Pole - Zero Plots

Consider a rational function of the form

$$F(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

The function in factored form becomes

$$F(s) = \frac{K(s - z_1)(s - z_2) \cdots (s - z_m)}{(s - p_1)(s - p_2) \cdots (s - p_n)}$$

where $z_1, z_2, \ldots, z_m$ are its _zeroes_ and $p_1, p_2, \ldots, p_n$ are its _poles_. The constant

$$K = \frac{b_m}{a_n}$$

is the _multiplying constant_ of the function.

A _pole-zero plot_ of a function, where the zero locations are indicated by "O" and the pole locations are indicated by "X," reveals several important properties of the function. To simplify the plot, the multiplying constant will be placed in a box at the right of the pole-zero plot.

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Example:

Construct a pole-zero plot for the following function.

$$F(s) = \frac{5s^3 + 65s^2 + 405s + 1025}{s^4 + 11s^3 + 41s^2 + 91s}$$

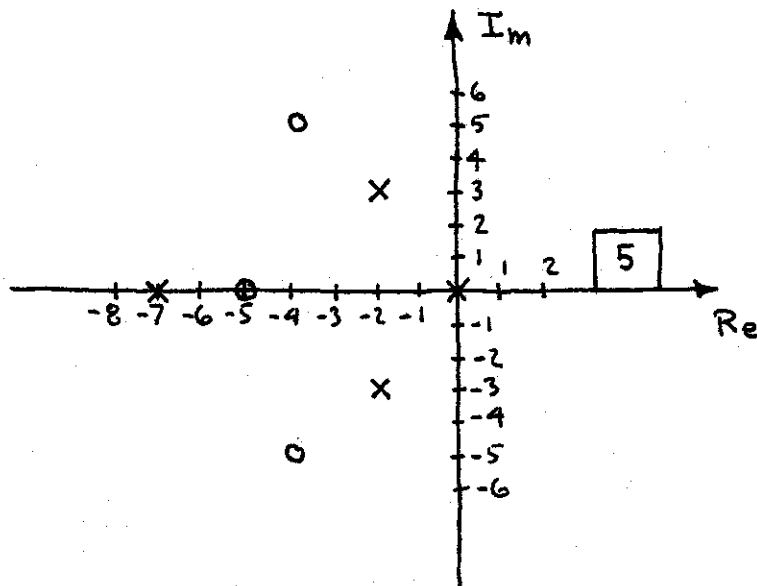Using the following MATLAB code

```
num = [5, 65, 405, 1025];
den = [1, 11, 41, 91, 0];
[z, p, K] = tf2zp(num, den)
```

the factored form of the function is

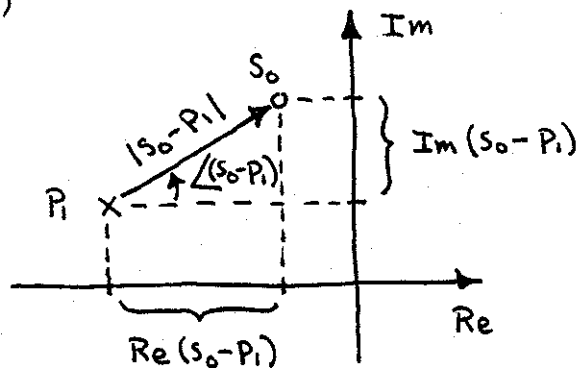$$F(s) = \frac{5(s+5)(s+4+j5)(s+4-j5)}{s(s+7)(s+2+j3)(s+2-j3)}$$

The pole-zero plot is

A rational function $F(s)$ can be graphically evaluated at a specific value of s. Letting $s = s_0$, the function becomes

$$F(s_0) = \frac{k(s_0 - z_1)(s_0 - z_2)\ldots(s_0 - z_m)}{(s_0 - p_1)(s_0 - p_2)\ldots(s_0 - p_n)}$$

Now consider an arbitrary factor of $F(s_0)$. For example,

$$(s_0 - p_1) = \text{Re}(s_0 - p_1) + j\,\text{Im}(s_0 - p_1)$$

$$= |s_0 - p_1| \underline{/(s_0 - p_1)}$$

$$= |s_0 - p_1|\, e^{j\underline{/(s_0 - p_1)}}$$

Graphically,

Therefore, the function $F(s_0)$ can be expressed as

$$F(s_0) = \frac{k\left[|s_0 - z_1|e^{j\underline{/(s_0 - z_1)}}\right]\left[|s_0 - z_2|e^{j\underline{/(s_0 - z_2)}}\right]\ldots}{\left[|s_0 - p_1|e^{j\underline{/(s_0 - p_1)}}\right]\left[|s_0 - p_2|e^{j\underline{/(s_0 - p_2)}}\right]\ldots}$$

The magnitude and angle are

$$|F(s_0)| = \frac{|K| \, |s_0 - z_1| \, |s_0 - z_2| \cdots}{|s_0 - p_1| \, |s_0 - p_2| \cdots}$$

$$\angle F(s_0) = \angle(s_0 - z_1) + \angle(s_0 - z_2) + \cdots$$

$$-\angle(s_0 - p_1) - \angle(s_0 - p_2) - \cdots$$
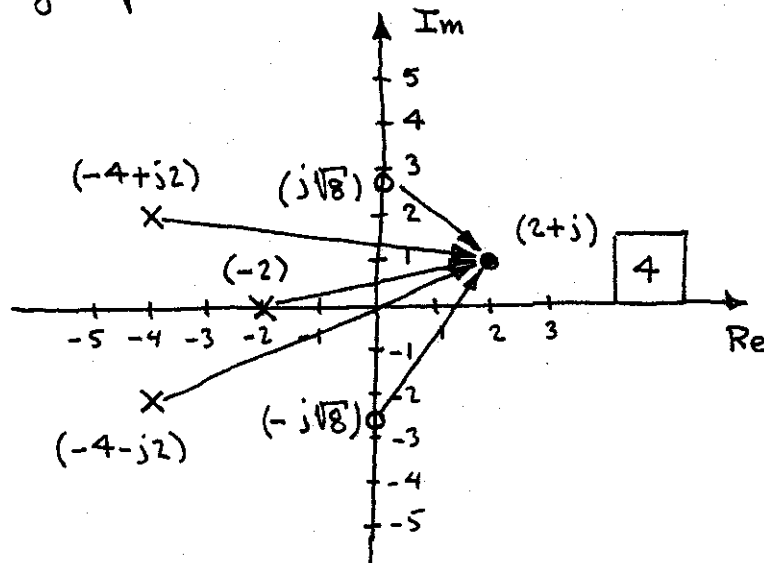
$$+180° \text{ if } K \text{ is negative}$$

Example:

Sketch the pole-zero plot and graphically evaluate the following function at $s = 2 + j$.

$$F(s) = \frac{4s^2 + 32}{(s^2 + 8s + 20)(s+2)}$$

The function in factored form is

$$F(s) = \frac{4(s + j\sqrt{8})(s - j\sqrt{8})}{(s + 4 + j2)(s + 4 - j2)(s + 2)}$$
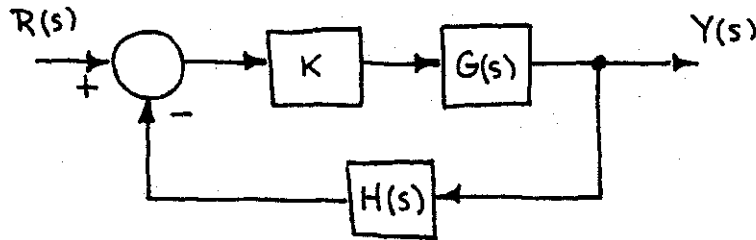
The pole-zero plot is

Evaluating,

$$F(s_0) = \frac{4\,(2.71\,\angle\,{-42.43°})(4.32\,\angle\,62.42°)}{(6.08\,\angle\,{-9.46°})(4.12\,\angle\,14.04°)(6.71\,\angle\,26.57°)}$$

$$= 0.28\,\angle\,{-11.16°} \quad = \quad 0.28e^{-j11.16°}$$

Root Locus for Feedback Systems

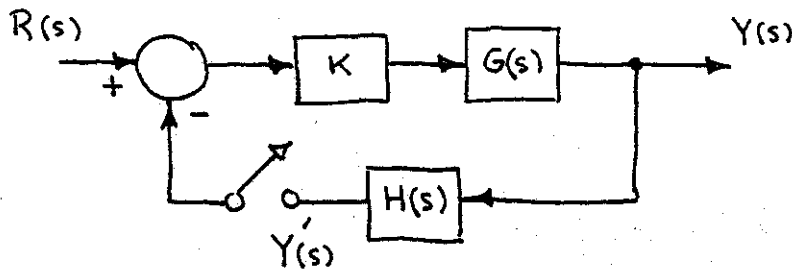Consider a simple feedback system with adjustable gain K.

The transfer function is

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$

and the poles of the transfer function are the roots of

$$1 + KG(s)H(s) = 0$$

The product $KG(s)H(s)$ is termed the <u>open-loop transmittance</u> (or gain) of the system, which results from opening the feedback loop.

The poles and zeroes of $G(s)H(s)$ are called the open-loop poles and zeroes, and the poles and zeroes of $T(s)$ are called the closed-loop poles and zeroes.

A <u>root locus plot</u> is a graph of the loci of the poles of a rational function as some system parameter is varied. For the feedback system above, the rational function is

$$1 + K G(s) H(s) = 0$$

which can be expressed as

$$G(s) H(s) = -\frac{1}{K}$$

When $0 \leq K < \infty$, a pole of $T(s)$ makes

$$|G(s) H(s)| = \frac{1}{K}$$

and

$$\angle G(s) H(s) = \pm n \, 180°, \quad n = 1, 3, 5, \ldots$$

Thus any point s for which $\angle G(s) H(s) = 180°$ is a point of the root locus, for some positive value of K.

Basic root locus properties include:

1. Locus segments are symmetric about the real axis.

2. As $K \to 0$, $|G(s) H(s)| \to \infty$; for small K, the poles of $T(s)$ are near the <u>poles</u> of $G(s) H(s)$.

3. As $K \to \infty$, $|G(s) H(s)| \to 0$; for large K, the poles of $T(s)$ are near the <u>zeroes</u> of $G(s) H(s)$.

Table 4.1 *Basic Root Locus Principles*

1. The branches of the locus are continuous curves that start at each of the n poles of GH, for $K = 0$. As K approaches $+\infty$, the locus branches approach the m zeros of GH. Locus branches for excess poles extend infinitely far from the origin; for excess zeros, locus segments extend from infinity.

2. The locus includes all points along the real axis to the left of an odd number of poles plus zeros of GH.

3. As K approaches $+\infty$, the branches of the locus become asymptotic to straight lines with angles

   $$\theta = \frac{180° + i360°}{n - m}$$

   for $i = 0, \pm 1, \pm 2, \ldots$, until all $n - m$ or $m - n$ angles are obtained, where n is the number of poles and m is the number of zeros of GH.

4. The starting point of the asymptotes, the centroid of the pole-zero plot, is on the real axis at

   $$\sigma = \frac{\Sigma \text{ pole values of } GH - \Sigma \text{ zero values of } GH}{n - m}$$

5. Loci leave the real axis at a gain K that is the maximum K in that region of the real axis. Loci enter the real axis at the minimum value of K in that region of the real axis. These points are termed *breakaway points* and *entry points*, respectively. A pair of locus segments leave or enter the real axis at angles of $\pm 90°$.

6. The angle of departure ϕ of a locus branch from a complex pole is given by

   $$\phi = -\Sigma \text{ other } GH \text{ pole angles} + \Sigma \text{ } GH \text{ zero angles} + 180° \quad \text{ZERO}$$

   The angle of approach ϕ' of a locus branch to a complex pole is given by

   $$\phi' = \Sigma \text{ } GH \text{ pole angles} - \Sigma \text{ other } GH \text{ zero angles} - 180°$$

   where each GH pole angle and GH zero angle is calculated to the complex pole for ϕ and to the complex zero for ϕ'.

   If the complex pole or zero is of order m, the m angles of arrival and approach are given by

   $$\phi = [-\Sigma \text{ other } GH \text{ pole angles} + \Sigma \text{ } GH \text{ zero angles} + (1 + 2i)180°]/m$$
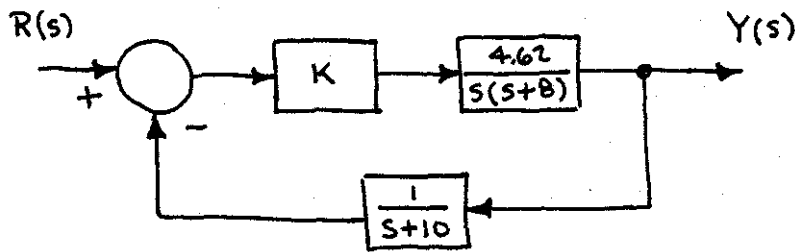
   $$\phi' = [\Sigma \text{ } GH \text{ pole angles} - \Sigma \text{ other } GH \text{ zero angles} - (1 + 2i)180°]/m$$

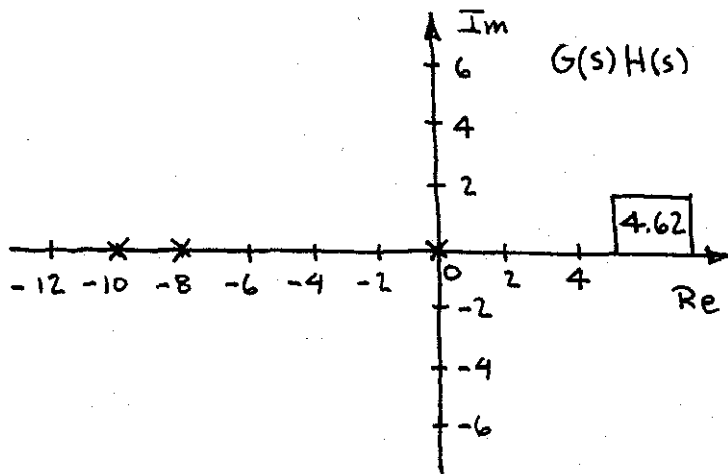   for $i = 0, 1, 2, \ldots, (m - 1)$.

Example :

Perform a root locus analysis on the following control system.

The open - loop transmittance of the system is

$$KG(s)H(s) = \frac{4.62K}{S(s+8)(s+10)}$$

and a plot of the open-loop poles and zeroes is

Because of the excess poles, locus branches will extend infinitely far from the origin.

The real axis segments of the root locus are $-8 \le s \le 0$ and $s \le -10$. The asymptotic angles are

$$\Theta = \frac{180° + i360°}{n-m} = \frac{180° + i360°}{3-0} = \pm 60°, \pm 180°$$

and the centroid of the asymptotes is

$$\sigma = \frac{\sum GH \text{ poles} - \sum GH \text{ zeroes}}{n-m} = \frac{0-8-10}{3-0} = -6$$

The real axis breakaway point is determined by evaluating

$$K = \left| \frac{1}{G(s)H(s)} \right| = \left| \frac{s(s+8)(s+10)}{4.62} \right|$$

to yield

Value of s	Value of K
-2.0	20.7792
-2.8	22.6909
-2.9	22.7292
-3.0	22.7273
-3.1	22.6864
-4.0	20.7792

FROM BEFORE WE HAVE:

$$1 + K\,G(s)H(s) = 0$$

$$\Rightarrow \quad K = -\frac{1}{G(s)H(s)}$$

WITH

$$G(s)H(s) = \frac{4.62}{s(s+8)(s+10)}$$

$$\Rightarrow \quad K = -\frac{s(s+8)(s+10)}{4.62} = -\frac{1}{4.62}\left(s^3 + 18s^2 + 80s\right)$$

TO FIND MAXIMUM K:

$$\frac{dK}{ds} = -\frac{1}{4.62}\left(3s^2 + 36s + 80\right) = 0$$

SOLUTIONS: $\quad s = -9.055 \quad$ and $\quad \underline{\underline{s = -2.94}}$

SOLUTION $s = -9.055$ DOES **NOT** LIE ON THE LOCUS

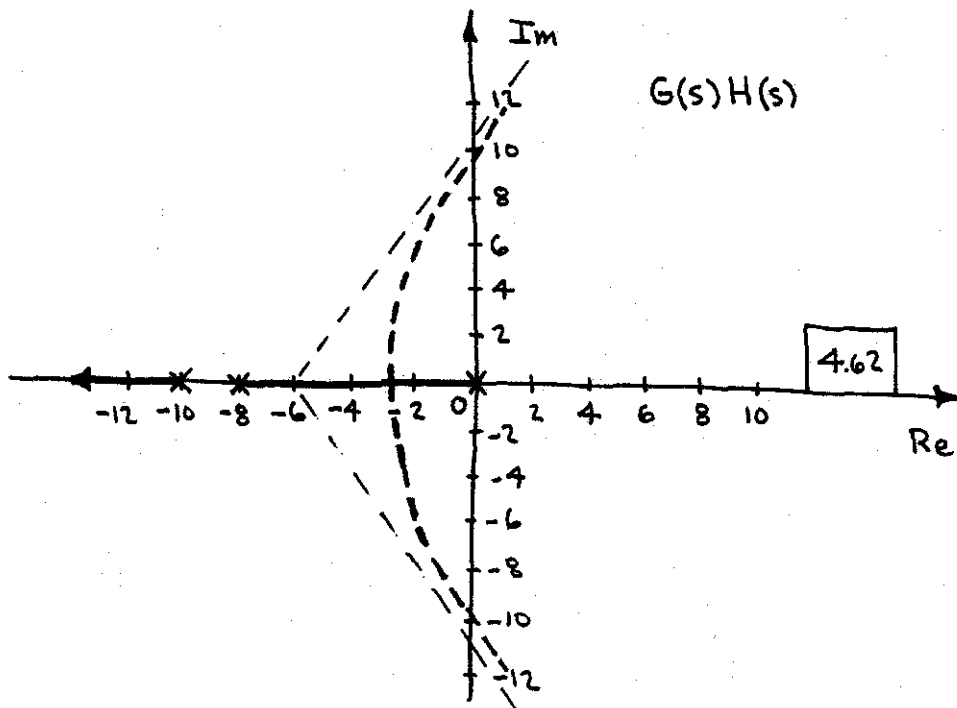SOLUTION $s = -2.94$ DOES LIE ON THE LOCUS, AND THUS
IS THE WANTED RESULT

$$\left(\begin{array}{l}
\text{ALSO NOTE:} \quad \dfrac{d^2K}{ds^2} = -\dfrac{1}{4.62}(6s + 36) \\[2mm]
\text{at } s = -2.9 \\[2mm]
\left.\dfrac{d^2K}{ds^2}\right|_{s = -2.9} = -4.03 < 0 \\[3mm]
\dfrac{d^2K}{ds^2} < 0 \quad \Rightarrow \quad \text{MAXIMUM POINT}
\end{array} \right)$$

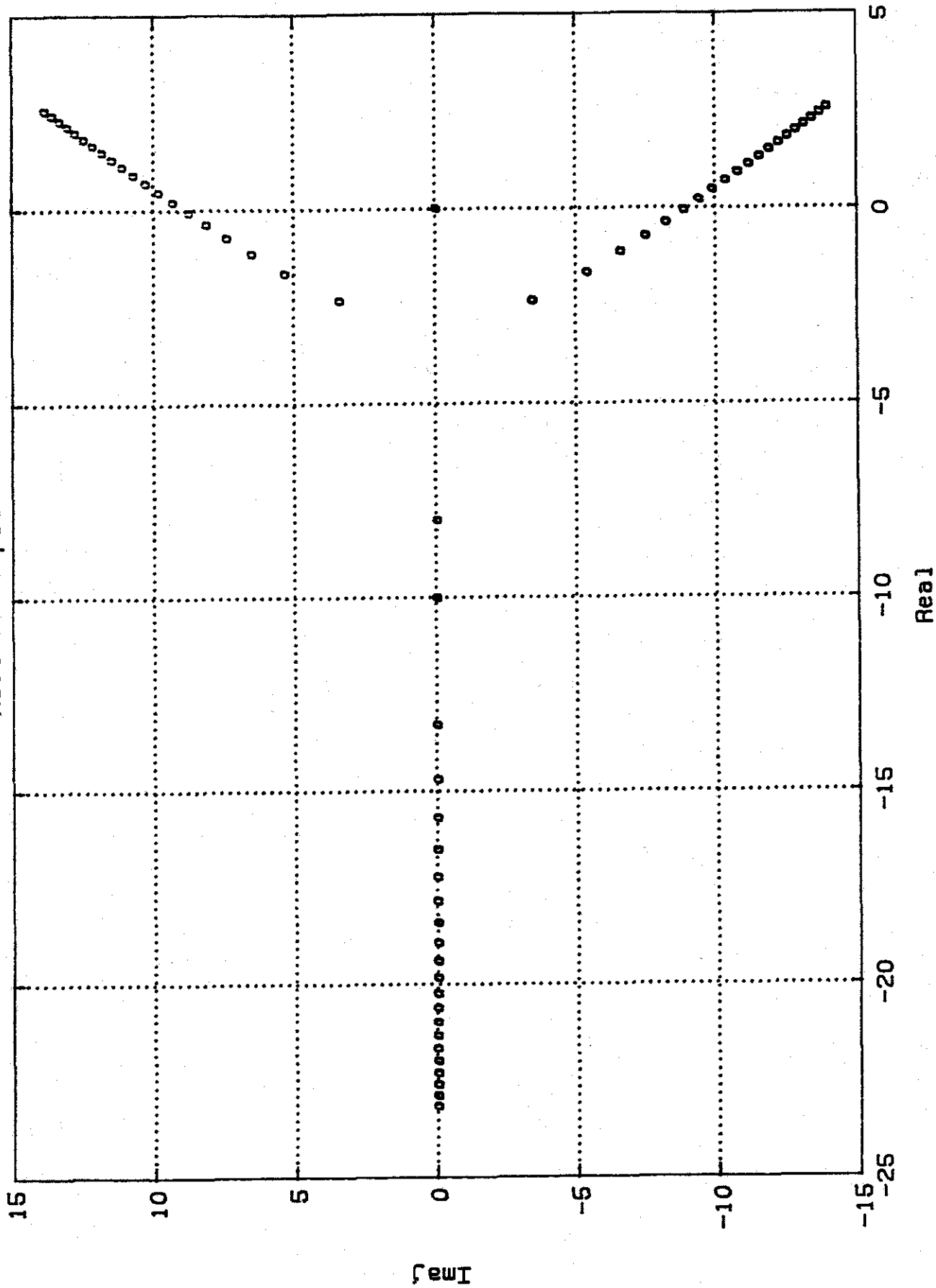A plot of these data is

A MATLAB program to plot the root locus of this system is

```
clear ; clc; clg
num = [4.62];
den1 = [1, 8, 0];
den2 = [1, 10];
den = conv (den1, den2);
K = 0 : 50 : 1000;
r = rlocus (num, den, K);
plot (r, 'o'), title ('Root locus plot'), xlabel ('Real'),
ylabel ('Imaj'); grid
clc; home
```
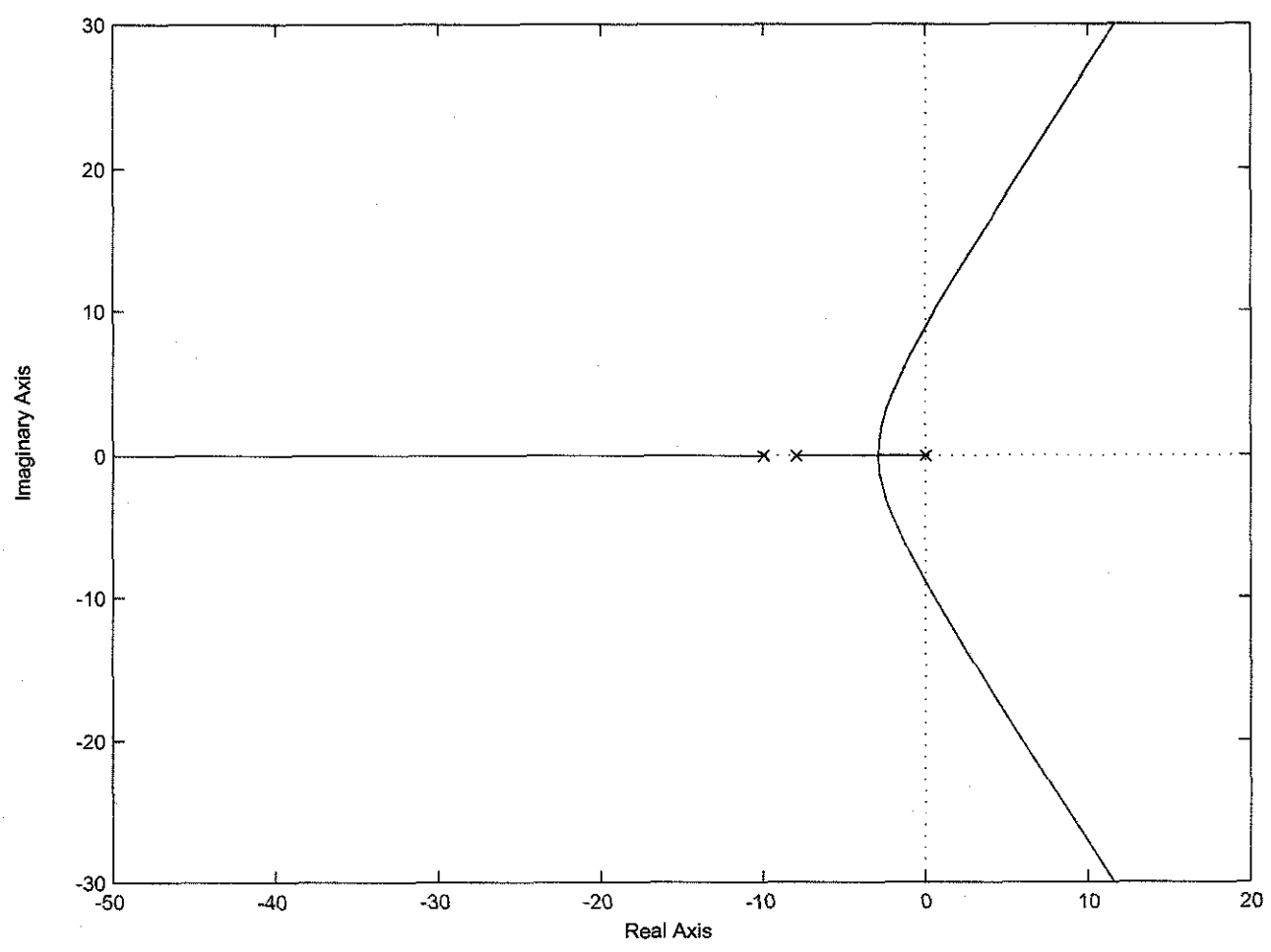
Root-locus plot

$$G(s) \, H(s) = \frac{4.62}{s(s+8)(s+10)}$$

$\longrightarrow$

MATLAB:

>> sys = zpk([], [0 -8 -10], 4.62)

Zero/pole/gain:

   4.62
----
s (s+8) (s+10)

>> rlocus(sys)

Root Locus

The root locus method may also be applied to nonstandard systems by first converting them to the standard form

$$T(s) = \frac{\text{num. poly.}}{1 + K \dfrac{a(s)}{b(s)}}$$

In addition, root locus plots can be obtained for negative values of the parameter K.

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Example:

Determine the root locus of the following function as K is varied

$$T(s) = \frac{6s^2 + Ks + 2}{s^3 + 2s^2 + (5+2K)s + 2K}$$

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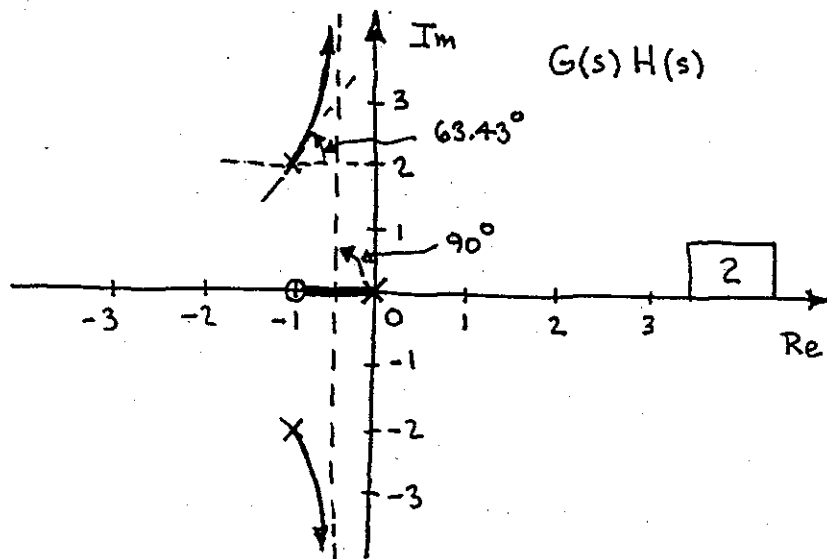Converting $T(s)$ to the standard form,

$$T(s) = \frac{6s^2 + Ks + 2}{s^3 + 2s^2 + 5s + K(2s+2)}$$

$$= \frac{\dfrac{6s^2 + Ks + 2}{s^3 + 2s^2 + 5s}}{1 + K \dfrac{2s+2}{s^3 + 2s^2 + 5s}}$$

The equivalent $G(s)H(s)$ for the root locus is

$$G(s)\,H(s) = \frac{2(s+1)}{s(s^2+2s+5)}$$

For $0 \le K < \infty$,

Applying basic root locus principles,

$$\Theta = \frac{180° \pm i360°}{2} = \underline{\pm 90°}$$

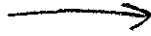$$\sigma = \frac{-1+j2-1-j2-(-1)}{2} = \underline{-\frac{1}{2}}$$

$$116.57° + 90° + \phi - 90° = 180°$$

$$\phi = \underline{63.43°}$$

$$G_{(s)} \, H_{(s)} = \frac{2(s+1)}{s(s^2 + 2s + 5)}$$

$\longrightarrow$

MATLAB:

>> sys = tf(2*[1 1], [1 2 5 0])

Transfer function:

  2 s + 2

----

s^3 + 2 s^2 + 5 s

>> rlocus(sys)

Root Locus