Chapter 10: Sinusoidal Steady-State Analysis

10.1 Basic Approach
10.2 Nodal Analysis
10.3 Mesh Analysis
10.4 Superposition Theorem
10.5 Source Transformation
10.6 Thevenin & Norton Equivalent Circuits
10.7 Op Amp AC Circuits
10.8 Applications
10.9 Summary
10.1 Basic Approach

- 3 Steps to Analyze AC Circuits:
  1. **Transform** the circuit to the phasor or frequency domain.
  2. **Solve** the problem using circuit techniques (nodal analysis, mesh analysis, superposition, etc.).
  3. **Transform** the resulting phasor to the time domain.

- **Sinusoidal Steady-State Analysis:**
  Frequency domain analysis of AC circuit via phasors is much easier than analysis of the circuit in the time domain.
10.2 Nodal Analysis

The basic of Nodal Analysis is **KCL**.

**Example:** Using nodal analysis, find $v_1$ and $v_2$ in the figure.
The basic of Mesh Analysis is \textit{KVL}.

\textbf{Example:} Find $I_o$ in the following figure using mesh analysis.
Table

Node Voltage Analysis Using the Phasor Concept to Find the Sinusoidal Steady-State Node Voltages

1. Convert the independent sources to phasor form.
2. Select the nodes and the reference node and label the node voltages in the time domain, \( v_n \), and their corresponding phasor voltages, \( V_n \).
3. If the circuit contains only independent current sources, proceed to step 5; otherwise, proceed to step 4.
4. If the circuit contains a voltage source, select one of the following three cases and the associated method:

   \begin{align*}
   \text{CASE} & & \text{METHOD} \\
   \text{a. The voltage source connects node } q \text{ and the reference node.} & & \text{Set } V_q = V_s \text{ and proceed.} \\
   \text{b. The voltage source lies between two nodes.} & & \text{Create a supernode including both nodes.} \\
   \text{c. The voltage source in series with an impedance lies between node } & & \text{Replace the voltage source and series impedance with a parallel} \\
   d \text{ and the ground, with its positive terminal at node } d. & & \text{combination of an admittance } Y_1 = 1/Z_1 \text{ and a current source} \\
   & & I_1 = V_S Y_1 \text{ entering node } d. \\
   \end{align*}

5. Using the known frequency of the sources, \( \omega \), find the impedance of each element in the circuit.
6. For each branch at a given node, find the equivalent admittance of that branch, \( Y_n \).
7. Write KCL at each node.
8. Solve for the desired node voltage \( V_n \), using Cramer’s rule.
9. Convert the phasor voltage \( V_n \) back to the time-domain form.

Table

Mesh Current Analysis Using the Phasor Concept to Find the Sinusoidal Steady-State Mesh Currents

1. Convert the independent sources to phasor form.
2. Select the mesh currents and label the currents in the time domain, \( i_n \), and the corresponding phasor currents, \( I_n \).
3. If the circuit contains only independent voltage sources, proceed to step 5; otherwise, proceed to step 4.
4. If the circuit contains a current source, select one of the following two cases and the associated method:

   \begin{align*}
   \text{CASE} & & \text{METHOD} \\
   \text{a. The current source appears as an element of only one mesh, } n. & & \text{Equate the mesh current } I_n \text{ to the current of the current source, accounting for the direction of the source current.} \\
   \text{b. The current source is common to two meshes.} & & \text{Create a supermesh as the periphery of the two meshes. In step 6, write one KVL equation around the periphery of the supermesh. Also record the constraining equation incurred by the current source.} \\
   \end{align*}

5. Using the known frequency of the sources, \( \omega \), find the impedance of each element in the circuit.
6. Write KVL for each mesh.
7. Solve for the desired mesh current \( I_n \), using Cramer’s rule.
8. Convert the phasor current \( I_n \) back to the time-domain form.
When a circuit has sources operating at different frequencies,

- The separate phasor circuit for each frequency must be solved independently, and
- The total response is the sum of time-domain responses of all the individual phasor circuits.

**Example:** Calculate $v_o$ in the circuit using the superposition theorem.
4.3 Superposition Theorem (1)

- **Superposition** states that the voltage across (or current through) an element in a linear circuit is the algebraic sum of the voltage across (or currents through) that element due to **EACH independent source acting alone**.

- The principle of superposition helps us to analyze a linear circuit with more than one independent source by calculating the contribution of each independent source separately.

- **Steps to Apply Superposition Principle**:
  1. Turn off all indep. sources except one source. Find the output ($v$ or $i$) due to that active source using techniques in Chapters 2 & 3.
  2. Repeat Step 1 for each of the other indep. sources.
  3. Find total contribution by adding all contributions from indep. sources.

**Note**: In Step 1, this implies that we replace every voltage source by 0 V (or a short circuit), and every current source by 0 A (or an open circuit). Dependent sources are left intact because they are controlled by others.
Example: Find $I_o$ using the concept of source transformation.
10.5 Source Transformation (2)

Method

Set $I_s = \frac{V_s}{Z_s}$

$Z_s$ is the same in both circuits

Method

Set $V_s = I_s Z_s$

$Z_s$ is the same in both circuits
- Like series-parallel combination and wye-delta transformation, source transformation is another tool for simplifying circuits.
- An **equivalent circuit** is one whose $v-i$ characteristics are identical with the original circuit.
- A **source transformation** is the process of replacing a voltage source $v_s$ in series with a resistor $R$ by a current source $i_s$ in parallel with a resistor $R$, and vice versa.

• Transformation of **independent sources**

• Transformation of **dependent sources**

- The arrow of the current source is directed toward the positive terminal of the voltage source.
- The source transformation is not possible when $R = 0$ for voltage source and $R = \infty$ for current source.
A voltage source $v_s$ connected in series with a resistor $R_s$ and a current source $i_s$ is connected in parallel with a resistor $R_p$ are equivalent circuits provided that

$$R_p = R_s \quad \text{and} \quad v_s = R_s i_s$$
10.6 Thevenin & Norton Equivalent Circuits

\[
V_{Th} = Z_N I_N, \quad Z_{Th} = Z_N
\]

**Thevenin Equivalent**

Linear circuit \[\rightarrow V_{Th}\]

**Norton Equivalent**

Linear circuit \[\rightarrow I_N\]

**Example:** Find the Thevenin equivalent at terminals \(a\)–\(b\).
4.5 Thevenin’s Theorem (1)

It states that a linear two-terminal circuit (Fig. a) can be replaced by an equivalent circuit (Fig. b) consisting of a **voltage** source $V_{Th}$ in series with a resistor $R_{Th}$, where

- $V_{Th}$ is the open-circuit voltage at the terminals.
- $R_{Th}$ is the input or equivalent resistance at the terminals when the independent sources are turned off.

\[ V_{Th} = v_{oc} \]

\[ R_{Th} = R_{in} \]
4.5 Thevenin’s Theorem (2)

To find $R_{Th}$:

**Case 1**: If the network has no dependent sources, we turn off all indep. Source. $R_{Th}$ is the input resistance of the network looking btw terminals $a$ & $b$.

**Case 2**: If the network has depend. Sources. Depend. sources are not to be turned off because they are controlled by circuit variables. (a) Apply $v_o$ at $a$ & $b$ and determine the resulting $i_o$. Then $R_{Th} = v_o/i_o$. Alternatively, (b) insert $i_o$ at $a$ & $b$ and determine $v_o$. Again $R_{Th} = v_o/i_o$.

$$R_{Th} = \frac{v_o}{i_o}$$
4.6 Norton’s Theorem (1)

It states that a linear two-terminal circuit (Fig. a) can be replaced by an equivalent circuit (Fig. b) consisting of a current source $I_N$ in parallel with a resistor $R_N$,

$\begin{align*}
\text{(a)} & \quad \text{Linear two-terminal circuit} \\
\text{(b)} & \quad \text{Current source } I_N \text{ in parallel with a resistor } R_N
\end{align*}$

where

- $I_N$ is the short-circuit current through the terminals.
- $R_N$ is the input or equivalent resistance at the terminals when the independent sources are turned off.

$\begin{align*}
\text{Linear circuit with all independent sources set equal to zero} \\
R_N = R_{Th} \quad \text{and} \quad R_{Th} = R_{in}
\end{align*}$
10.7 Op Amp AC Circuits (1)

The key to analyzing op amp circuits is to keep two important properties of an ideal op amp in mind:
• No current enters either of its input terminals.
• The voltage across its input terminals is zero.

**Example:** Compute the closed-loop gain and phase shift. Assume that \( R_1 = R_2 = 10 \, \text{k}\Omega \), \( C_1 = 2 \, \mu\text{F} \), \( C_2 = 1 \, \mu\text{F} \), and \( \omega = 200 \, \text{rad/s} \).

\[
Z_f = R_2 \left( \frac{1}{j\omega C_2} \right) = \frac{R_2}{1 + j\omega R_2 C_2}
\]

\[
Z_i = R_1 + \frac{1}{j\omega C_1} = \frac{1}{j\omega C_1}
\]

\[
G = \frac{V_o}{V_s} = -\frac{Z_f}{Z_i} = -\frac{-j\omega C_1 R_2}{(1 + j\omega R_1 C_1)(1 + j\omega R_2 C_2)}
\]

Substituting the given values of \( R_1, R_2, C_1, C_2 \), and \( \omega \), we obtain

\[
G = \frac{-j4}{(1 + j4)(1 + j2)} = 0.434/130.6^\circ
\]

Thus the **closed-loop gain** is 0.434 and the phase shift is 130.6°.
5.2 Ideal Op Amp

An ideal op amp has the following characteristics:

1. Infinite open-loop gain, \( A \approx \infty \)
2. Infinite input resistance, \( R_i \approx \infty \)
3. Zero output resistance, \( R_o \approx 0 \)

**Example:** Determine the value of \( i_o \).

\[ v_2 = v_s \]
\[ v_1 = \frac{5}{5 + 40} v_o = \frac{v_o}{9} \]
\[ i_o = 0.2 + 0.45 = 0.65 \text{ mA} \]

when \( v_s = 1 \text{ V} \)
5.3 Configurations of Op amp (1)

- **Inverting Amplifier:** reverses the polarity of the input signal while amplifying it.

Negative feedback btw the inverting input \( (v_i) \) & output \( (v_o) \)

\( v_i \) is connected to the inverting input via \( R_1 \)

To find the relationship btw \( v_i \) & \( v_o \):

By KCL at node 1,

\[
i_1 = i_2 \Rightarrow \frac{v_i - v_1}{R_1} = \frac{v_1 - v_o}{R_f}
\]

\( \therefore v_1 = v_2 = 0 \) for an ideal op amp since the noninverting terminal is grounded.

Closed-loop voltage gain is

\[
A_v = \frac{v_o}{v_i} = -\frac{R_f}{R_1}
\]

**Example:** Find \( v_o \) & \( i \) in \( R_1 \) if \( v_i = 0.5 \text{ V}, R_1 = 10 \text{ k}\Omega, \& R_f = 25 \text{ k}\Omega. \)
5.3 Configurations of Op amp (3)

- **Noninverting Amplifier**: designed to produce positive voltage gain.

To find the relationship btw $v_i$ & $v_o$:

By KCL at inverting terminal,

$$i_1 = i_2 \Rightarrow \frac{0 - v_1}{R_1} = \frac{v_1 - v_o}{R_f}$$

$$\therefore v_1 = v_2 = v_i \Rightarrow v_o = \left(1 + \frac{R_f}{R_1}\right)v_i$$

Voltage gain: $A_v = v_o/v_i = 1 + R_f/R_1$

$\cdot$ **To isolate two cascaded stages:**

$y_i$ connected to noninverting input terminal

$R_f = 0$ (short circuit) or $R_1 = \infty$ (open circuit)

$\Rightarrow v_o = v_i$

as voltage follower

Has a very high input impedance and thus eliminate interstage loading
**Oscillator:** a circuit produces **an ac waveform** as output when powered by dc input.

- In order for a sine wave oscillator to sustain oscillations, it must meet the **Barkhausen criteria:**
  1. The overall gain of the oscillator must be unity or greater. Thus losses must be compensated for by an amplifying device.
  2. The overall phase shift (from the output and back to the input) must be zero.
- Three common types of sine wave oscillators are phase-shift, twin T, and **Wein-bridge** oscillators.

\[
\boxed{f_o = \frac{1}{2\pi RC}}
\]

1. Negative feedback path to control gain
2. Positive feedback path to create oscillations
10.7 Op Amp AC Circuits (3)

Capacitance Multiplier: to create a large capacitance.

\[ C_{eq} = \left( 1 + \frac{R_2}{R_1} \right) C \]

Example:
Fig. (a) shows two sinusoidal voltages, one labeled as input and the other labeled as output. We want to design a circuit that will transform the input sinusoid into the output sinusoid. Fig. (b) shows a candidate circuit. We must first determine whether this circuit can do the job. Then, if it can, we will design the circuit, that is, specify the required values of $R_1$, $R_2$, and $C$. 

\[ v_1(t) = \sin(2\pi1000t) \text{ V} \]

\[ v_2(t) = 2\sin(2\pi1000t + 120^\circ) \text{ V} \]

(b)
The input sinusoid is \( v_1(t) = \sin(2\pi 1000t) = \cos(2\pi 1000t - 90^\circ) \) V and the corresponding phasor is \( V_1 = 1e^{-j90^\circ} = 1\angle -90^\circ \) V.

The output sinusoid is
\[
v_2(t) = 2\sin(2\pi 1000t + 120^\circ) = 2\cos(2\pi 1000t + 30^\circ) \text{ V}
\]
and the corresponding phasor is \( V_2 = 2e^{j30^\circ} \) V.

The ratio of these phasors is
\[
\frac{V_2}{V_1} = \frac{2e^{j30^\circ}}{1e^{-j90^\circ}} = 2e^{j120^\circ}
\]

The magnitude of this ratio, called the gain, \( G \), of the circuit used to transform the input sinusoid into the output sinusoid is \( G = \left| \frac{V_2}{V_1} \right| = 2 \)

The angle of this ratio is called the phase shift, \( \theta \), of the required circuit:
\[
\theta = \angle \frac{V_2}{V_1} = 120^\circ
\]

Therefore, we need a circuit that has a gain of 2 and a phase shift of 120°.
**State the Goal:** Determine whether it is possible to design the circuit shown in Fig. (b) to have a gain of 2 and a phase shift of 120°. If it is possible, specify the appropriate values of $R_1$, $R_2$, & $C$.

**Generate a Plan:** Analyze the circuit shown in Fig. (b) to determine the ratio of the output phasor to the input phasor, $\frac{V_2}{V_1}$. Determine whether this circuit can have a gain of 2 and a phase shift of 120°. If so, determine the required values of $R_1$, $R_2$, & $C$.

**Act on the Plan:** The impedance $Z_1$ corresponds to the resistor $R_1$ in Fig. (b), and impedance $Z_2$ corresponds to the parallel combination of resistor $R_2$ and capacitor $C$. That is,

$$Z_1 = R_1 \quad \& \quad Z_2 = \frac{R_2 (1/ j\omega C)}{R_2 + 1/ j\omega C} = \frac{R_2}{1 + j\omega CR_2}$$

$$\therefore \quad \frac{V_2}{V_1} = - \frac{Z_2}{Z_1} = - \frac{R_2}{(1 + j\omega CR_2)} = - \frac{R_2 / R_1}{1 + j\omega CR_2}$$
The phase shift: $\theta = \angle \frac{V_2}{V_1} = \angle - \frac{R_2 / R_1}{1 + j\omega CR_2} = 180^\circ - \tan^{-1} \omega CR_2$

What values of phase shift are possible? Notice that $\omega$, $C$, and $R_2$ are all positive, which means that

$$0^\circ \leq \tan^{-1} \omega CR_2 \leq 90^\circ$$

Therefore, the circuit shown in Fig. (b) can be used to obtain phase shifts between $90^\circ$ and $180^\circ$. Hence, we can use this circuit to produce a phase shift of $120^\circ$.

The gain: $G = \left| \frac{V_2}{V_1} \right| = \left| - \frac{R_2 / R_1}{1 + j\omega CR_2} \right| = \frac{R_2 / R_1}{\sqrt{1 + \omega^2 C^2 R_2^2}} = \frac{R_2 / R_1}{\sqrt{1 + \tan^2 (180^\circ - \theta)}}$

To find $R_2$

$$R_2 = \frac{\tan(180^\circ - \theta)}{\omega C} \quad \Rightarrow \quad R_1 = \frac{R_2 / G}{\sqrt{1 + \tan^2 (180^\circ - \theta)}}$$

These equations can be used to design the circuit. For $\omega = 6283$ rad/s, $C = 0.02 \ \mu F$, $G = 2$, and $\theta = 120^\circ$, we calculate

$$R_1 = 3446 \ \Omega \quad \text{and} \quad R_2 = 13.78 \ \text{k}\Omega$$
10.9 Summary (1)

- With the pervasive use of ac electric power in the home and industry, it is important for engineers to analyze circuits with sinusoidal independent sources.
- The steady-state response of a linear circuit to a sinusoidal input is itself a sinusoid having the same frequency as the input signal.
- Circuits that contain inductors and capacitors are represented by differential equations. When the input to the circuit is sinusoidal, the phasors and impedances can be used to represent the circuit in the frequency domain. In the frequency domain, the circuit is represented by algebraic equations.
- The steady-state response of a linear circuit with a sinusoidal input is obtained as follows:
  1. Transform the circuit into the frequency domain, using phasors and impedances.
2. Represent the frequency-domain circuit by algebraic equation, for example, mesh or node equations.
3. Solve the algebraic equations to obtain the response of the circuit.
4. Transform the response into the time domain, using phasors.

• A circuit contains several sinusoidal sources, two cases:
  ✓ When all of the sinusoidal sources have the same frequency, the response will be a sinusoid with that frequency, and the problem can be solved in the same way that it would be if there was only one source.
  ✓ When the sinusoidal sources have different frequencies, superposition is used to break the time-domain circuit up into several circuits, each with sinusoidal inputs all at the same frequency. Each of the separate circuits is analyzed separately and the responses are summed in the time domain.