{ W} \]

The percent reduction in power loss is

\[ \Delta \% P_L = \frac{31.02 - 20.34}{31.02} \times 100\% = 34.43\% \]
Complex Power

Average or real power can be generalized with the notion of complex power. Beginning with the following definition of average power,

\[ P = V_{\text{eff}} I_{\text{eff}} \cos \theta = V_{\text{eff}} I_{\text{eff}} \text{Re}[e^{j\theta}] \]

\[ = V_{\text{eff}} I_{\text{eff}} \text{Re}[e^{j(\omega t - \phi)}] = V_{\text{eff}} I_{\text{eff}} \text{Re}[e^{j\omega t}e^{-j\theta}] \]

\[ = \text{Re}[V_{\text{eff}} I_{\text{eff}} e^{j\theta} e^{-j\theta}] = \text{Re}[(V_{\text{eff}} e^{j\omega})(I_{\text{eff}} e^{-j\omega})] \]

\[ = \text{Re}[(V_{\text{eff}} e^{j\omega})(I_{\text{eff}} e^{j\omega})^*] = \text{Re}[\overline{V}_{\text{eff}} \overline{I}_{\text{eff}}^*] \]

Therefore, the average power absorbed is the real part of the complex power absorbed by the load, where complex power is defined as

\[ S = \overline{V}_{\text{eff}} \overline{I}_{\text{eff}}^* \]

To distinguish complex power from either real or reactive power, the term "volt-amperes" is used.

---

Rewriting complex power,

\[ S = \overline{V}_{\text{eff}} \overline{I}_{\text{eff}}^* = (V_{\text{eff}} e^{j\omega})(I_{\text{eff}} e^{-j\omega}) \]

\[ = V_{\text{eff}} I_{\text{eff}} e^{j(\omega t - \phi)} = V_{\text{eff}} I_{\text{eff}} e^{j\theta} \]

Using Euler's identity,

\[ S = V_{\text{eff}} I_{\text{eff}} (\cos \theta + j \sin \theta) \]

\[ = V_{\text{eff}} I_{\text{eff}} \cos \theta + j V_{\text{eff}} I_{\text{eff}} \sin \theta \]

\[ = \text{(real power)} + j \text{(reactive power)} = P + j Q \]

The term "var" is used for reactive power. It stands for volt-amperes reactive.
The magnitude of complex power is

\[ |S| = \sqrt{P^2 + Q^2} \]

\[ = \sqrt{\text{real power}^2 + \text{reactive power}^2} \]

\[ = \text{apparent power} \]

Expressed as a (power) triangle,

\[ P = \text{real power} \]
\[ Q = \text{reactive power} \]
\[ I_m = \text{apparent power} \]

where

\[ \text{pf} = \cos \theta = \frac{P}{|S|} = \frac{\text{real power}}{\text{apparent power}} \]

**Example:**

Consider the following circuit where \( \bar{I}_1 = 2\sqrt{2}/-105^\circ \) A and \( \bar{I}_2 = \sqrt{2}/-105^\circ \) A.

\[ \frac{36/60^\circ}{\omega = 3} \]

Determine:

(a) The complex power absorbed by the capacitor.

(b) The real power absorbed by the capacitor.

(c) The complex power absorbed by the resistor.

(d) The real power absorbed by the resistor.
(a) 
\[ \bar{V}_c = -j 6 \bar{I}_2 = (6 \angle -90^\circ)(6 \angle 105^\circ) = 6 \sqrt{2} \angle 195^\circ = 6 \sqrt{2} \angle 120^\circ \]
\[ S_c = \frac{1}{2} \bar{V}_c \bar{I}_2^* = \frac{1}{2} (6 \sqrt{2} \angle 120^\circ)(6 \angle 105^\circ) = 6 \angle 270^\circ = 6 \angle 90^\circ \text{ VA} \]

(b) 
\[ P_c = V_{\text{eff}} I_{\text{eff}} \cos \theta = V_{\text{eff}} I_{\text{eff}} \cos (-90^\circ) = 0 \text{ W} \]

(c) 
\[ \bar{V}_r = 9 \bar{I}_1 = 9 (2 \sqrt{2} \angle -105^\circ) = 18 \sqrt{2} \angle -105^\circ \]
\[ S_r = \frac{1}{2} \bar{V}_r \bar{I}_1^* = \frac{1}{2} (18 \sqrt{2} \angle -105^\circ)(2 \sqrt{2} \angle 105^\circ) = 36 \angle 0^\circ \text{ VA} \]

(d) 
\[ P_r = \frac{1}{2} |\bar{I}_1|^2 R = \frac{1}{2} (2 \sqrt{2})^2 9 = 36 \text{ W} \]
Self-Inductance

Consider the following N-turn coil.

The right-hand rule determines the orientation of the magnetic field related to the direction of the current. Induced voltage can be expressed by Faraday's law,

\[ v = \frac{\Delta \phi}{\Delta t} \]

where \( \phi \) is the flux linkage in weber-turns. Flux linkage is the product of the magnetic field \( \phi \) in webers and the number of turns linked by the field \( N \).

\[ \phi = N \phi \]

The magnitude of the flux \( \phi \) can be written as

\[ \phi = \mathcal{P} N i \]

where \( \mathcal{P} \) is the permeance of the space occupied by the field. Combining these relationships,

\[ v = \frac{\Delta \phi}{\Delta t} = \frac{\Delta (N \phi)}{\Delta t} = N \frac{\Delta \phi}{\Delta t} = N \mathcal{P} \frac{\Delta i}{\Delta t} = \mathcal{L} \frac{\Delta i}{\Delta t} \]

thus self-inductance is proportional to the square of the number of turns of the coil.

\[ L = N^2 \mathcal{P} \]
Mutual inductance is the circuit parameter that relates the voltage induced in one coil to the time-varying current in another coil. Consider the following two coils that are magnetically coupled.

\[ \phi_1 = \phi_{11} + \phi_{21} \]

Letting \( \phi_1 \) represent the total flux linking coil 1,

\[ \phi_{11} = \mathcal{P}_{11} N_1 i_1 \]
\[ \phi_{21} = \mathcal{P}_{21} N_2 i_1 \]

Substituting,

\[ \mathcal{P}_1 = \mathcal{P}_{11} + \mathcal{P}_{21} \]

Using Faraday's law,

\[ v_1 = \frac{d\phi_1}{dt} = \frac{d}{dt} (N_1 \phi_1) = N_1 \frac{d}{dt} (\phi_{11} + \phi_{21}) \]

\[ = v_1^2 (\mathcal{P}_{11} + \mathcal{P}_{21}) \frac{di_1}{dt} = N_1 i_1 \mathcal{P}_{11} \frac{di_1}{dt} = L_1 \frac{di_1}{dt} \]

where \( L_1 \) is the self-inductance of coil 1.
\[ v_2 = \frac{d\psi_2}{dt} = \frac{d}{dt} (N_2 \phi_2) = N_2 \frac{d}{dt} (\beta_{21} N_1 i_1) \]
\[ = N_2 N_1 \beta_{21} \frac{di_1}{dt} = M_{21} \frac{di_1}{dt} \]

The mutual inductance \( M_{21} \) relates the voltage induced in coil 2 to the current in coil 1.

\( M_{21} = N_2 N_1 \beta_{21} \)

If coil 2 is excited and coil 1 is left open, a similar procedure yields \( M_{12} = N_1 N_2 \beta_{12} \).

For nonmagnetic materials, \( \beta_{12} \) and \( \beta_{21} \) are equal. Therefore, \( M_{12} = M_{21} = M \).

Mutual inductance is also a function of the self-inductances. Beginning with
\[ L_1 = N_1^2 \beta_1 \]
\[ L_2 = N_2^2 \beta_2 \]

it follows that
\[ L_1 L_2 = N_1^2 N_2^2 \beta_1 \beta_2 = N_1^2 N_2 \left( \beta_{11} + \beta_{12} \right) \left( \beta_{21} + \beta_{22} \right) \]
\[ = N_1^2 N_2 \left( \frac{\beta_{11} \beta_{22} + \beta_{11} \beta_{12} + \beta_{21} \beta_{22} + \beta_{21} \beta_{12}}{\beta_{11}} \right) \]
\[ = \left( N_1 N_2 \beta_{12} \right)^2 \left( 1 + \frac{\beta_{11}}{\beta_{12}} \right) \left( 1 + \frac{\beta_{22}}{\beta_{12}} \right) \]

assuming \( \beta_{21} = \beta_{12} \).
Making the following substitution,

\[ \frac{1}{k^2} = \left(1 + \frac{\mathcal{P}_{11}}{\mathcal{P}_{12}}\right) \left(1 + \frac{\mathcal{P}_{12}}{\mathcal{P}_{11}}\right) \]

results in

\[ M^2 = k^2 L_1 L_2 \]

\[ M = k \sqrt{L_1 L_2} \]

where the constant \( k \) is termed the coefficient of coupling. The limits of \( k \) are

\[ 0 \leq k \leq 1 \]

The coefficient of coupling can also be expressed in the following manner.

\[ k = \frac{\text{flux linkages between } L_1 \text{ and } L_2}{\text{flux produced by } L_1} \]

Pictorially,

which represents \( k = \frac{1}{4} = 0.25 \).
Polarity of Mutually Induced Voltages

Consider two coils with self-inductances $L_1$ and $L_2$ that are magnetically coupled.

\[ i_1 \quad \begin{array}{c} \nwarrow \vspace{1cm} \downarrow M_{12} \vspace{1cm} \nearrow \end{array} \quad i_2 \]

\[ V_1 \quad \begin{array}{c} \nwarrow \vspace{1cm} \downarrow L_1 \vspace{1cm} \nearrow \end{array} \quad V_2 \]

Furthermore,

\[ V_2 = M_{21} \frac{di_1}{dt}, \quad i_1 \neq 0 \text{ and } i_2 = 0 \]

\[ V_1 = M_{12} \frac{di_2}{dt}, \quad i_2 \neq 0 \text{ and } i_1 = 0 \]

where $M_{12}$ and $M_{21}$ are the coefficients of mutual inductance or mutual inductance, for short.

The dots describe the physical orientation of the coils and indicate the phase relationship of the current in one inductor and the resulting induced voltage in the other inductor. Current flowing into one dot of an inductor results in a positive induced voltage at the dot of the other inductor.
Such an arrangement of coils produces a four-terminal device called a transformer. One side of the transformer is often called the primary winding and the other side is the secondary winding. If $M_{12} = M_{21} = M$, the symbol becomes

![Transformer symbol]

The equations that describe this circuit element in the time domain are:

\[ v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \]
\[ v_2 = M \frac{di_1}{dt} + L_2 \frac{di_2}{dt} \]

In the phasor domain:

\[ \bar{V}_1 = j\omega L_1 \bar{I}_1 + j\omega M \bar{I}_2 \]
\[ \bar{V}_2 = j\omega M \bar{I}_1 + j\omega L_2 \bar{I}_2 \]

Example:

Write a set of mesh-current equations for the following circuit.

![Mesh circuit diagram]
Around loop 1,

\[ V_g = 2(i_1 - i_2) + 7 \frac{d}{dt} (i_1 - i_3) - 2 \frac{di_2}{dt} \]

Loop 2,

\[ 0 = 3 \frac{di_2}{dt} + 5(i_2 - i_3) + 2(i_2 - i_3) + 2 \frac{d}{dt} (i_3 - i_1) \]

Loop 3,

\[ 0 = 5(i_3 - i_2) + 8i_3 + 7 \frac{d}{dt} (i_3 - i_1) + 2 \frac{di_3}{dt} \]
The Ideal Transformer

Consider the following ideal transformer in the phasor domain.

\[
\begin{array}{c}
\frac{\bar{I}_1}{\bar{I}_2} \\
\bar{V}_1 = j\omega L_1 \bar{I}_1 + j\omega M \bar{I}_2 \\
\bar{I}_1 = \frac{\bar{V}_1 - j\omega M \bar{I}_2}{j\omega L_1}
\end{array}
\]

Also,

\[
\begin{align*}
\bar{V}_2 &= j\omega M \bar{I}_1 + j\omega L_2 \bar{I}_2 \\
&= j\omega M \left( \frac{\bar{V}_1 - j\omega M \bar{I}_2}{j\omega L_1} \right) + j\omega L_2 \bar{I}_2 \\
&= \frac{M \bar{V}_1}{L_1} - \frac{j\omega M^2 \bar{I}_2}{L_1} + j\omega L_2 \bar{I}_2
\end{align*}
\]

For perfect coupling,

\[
K = \frac{M}{\sqrt{L_1 L_2}} = 1 \Rightarrow M^2 = L_1 L_2
\]

Therefore,

\[
\begin{align*}
\bar{V}_2 &= \frac{V_1}{L_1} - \frac{j\omega L_2 \bar{I}_2}{L_1} + j\omega L_2 \bar{I}_2 = \sqrt{\frac{L_2}{L_1}} \bar{V}_1 \\
\text{and,} \\
\frac{\bar{V}_2}{\bar{V}_1} &= \sqrt{\frac{L_2}{L_1}} = \sqrt{\frac{N_2^2 P}{N_1^2 P}} = \frac{N_2}{N_1} = \alpha = \text{turns ratio}
\end{align*}
\]

Solving for the current ratio produces

\[
\frac{\bar{I}_1}{\bar{I}_2} = -\sqrt{\frac{L_2}{L_1}} = -\frac{N_2}{N_1} = -\alpha
\]
In general, for an ideal transformer,

\[
\begin{align*}
    v_2 &= a \cdot v_1 \iff v_1 = \frac{v_2}{a} \\
    i_2 &= -\frac{i_1}{a} \iff i_1 = -a \cdot i_2
\end{align*}
\]

In the phasor domain,

\[
\begin{align*}
    \bar{v}_2 &= a \cdot \bar{v}_1 \iff \bar{v}_1 = \frac{\bar{v}_2}{a} \\
    \bar{i}_2 &= -\frac{\bar{i}_1}{a} \iff \bar{i}_1 = -a \cdot \bar{i}_2
\end{align*}
\]

The instantaneous energy stored in an ideal transformer is

\[
\begin{align*}
    \omega(t) &= \frac{1}{2} L_1 i_1^2 + \frac{1}{2} L_2 i_2^2 + M_i i_1 i_2 \\
    &= \frac{1}{2} L_1 (-ai_2)^2 + \frac{1}{2} L_2 i_2^2 + M (-ai_2) i_2 \\
    &= \left( \frac{1}{2} L_1 a^2 + \frac{1}{2} L_2 - Ma \right) i_2^2 \\
    &= \left[ \frac{1}{2} L_1 \left( \frac{L_2}{L_1} \right) + \frac{1}{2} L_2 - \sqrt{L_1 L_2} \cdot \sqrt{\frac{L_2}{L_1}} \right] i_2^2 \\
    &= \left( \frac{1}{2} L_2 + \frac{1}{2} L_2 - L_2 \right) i_2^2 = 0 J
\end{align*}
\]

Therefore, the instantaneous and average powers are also zero.
Example:

The following transformer is ideal.

Determine:

(a) The impedance seen by the voltage source,
(b) The voltage gain \( V_2 / V_1 \),
(c) The new load that will absorb maximum power, and
(d) The value of a that will result in the 16Ω resistor absorbing maximum power.

(a)

\[ Z_{int} = R_1 + Z_r = R_1 + R_L \left( \frac{V_i}{V_1} \right)^2 = 3 + 16 \left( \frac{1}{4} \right)^2 = 4Ω \]

(b)

\[ \bar{V}_i = \frac{1}{1+3} \bar{V}_g = \frac{1}{4} \bar{V}_g \]
\[ \bar{V}_2 = a \bar{V}_i = 4 \left( \frac{1}{4} \bar{V}_g \right) = \bar{V}_g \]
\[ \frac{\bar{V}_2}{\bar{V}_g} = 1 \]

(c)

\[ Z_0 = R_1 \left( \frac{V_i}{V_1} \right)^2 = 3 \left( \frac{1}{4} \right)^2 = 4Ω \]

(d)

\[ z_0 = 3a^2 = R_L = 16 \Rightarrow a^2 = \frac{16}{3} \Rightarrow a = \frac{4}{\sqrt{3}} = 2.31 \]