Frequency Response

The transmittance function can be used to find the steady-state response of a system that is driven by a sinusoidal source. Beginning with

$$r(t) = B\cos(\omega t + \beta)$$

the Laplace transform is

$$R(s) = \mathcal{L}\{r(t)\} = \frac{(B\cos\beta)s}{s^2+\omega^2} - \frac{(B\sin\beta)\omega}{s^2+\omega^2}$$

$$= \frac{B(s\cos\beta - \omega\sin\beta)}{s^2+\omega^2}$$

If $F(s)$ represents the transmittance of the system,

$$Y(s) = F(s)R(s) = F(s)\frac{B(s\cos\beta - \omega\sin\beta)}{s^2+\omega^2}$$

$$= \frac{K_1}{s-j\omega} + \frac{K_1^*}{s+j\omega} + \sum \text{terms due to } F(s) \text{ poles}$$

Now, assuming that all poles of $F(s)$ lie in the left half-plane, the terms generated by these poles will not contribute to the steady-state response of $y(t)$.

Solving for K_1,

$$K_1 = (s - j\omega)\, Y(s)\Big|_{s=j\omega} = \frac{F(s)\, B(s\cos\beta - \omega\sin\beta)}{s + j\omega}\Bigg|_{s=j\omega}$$

$$= \frac{F(j\omega)\, B(j\omega\cos\beta - \omega\sin\beta)}{2j\omega} = \frac{F(j\omega)\, B(\cos\beta + j\sin\beta)}{2}$$

$$= \frac{1}{2} F(j\omega)\, B e^{j\beta}$$

In general, $F(j\omega)$ will be a complex quantity; therefore

$$F(j\omega) = |F(j\omega)|\, e^{j\phi(\omega)}$$

and K_1 becomes

$$K_1 = \frac{B}{2} |F(j\omega)|\, e^{j[\beta + \phi(\omega)]}$$

Finally,

$$y_{forced}(t) = B\, |F(j\omega)| \cos[\omega t + \beta + \phi(\omega)]$$

The evaluation of $F(j\omega)$ and $\phi(\omega)$ is best done by a computer.

For $n=4$,

$$F(s) = \frac{a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_\emptyset}{b_4 s^4 + b_3 s^3 + b_2 s^2 + b_1 s + b_\emptyset}$$

Letting $s \to j\omega$,

$$F(j\omega) = \frac{a_4 (j\omega)^4 + a_3 (j\omega)^3 + a_2 (j\omega)^2 + a_1 j\omega + a_\emptyset}{b_4 (j\omega)^4 + b_3 (j\omega)^3 + b_2 (j\omega)^2 + b_1 j\omega + b_\emptyset}$$

$$= \frac{(a_4\omega^4 - a_2\omega^2 + a_\emptyset) + j(-a_3\omega^3 + a_1\omega)}{(b_4\omega^4 - b_2\omega^2 + b_\emptyset) + j(-b_3\omega^3 + b_1\omega)}$$

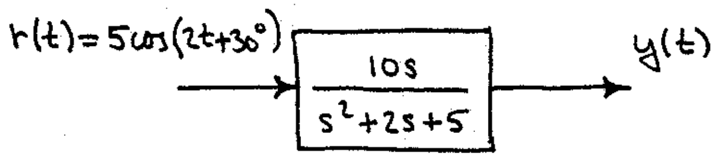
$$= \frac{N \angle \phi_N}{D \angle \phi_D} = \frac{N}{D} \angle \phi = |F(j\omega)| e^{j\phi(\omega)}$$

In decibels,

$$|F(j\omega)|\,db = 20 \log_{10} \frac{N}{D}$$

Example :

Find the forced sinusoidal response of the following system.



The transmittance is

$$F(s) = \frac{10s}{s^2+2s+5}$$

Therefore,

$$F(j\omega) = F(j2) = \frac{10(j2)}{(j2)^2+2(j2)+5}$$

$$= \frac{j20}{1+j4} = \frac{20\angle 90^\circ}{4.12\angle 75.96^\circ} = 4.85\angle 14.04^\circ$$

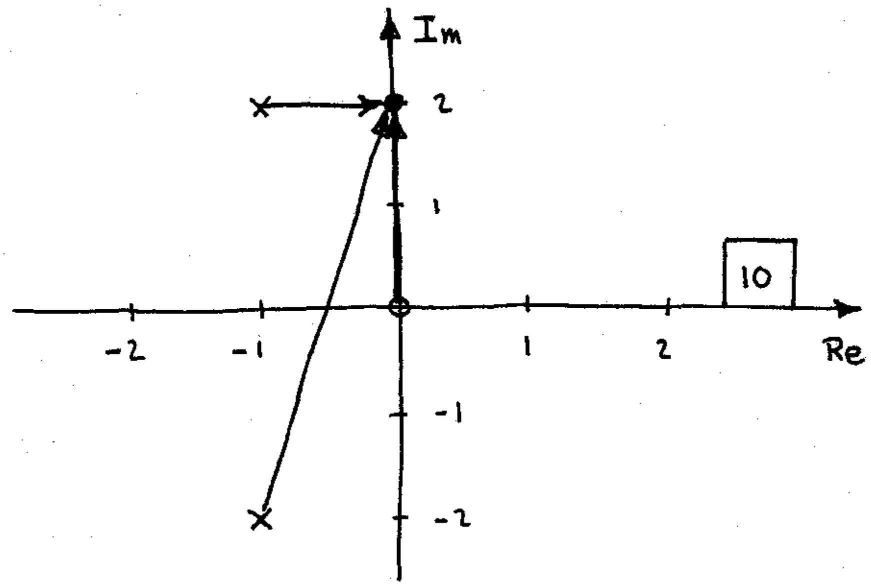
For $B = 5$ and $\beta = 30^\circ$,

$$y_{forced}(t) = B\,|F(j\omega)|\cos\left[\omega t + \beta + \phi(\omega)\right]$$

$$= 5(4.85)\cos(2t + 30^\circ + 14.04^\circ)$$

$$= \underline{\underline{24.25\cos(2t + 44.04^\circ)}}$$

F(jω) can also be evaluated graphically.



Therefore,

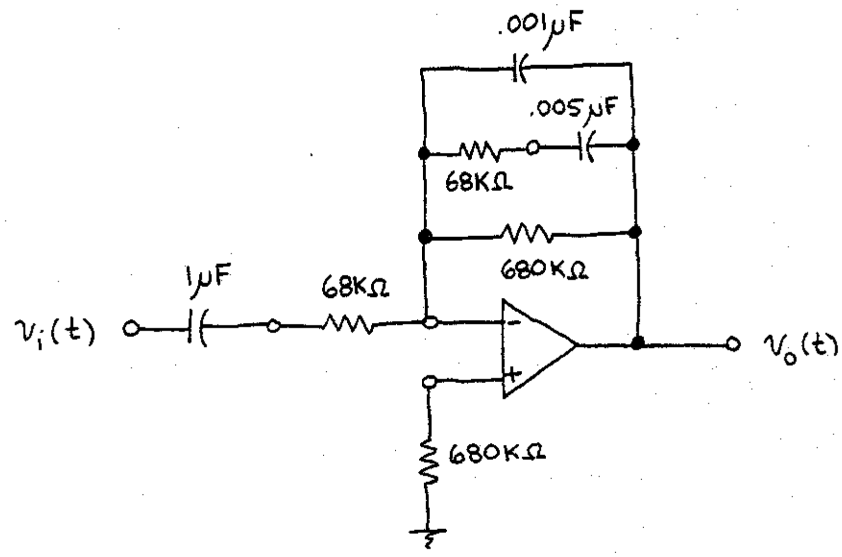
$$F(j\omega) = F(j2) = \frac{10\,(2\angle 90^\circ)}{(1\angle 0^\circ)(4.12\angle 75.96^\circ)}$$

$$= 4.85\angle 14.04^\circ \leftarrow$$

The remaining procedure to find $y_{forced}(t)$ is the same as above.

Example:

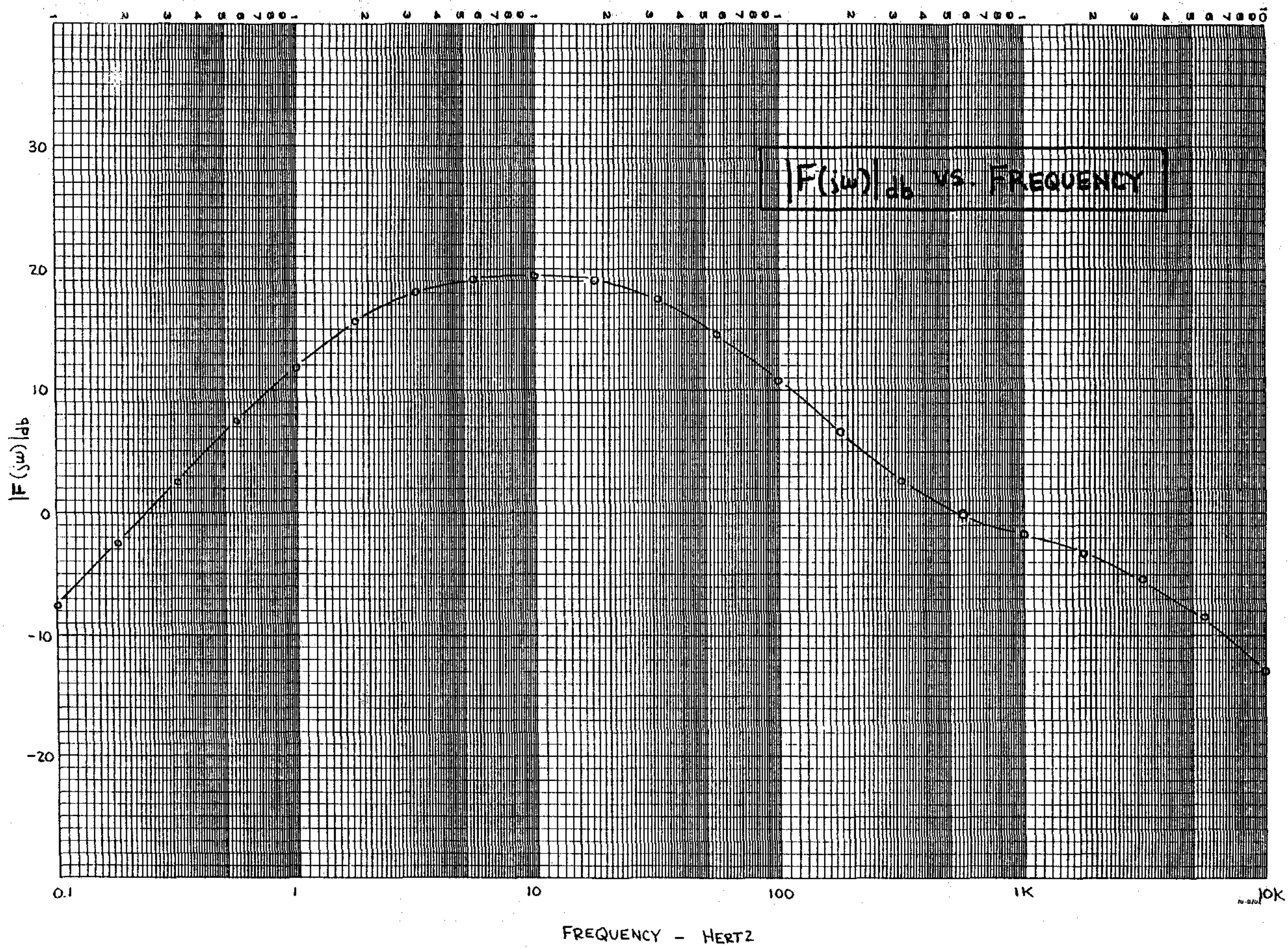
Evaluate $F(j\omega)$ and $\Phi(\omega)$ for the following circuit from $f = 0.1$ Hz to 10 KHz, assuming $v_i(t) = \cos \omega t$.

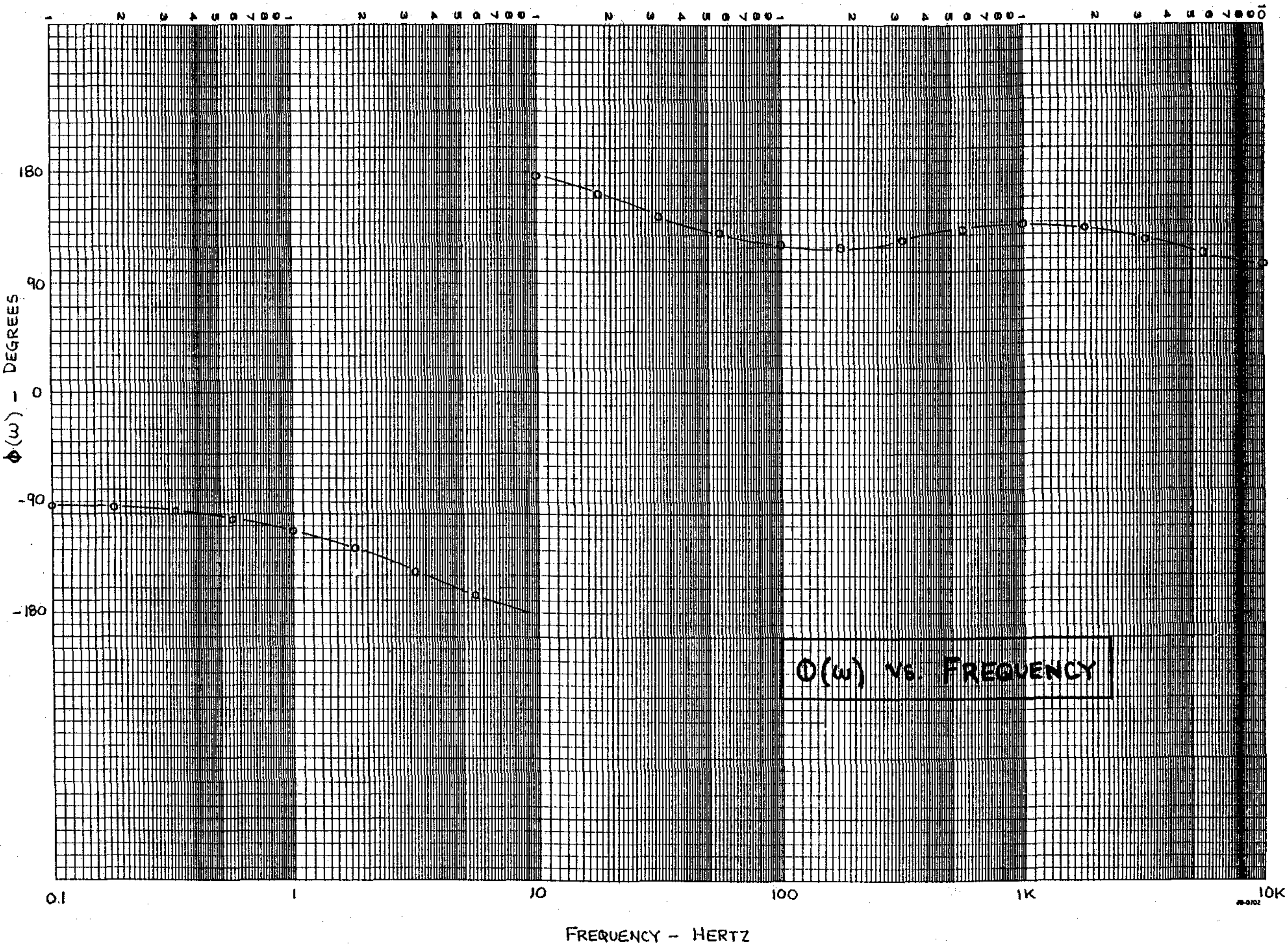


Given the transfer function,

$$F(s) = \frac{-1.47\times10^{4} s^{2} - 4.32\times10^{7} s}{s^{3} + 1.91\times10^{4} s^{2} + 4.61\times10^{6} s + 6.36\times10^{7}} ,$$

the coefficients are entered into an appropriate computer or calculator program.


|F(jω)|db VS. FREQUENCY
30
20
10
0
-10
-20
|F(jω)|db
0.1
1
10
100
1K
10K
FREQUENCY - HERTZ


Φ(ω) VS. FREQUENCY
Φ(ω) - DEGREES
180
90
0
-90
-180
0.1
1
10
100
1K
10K
FREQUENCY - HERTZ

MATLAB Function Reference

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unwrap

Correct phase angles to produce smoother phase plots

Syntax

```
Q = unwrap(P)
Q = unwrap(P,tol)
Q = unwrap(P,[],dim)
Q = unwrap(P,tol,dim)
```

Description

`Q = unwrap(P)` corrects the radian phase angles in a vector `P` by adding multiples of $\pm 2\pi$ when absolute jumps between consecutive elements of `P` are greater than or equal to the default jump tolerance of π radians. If `P` is a matrix, `unwrap` operates columnwise. If `P` is a multidimensional array, `unwrap` operates on the first nonsingleton dimension.

`Q = unwrap(P,tol)` uses a jump tolerance `tol` instead of the default value, π.

`Q = unwrap(P,[],dim)` unwraps along `dim` using the default tolerance.

`Q = unwrap(P,tol,dim)` uses a jump tolerance of `tol`.

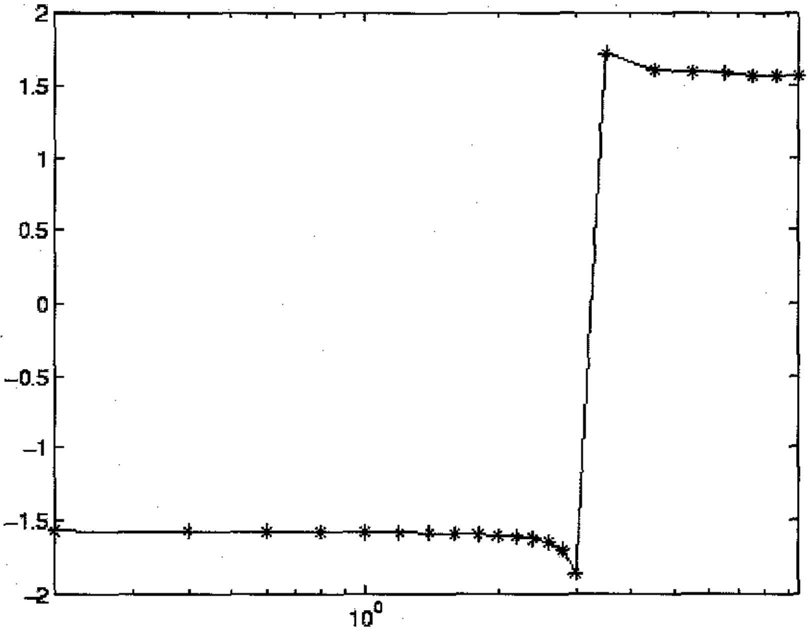
> **Note** A jump tolerance less than π has the same effect as a tolerance of π. For a tolerance less than π, if a jump is greater than the tolerance but less than π, adding $\pm 2\pi$ would result in a jump larger than the existing one, so `unwrap` chooses the current point. If you want to eliminate jumps that are less than π, try using a finer grid in the domain.

Examples

Example 1

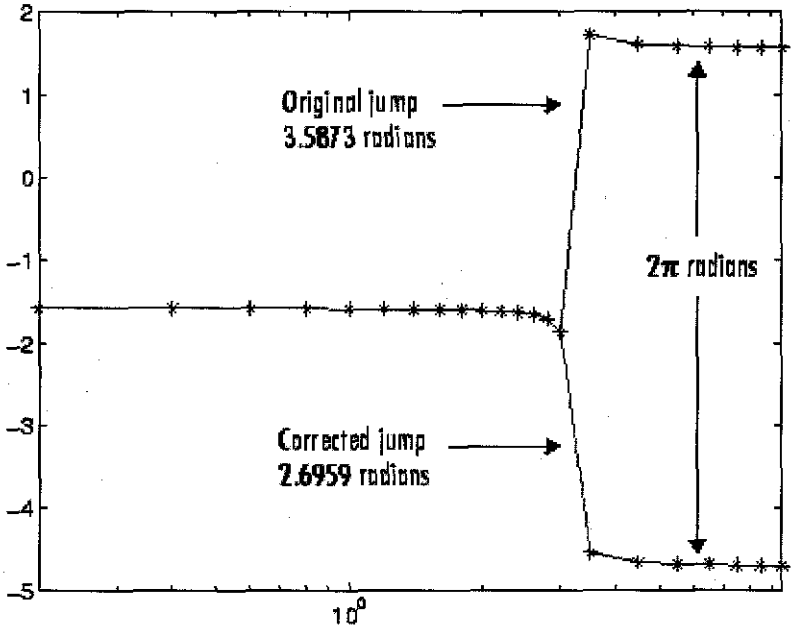
The following phase data comes from the frequency response of a third-order transfer function. The phase curve jumps `3.5873` radians between `w = 3.0` and `w = 3.5`, from `-1.8621` to `1.7252`.

```
w = [0:.2:3,3.5:1:10];
p = [     0
      -1.5728
      -1.5747
      -1.5772
      -1.5790
      -1.5816
      -1.5852
      -1.5877
      -1.5922
      -1.5976
      -1.6044
      -1.6129
      -1.6269
      -1.6512
      -1.6998
      -1.8621
       1.7252
       1.6124
       1.5930
       1.5916
       1.5708
       1.5708
       1.5708 ];
semilogx(w,p,'b*-'), hold
```



Using `unwrap` to correct the phase angle, the resulting jump is `2.6959`, which is less than the default jump tolerance **π**. This figure plots the new curve over the original curve.

```
semilogx(w,unwrap(p),'r*-')
```



> **Note** If you have the Control System Toolbox, you can create the data for this example with the following code.
>
> ```
> h = freqresp(tf(1,[1 .1 10 0]));
> p = angle(h(:));
> ```

Example 2

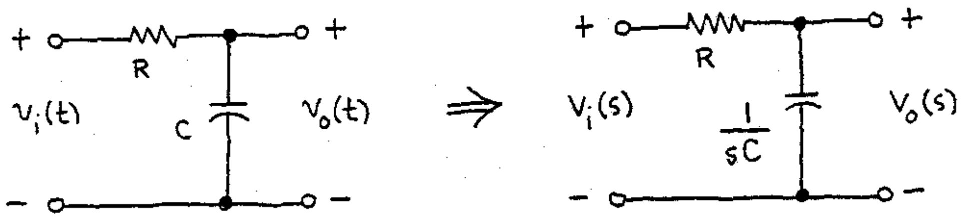
Array `P` features smoothly increasing phase angles except for discontinuities at elements `(3,1)` and `(1,2)`.

```
P = [      0    7.0686    1.5708    2.3562
      0.1963    0.9817    1.7671    2.5525
```

Bode Plots

The response curves of the amplitude and phase angle of $F(j\omega)$ are often approximated with a series of straight lines. Such straight-line approximations of amplitude and phase angle versus frequency, where frequency is plotted on a logarithmic scale, are called <u>idealized Bode diagrams</u>.

Consider the following circuit.



The transfer function is

$$F(s) = \frac{V_o(s)}{V_i(s)} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} = \frac{1}{sRC + 1}$$

Letting $s \to j\omega$,

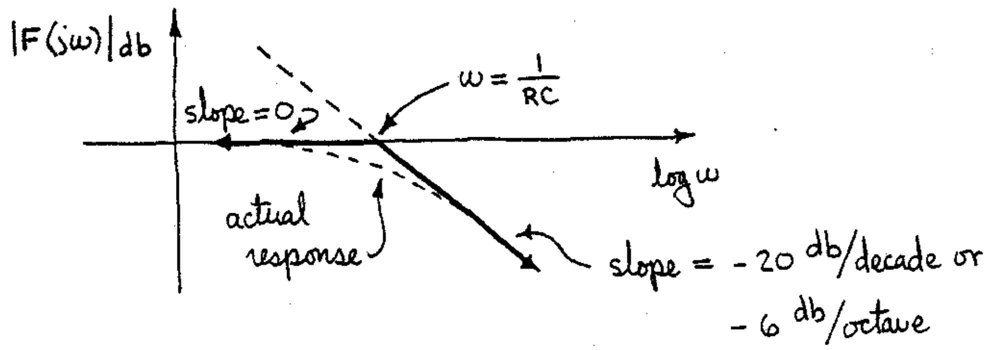
$$F(j\omega) = \frac{1}{j\omega RC + 1} = \frac{1}{1 + j\frac{\omega}{\frac{1}{RC}}} = \frac{1\angle 0^\circ}{\sqrt{1 + \left(\frac{\omega}{\frac{1}{RC}}\right)^2}\ \angle \tan^{-1}\frac{\omega}{\frac{1}{RC}}}$$

The magnitude in decibels is

$$|F(j\omega)|_{db} = \underbrace{20\log_{10} 1}_{0} - 20\log_{10}\sqrt{1 + \left(\frac{\omega}{\frac{1}{RC}}\right)^2}$$

$$= \begin{cases} 0 & \text{for } \omega \ll \frac{1}{RC} \\ -20\log_{10}\frac{\omega}{\frac{1}{RC}} & \text{for } \omega \gg \frac{1}{RC} \end{cases}$$

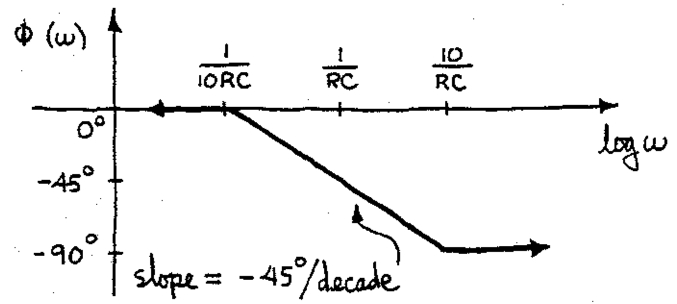
These two limiting functions are asymptotes of the original voltage transfer function in decibels. When plotted, they form an idealized Bode diagram. Note that they intersect when $\omega = \frac{1}{RC}$.



The phase angle is

$$\Phi(\omega) = \begin{cases} 0^\circ & \text{for } \omega << \frac{1}{RC} \\ -90^\circ & \text{for } \omega >> \frac{1}{RC} \end{cases}$$

The plot is



In general,

$$F(j\omega) = \frac{K\left(1+j\frac{\omega}{z_1}\right)\left(1+j\frac{\omega}{z_2}\right)\left(1+j\frac{\omega}{z_3}\right)\cdots}{\left(1+j\frac{\omega}{p_1}\right)\left(1+j\frac{\omega}{p_2}\right)\left(1+j\frac{\omega}{p_3}\right)\cdots}$$

where $z_1, z_2, z_3, \ldots, p_1, p_2, p_3, \ldots$ are called <u>break or corner frequencies</u>. The numerator corner frequencies cause the Bode amplitude plot to <u>increase</u> by 20 db/decade. The denominator corner frequencies cause the Bode amplitude plot to <u>decrease</u> by 20 db/decade (or -20 db/decade).

Example:

Consider the following transfer function.

$$F(s) = \frac{10^8 s^2 (s+100)}{(s+10)^2 (s+1{,}000)(s+10{,}000)}$$

Letting $s \to j\omega$,

$$F(j\omega) = \frac{10^8 (j\omega)^2 (j\omega+100)}{(j\omega+10)^2 (j\omega+1{,}000)(j\omega+10{,}000)}$$

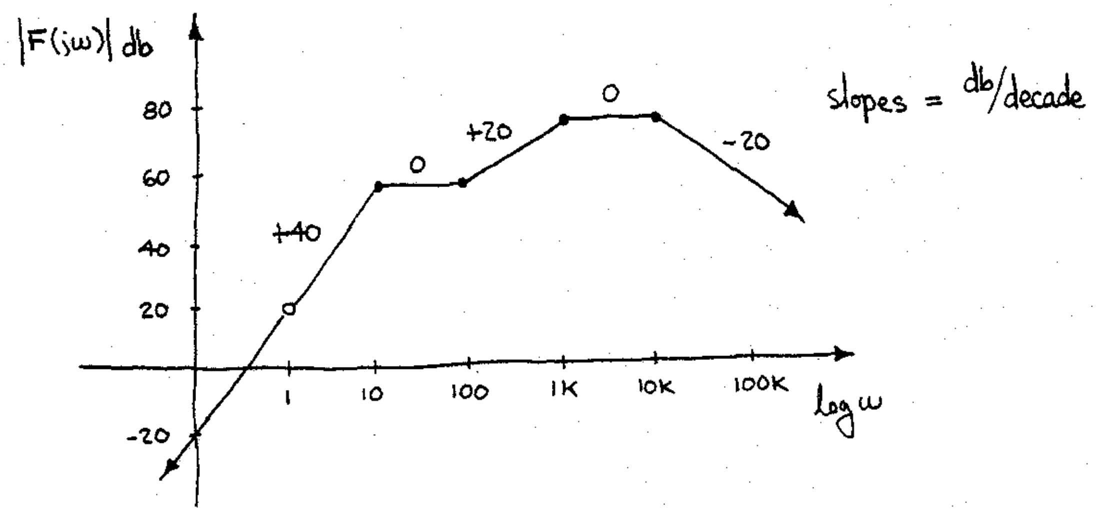
$$= \frac{-10^8 \omega^2 \left(1+j\frac{\omega}{100}\right)\left(\frac{100}{10\cdot 10\cdot 1{,}000\cdot 10{,}000}\right)}{\left(1+j\frac{\omega}{10}\right)\left(1+j\frac{\omega}{10}\right)\left(1+j\frac{\omega}{1{,}000}\right)\left(1+j\frac{\omega}{10{,}000}\right)}$$

$$= \frac{-10\omega^2 \left(1+j\frac{\omega}{100}\right)}{\left(1+j\frac{\omega}{10}\right)^2 \left(1+j\frac{\omega}{1{,}000}\right)\left(1+j\frac{\omega}{10{,}000}\right)}$$

At $\omega = 1$,

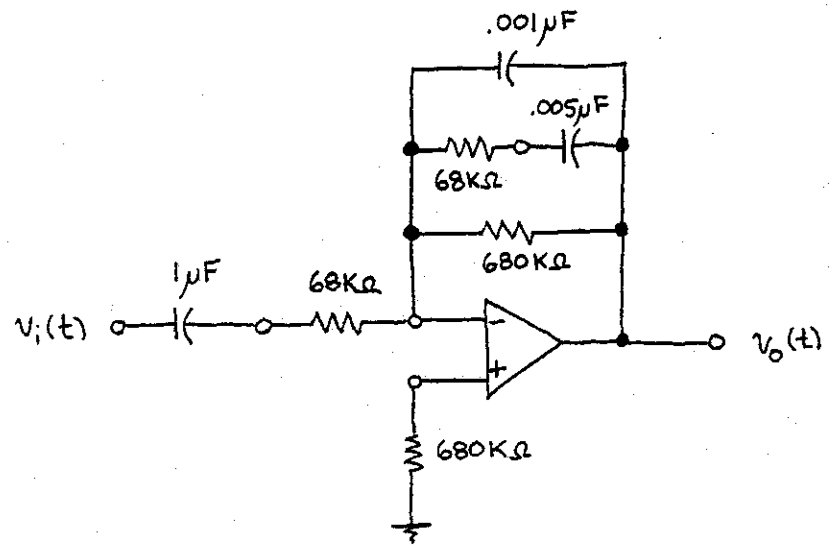
$$|F(j\omega)|_{db} = 20 \log_{10} 10 = 20 \text{ db}$$

The idealized Bode plot is



Example:

Plot the idealized Bode diagram $|F(j\omega)|_{db}$ for the following circuit from f = 0.1 Hz to 10 KHz.



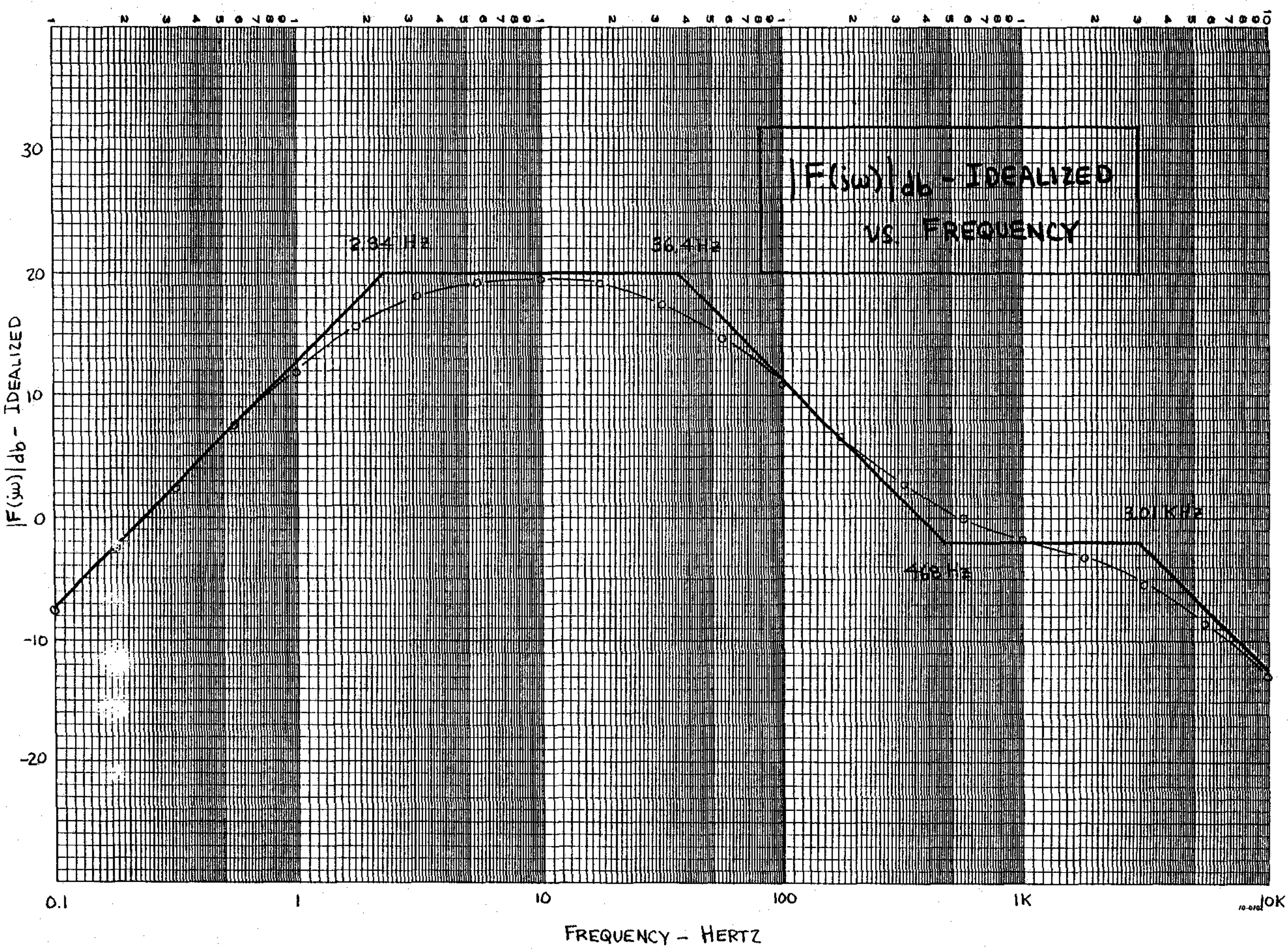
Obtaining the "factored form" of the transfer function,

$$F(s) = \frac{-1.47\times10^{4}\, s\,(s+2.94\times10^{3})}{(s+1.47\times10^{1})(s+2.29\times10^{2})(s+1.89\times10^{4})}$$

Letting $s \to j\omega$,

$$F(j\omega) = \frac{-j\,0.68\,\omega\left(1+j\frac{\omega}{2.94\times10^{3}}\right)}{\left(1+j\frac{\omega}{1.47\times10^{1}}\right)\left(1+j\frac{\omega}{2.29\times10^{2}}\right)\left(1+j\frac{\omega}{1.89\times10^{4}}\right)}$$

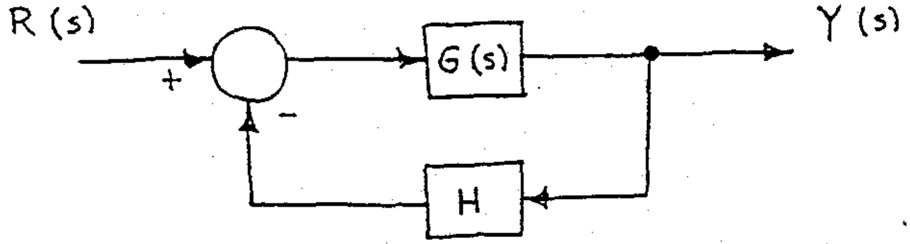
$$= \frac{-j\,4.27f\left(1+j\frac{f}{468}\right)}{\left(1+j\frac{f}{2.34}\right)\left(1+j\frac{f}{36.4}\right)\left(1+j\frac{f}{3.01\times10^{3}}\right)}$$



|F(jω)|db - IDEALIZED
VS. FREQUENCY
2.34 Hz
36.4 Hz
462 Hz
3.01 KHz
30
20
10
0
-10
-20
|F(jω)|db - IDEALIZED
0.1
1
10
100
1K
10K
FREQUENCY - HERTZ

Gain and Phase Margin

Consider a simple feedback system with constant H



where

$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + HG(s)}$$

Letting $s = j2\pi f$,

$$T(f) = \frac{G(f)}{1 + HG(f)}$$

Now suppose that as $f \rightarrow f_1$,

$$\lim_{f \rightarrow f_1} HG(f) = -1$$

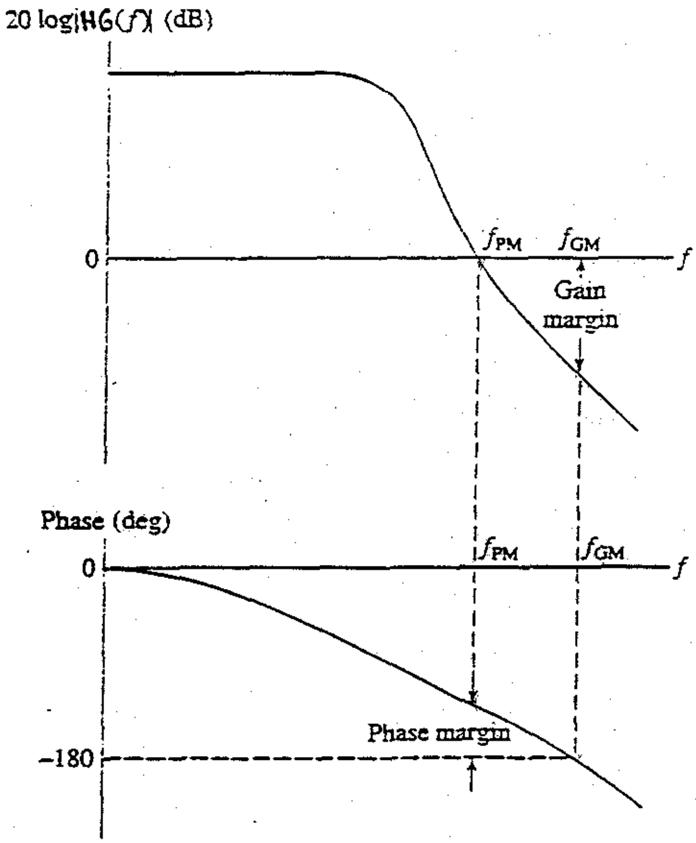
As a result,

$$\lim_{f \rightarrow f_1} T(f) = \lim_{f \rightarrow f_1} \frac{G(f)}{1 + HG(f)} = \infty$$

which corresponds to a pole on the $j\omega$-axis at $s = j2\pi f_1$. The resulting transient response would contain a constant-amplitude sinusoid.

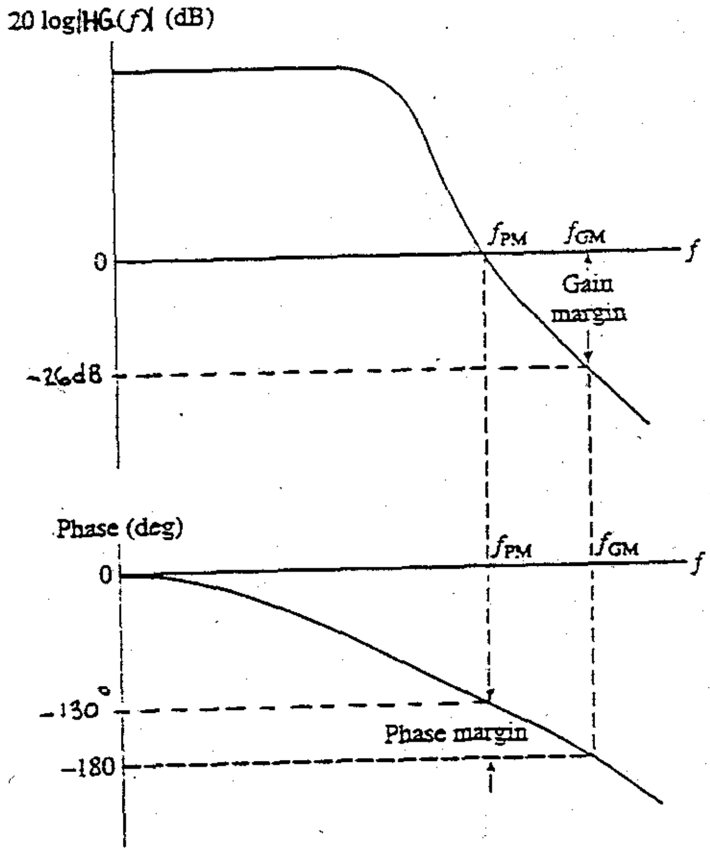
The Bode plot of the loop gain $HG(f)$ can be used to determine the stability of a system. First we examine the Bode plot of the phase shift of $HG(f)$ to determine the frequency f_{GM} for which the phase shift is 180°. If the magnitude of the loop gain is less than unity at this frequency, the system is stable. The amount that the gain magnitude is less then unity (or 0 db) is called the <u>gain margin</u>.

Another measure of stability that can be obtained from the Bode plots is the <u>phase margin</u>. Phase margin is determined at the frequency f_{PM} for which the loop gain $HG(f_{PM})$ is unity in magnitude (or 0 db). The phase margin is the difference between the actual phase and 180°.



Example:

Determine the gain and phase margin of a system with the following Bode plots.



By definition,

$$\text{Gain Margin} = 0 - (-26) = \underline{26\,\text{db}} \leftarrow$$

$$\text{Phase Margin} = -130° - (-180°) = \underline{50°} \leftarrow$$

A generally accepted rule-of-thumb is to design for a minimum gain margin of 10 db and a minimum phase margin of 45°.

The phase margin test

A test on $T(s)$, to determine whether $1/(1+T(s))$ contains RHP poles.

The crossover frequency f_c is defined as the frequency where

$$\| T(j2\pi f_c) \| = 1 \Rightarrow 0\text{dB}$$

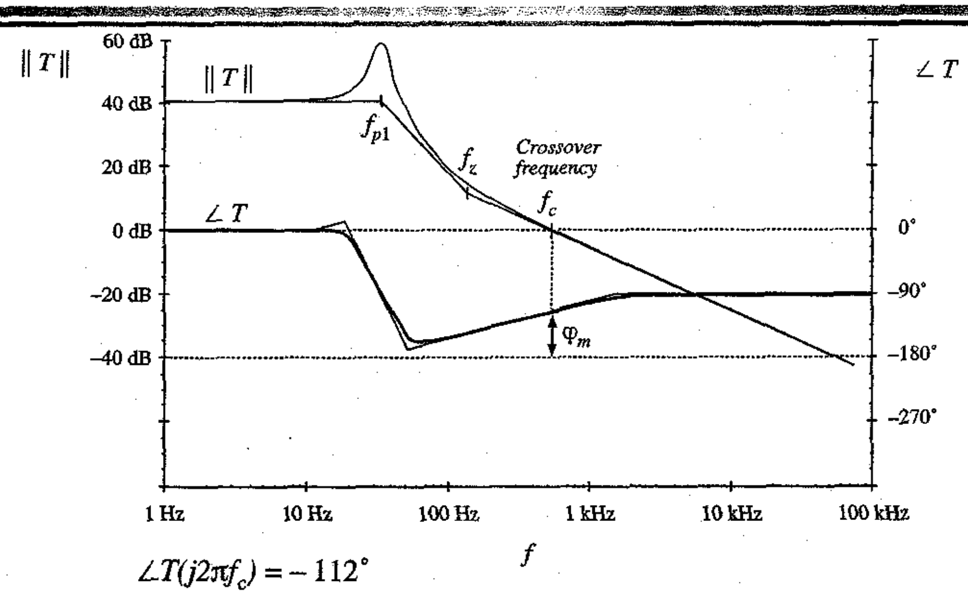
The phase margin φ_m is determined from the phase of $T(s)$ at f_c, as follows:

$$\varphi_m = 180^\circ + \angle T(j2\pi f_c)$$

If there is exactly one crossover frequency, and if $T(s)$ contains no RHP poles, then

the quantities $T(s)/(1+T(s))$ and $1/(1+T(s))$ contain no RHP poles whenever the phase margin φ_m is positive.

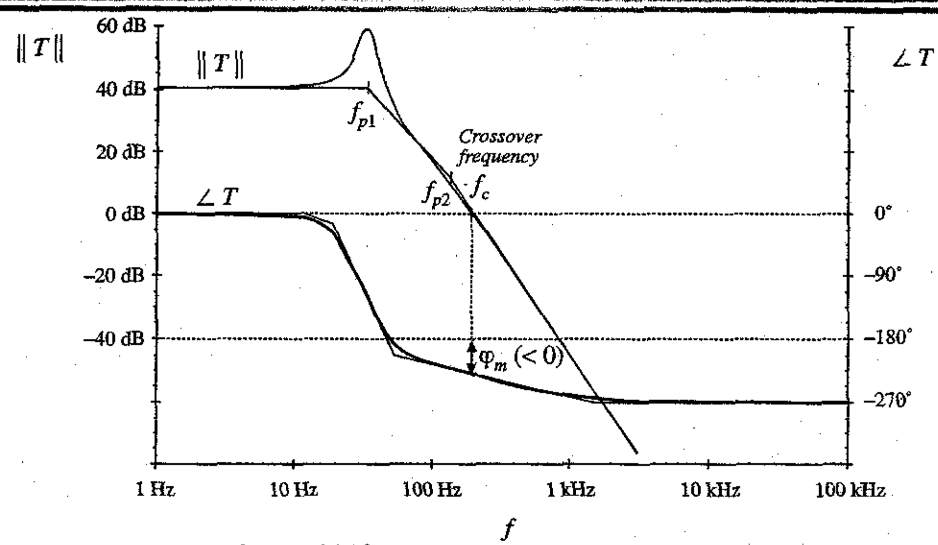
Example: a loop gain leading to a stable closed-loop system



$$\angle T(j2\pi f_c) = -112^\circ$$

$$\varphi_m = 180^\circ - 112^\circ = +68^\circ$$

Example: a loop gain leading to an unstable closed-loop system



$\angle T(j2\pi f_c) = -230^\circ$

$\varphi_m = 180^\circ - 230^\circ = -50^\circ$

Review of Bode plots

Decibels

$$\left\| G \right\|_{dB} = 20 \log_{10}\left(\left\| G \right\|\right)$$

Decibels of quantities having units (impedance example): normalize before taking log

$$\left\| Z \right\|_{dB} = 20 \log_{10}\left(\frac{\left\| Z \right\|}{R_{base}}\right)$$

Expressing magnitudes in decibels

Actual magnitude	*Magnitude in dB*
1/2	– 6dB
1	0 dB
2	6 dB
5 = 10/2	20 dB – 6 dB = 14 dB
10	20dB
$1000 = 10^3$	3 · 20dB = 60 dB

5Ω is equivalent to 14dB with respect to a base impedance of R_{base} = 1Ω, also known as 14dBΩ.

60dBμA is a current 60dB greater than a base current of 1μA, or 1mA.

Bode plot of f^n

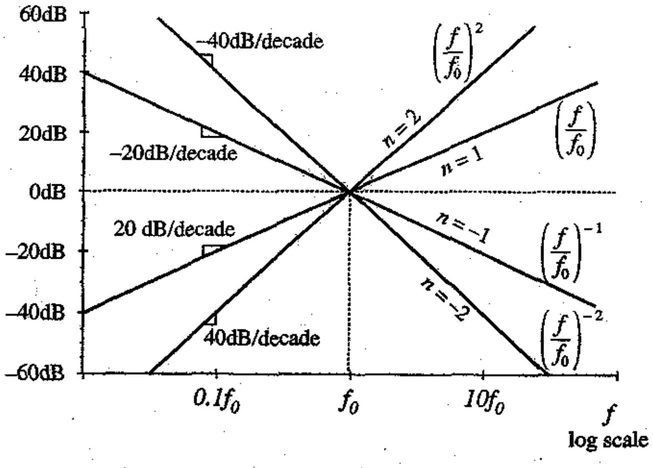
Bode plots are effectively log-log plots, which cause functions which vary as f^n to become linear plots. Given:

$$\left\| G \right\| = \left(\frac{f}{f_0}\right)^n$$

Magnitude in dB is

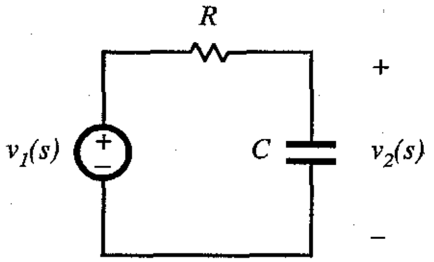
$$\left\| G \right\|_{dB} = 20 \log_{10}\left(\frac{f}{f_0}\right)^n = 20n \log_{10}\left(\frac{f}{f_0}\right)$$

- Slope is $20n$ dB/decade
- Magnitude is 1, or 0dB, at frequency $f = f_0$



Single pole response

Simple R-C example



Transfer function is

$$G(s) = \frac{v_2(s)}{v_1(s)} = \frac{\frac{1}{sC}}{\frac{1}{sC} + R}$$

Express as rational fraction:

$$G(s) = \frac{1}{1 + sRC}$$

This coincides with the normalized form

$$G(s) = \frac{1}{\left(1 + \frac{s}{\omega_0}\right)}$$

with $\omega_0 = \frac{1}{RC}$

$G(j\omega)$ and $\| G(j\omega) \|$

Let $s = j\omega$:

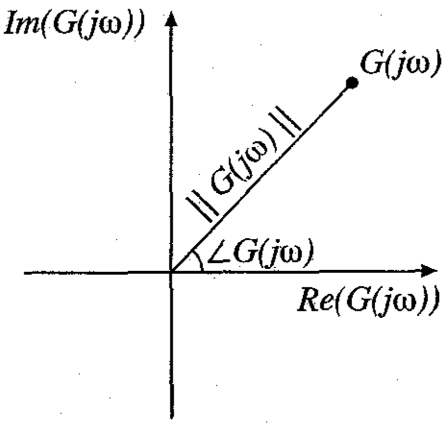
$$G(j\omega) = \frac{1}{\left(1 + j\frac{\omega}{\omega_0}\right)} = \frac{1 - j\frac{\omega}{\omega_0}}{1 + \left(\frac{\omega}{\omega_0}\right)^2}$$

Magnitude is

$$\left\| G(j\omega) \right\| = \sqrt{\left[\mathrm{Re}\,(G(j\omega))\right]^2 + \left[\mathrm{Im}\,(G(j\omega))\right]^2} = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}}$$

Magnitude in dB:

$$\left\| G(j\omega) \right\|_{\mathrm{dB}} = -20 \log_{10}\left(\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}\right) \ \mathrm{dB}$$



Asymptotic behavior: low frequency

For small frequency,
$\omega \ll \omega_0$ and $f \ll f_0$:

$$\left\| G(j\omega) \right\| = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}}$$

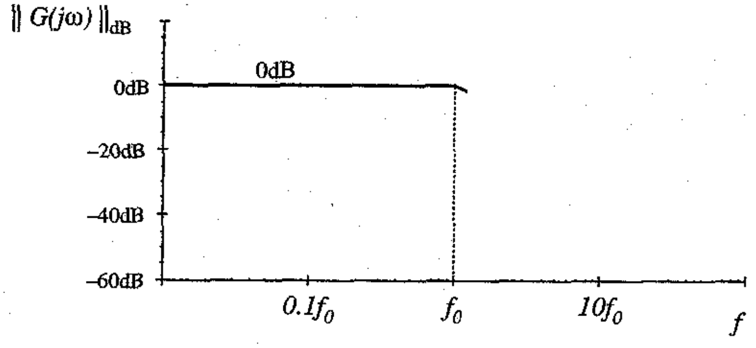
$$\left(\frac{\omega}{\omega_0}\right) \ll 1$$

Then $\| G(j\omega) \|$ becomes

$$\left\| G(j\omega) \right\| \approx \frac{1}{\sqrt{1}} = 1$$

Or, in dB,

$$\left\| G(j\omega) \right\|_{dB} \approx 0\text{dB}$$



This is the low-frequency asymptote of $\| G(j\omega) \|$

Asymptotic behavior: high frequency

For high frequency,
$\omega \gg \omega_0$ and $f \gg f_0$:

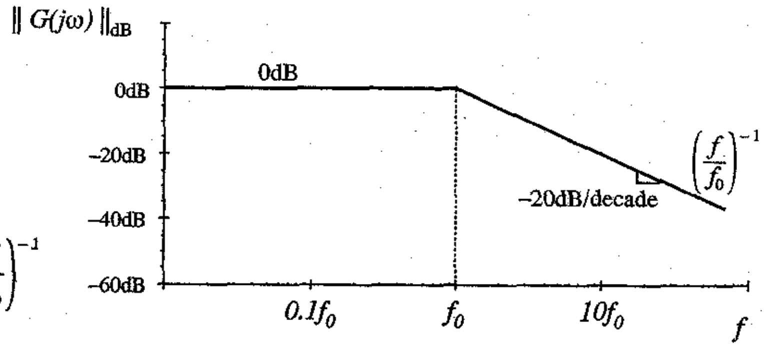
$$\left\| G(j\omega) \right\| = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}}$$

$$\left(\frac{\omega}{\omega_0}\right) \gg 1$$

$$1 + \left(\frac{\omega}{\omega_0}\right)^2 \approx \left(\frac{\omega}{\omega_0}\right)^2$$

Then $\| G(j\omega) \|$ becomes

$$\left\| G(j\omega) \right\| \approx \frac{1}{\sqrt{\left(\frac{\omega}{\omega_0}\right)^2}} = \left(\frac{f}{f_0}\right)^{-1}$$



The high-frequency asymptote of $\| G(j\omega) \|$ varies as f^{-1}. Hence, $n = -1$, and a straight-line asymptote having a slope of -20dB/decade is obtained. The asymptote has a value of 1 at $f = f_0$.

Deviation of exact curve near $f = f_0$

Evaluate exact magnitude:

at $f = f_0$:

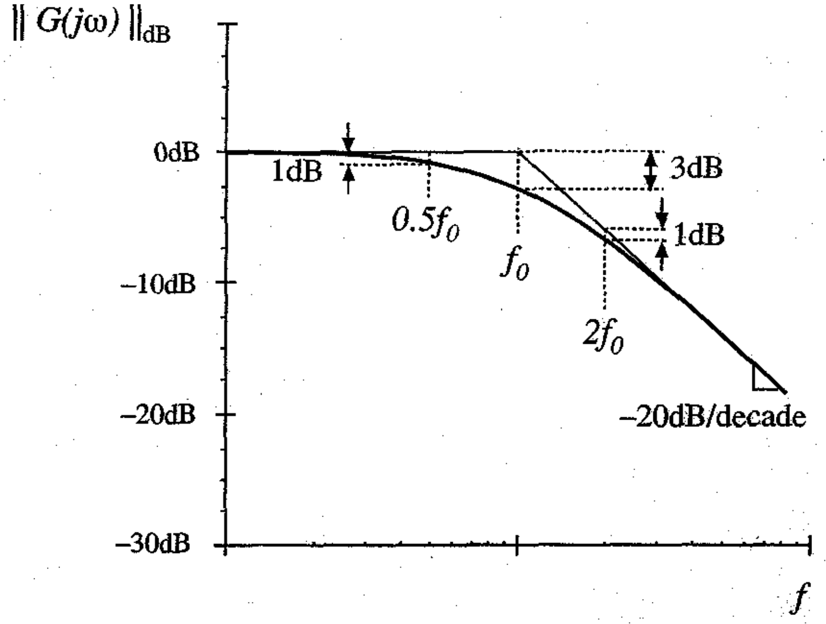
$$\left\| G(j\omega_0) \right\| = \frac{1}{\sqrt{1 + \left(\frac{\omega_0}{\omega_0}\right)^2}} = \frac{1}{\sqrt{2}}$$

$$\left\| G(j\omega_0) \right\|_{dB} = -20 \log_{10}\left(\sqrt{1 + \left(\frac{\omega_0}{\omega_0}\right)^2}\right) \approx -3 \text{ dB}$$

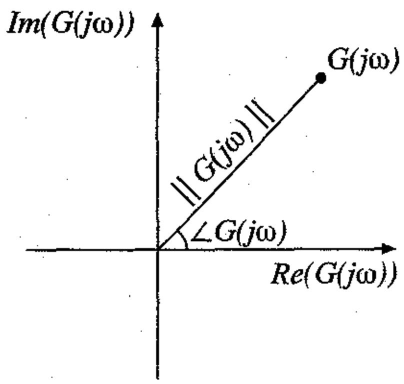
at $f = 0.5f_0$ *and* $2f_0$:

Similar arguments show that the exact curve lies 1dB below the asymptotes.

Summary: magnitude



Phase of $G(j\omega)$

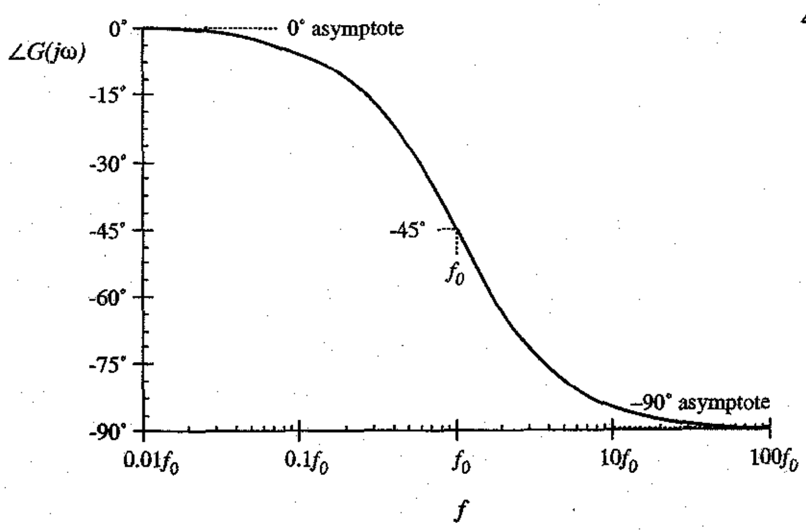


$$\angle G(j\omega) = \tan^{-1}\left(\frac{\mathrm{Im}\left(G(j\omega)\right)}{\mathrm{Re}\left(G(j\omega)\right)}\right)$$

$$G(j\omega) = \frac{1}{\left(1 + j\frac{\omega}{\omega_0}\right)} = \frac{1 - j\frac{\omega}{\omega_0}}{1 + \left(\frac{\omega}{\omega_0}\right)^2}$$

$$\angle G(j\omega) = -\tan^{-1}\left(\frac{\omega}{\omega_0}\right)$$

Phase of $G(j\omega)$



$$\angle G(j\omega) = -\tan^{-1}\left(\frac{\omega}{\omega_0}\right)$$

ω	$\angle G(j\omega)$
0	0°
ω_0	–45°
∞	–90°

Phase asymptotes

Low frequency: 0˚

High frequency: –90˚

Low- and high-frequency asymptotes do not intersect

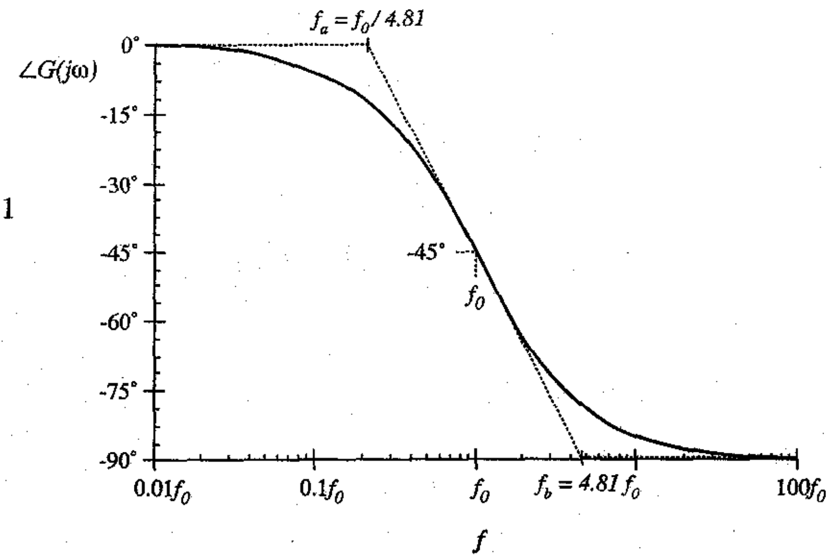
Hence, need a midfrequency asymptote

Try a midfrequency asymptote having slope identical to actual slope at the corner frequency f_0. One can show that the asymptotes then intersect at the break frequencies

$$f_a = f_0\, e^{-\pi/2} \approx f_0 / 4.81$$
$$f_b = f_0\, e^{\pi/2} \approx 4.81\, f_0$$

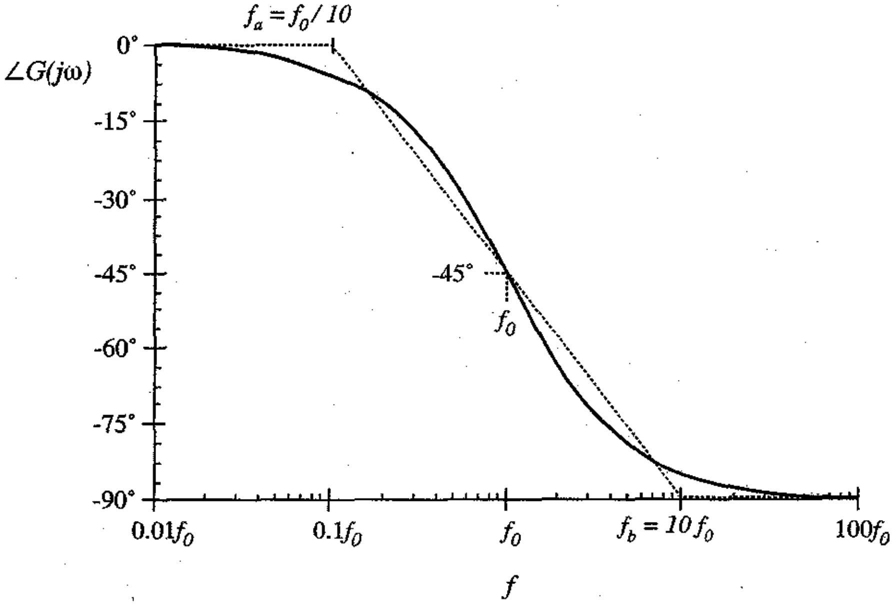
Phase asymptotes

$$f_a = f_0\, e^{-\pi/2} \approx f_0 / 4.81$$
$$f_b = f_0\, e^{\pi/2} \approx 4.81\, f_0$$



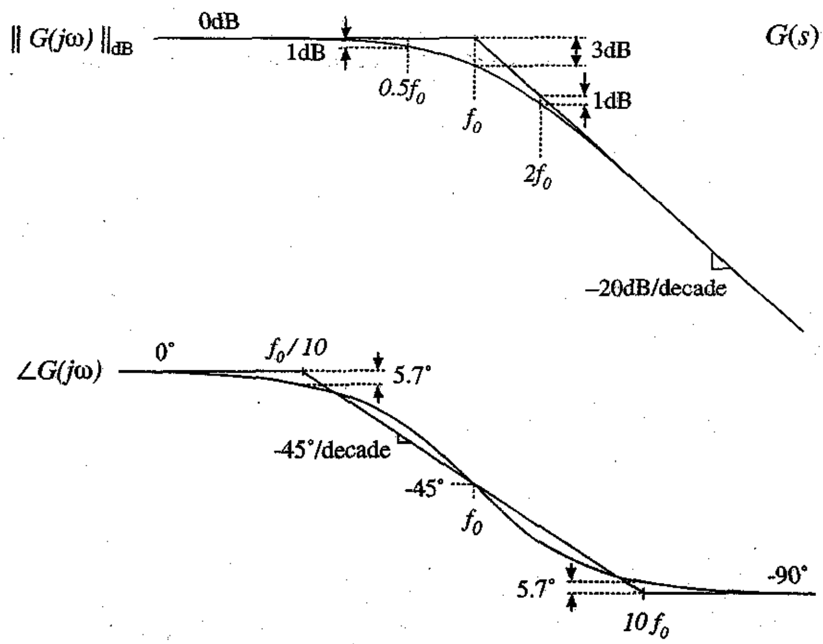
Phase asymptotes: a simpler choice

$f_a = f_0 / 10$
$f_b = 10\ f_0$



Summary: Bode plot of real pole

$$G(s) = \frac{1}{\left(1 + \frac{s}{\omega_0}\right)}$$



Single zero response

Normalized form:

$$G(s) = \left(1 + \frac{s}{\omega_0}\right)$$

Magnitude:

$$\left\| G(j\omega) \right\| = \sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}$$

Use arguments similar to those used for the simple pole, to derive asymptotes:

0dB at low frequency, $\omega \ll \omega_0$

+20dB/decade slope at high frequency, $\omega \gg \omega_0$

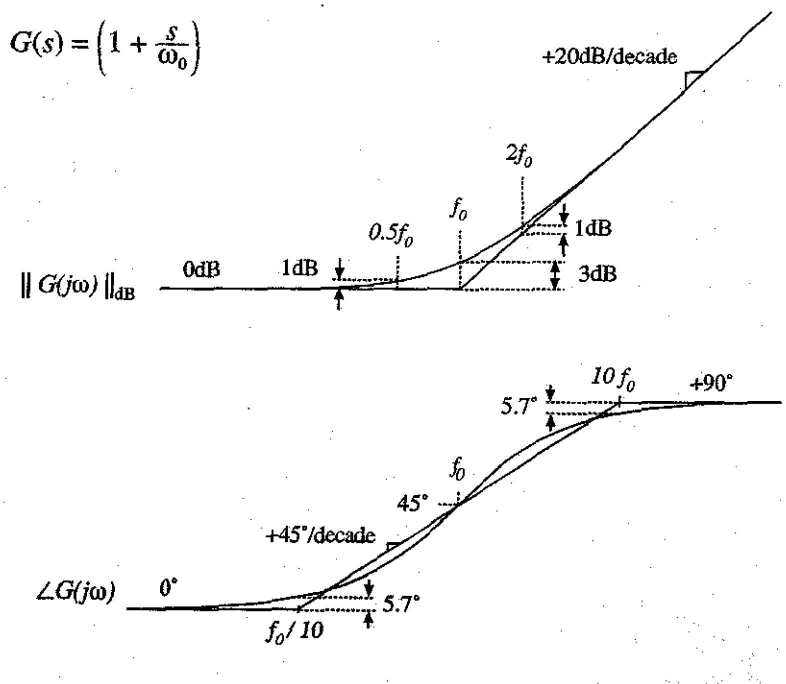
Phase:

$$\angle G(j\omega) = \tan^{-1}\left(\frac{\omega}{\omega_0}\right)$$

—with the exception of a missing minus sign, same as simple pole

Summary: Bode plot, real zero

$$G(s) = \left(1 + \frac{s}{\omega_0}\right)$$



Combinations

Suppose that we have constructed the Bode diagrams of two complex-values functions of frequency, $G_1(\omega)$ and $G_2(\omega)$. It is desired to construct the Bode diagram of the product, $G_3(\omega) = G_1(\omega)\, G_2(\omega)$.

Express the complex-valued functions in polar form:

$$G_1(\omega) = R_1(\omega)\, e^{j\theta_1(\omega)}$$
$$G_2(\omega) = R_2(\omega)\, e^{j\theta_2(\omega)}$$
$$G_3(\omega) = R_3(\omega)\, e^{j\theta_3(\omega)}$$

The product $G_3(\omega)$ can then be written

$$G_3(\omega) = G_1(\omega)\, G_2(\omega) = R_1(\omega)\, e^{j\theta_1(\omega)}\, R_2(\omega)\, e^{j\theta_2(\omega)}$$

$$G_3(\omega) = \left(R_1(\omega)\, R_2(\omega)\right) e^{j(\theta_1(\omega) + \theta_2(\omega))}$$

Combinations

$$G_3(\omega) = \left(R_1(\omega)\, R_2(\omega)\right) e^{j(\theta_1(\omega) + \theta_2(\omega))}$$

The composite phase is

$$\theta_3(\omega) = \theta_1(\omega) + \theta_2(\omega)$$

The composite magnitude is

$$R_3(\omega) = R_1(\omega)\, R_2(\omega)$$

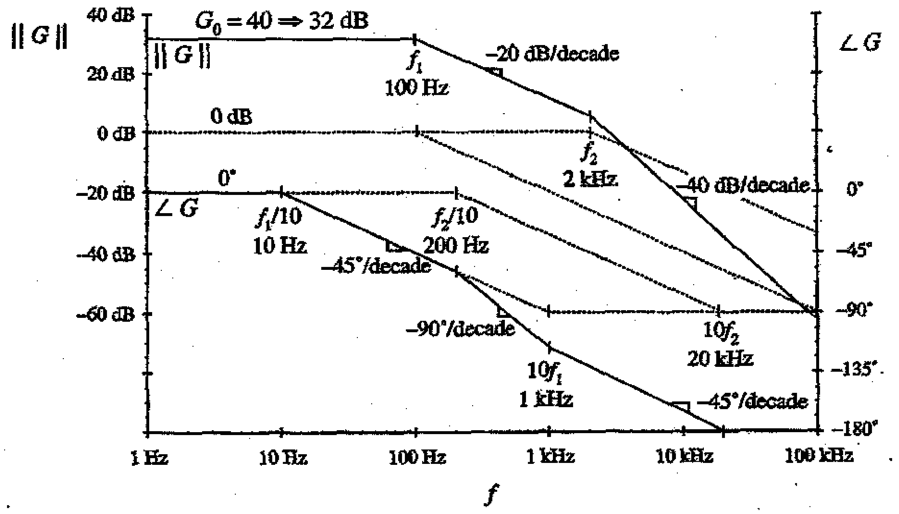
$$\left| R_3(\omega) \right|_{\mathrm{dB}} = \left| R_1(\omega) \right|_{\mathrm{dB}} + \left| R_2(\omega) \right|_{\mathrm{dB}}$$

Composite phase is sum of individual phases.

Composite magnitude, when expressed in dB, is sum of individual magnitudes.

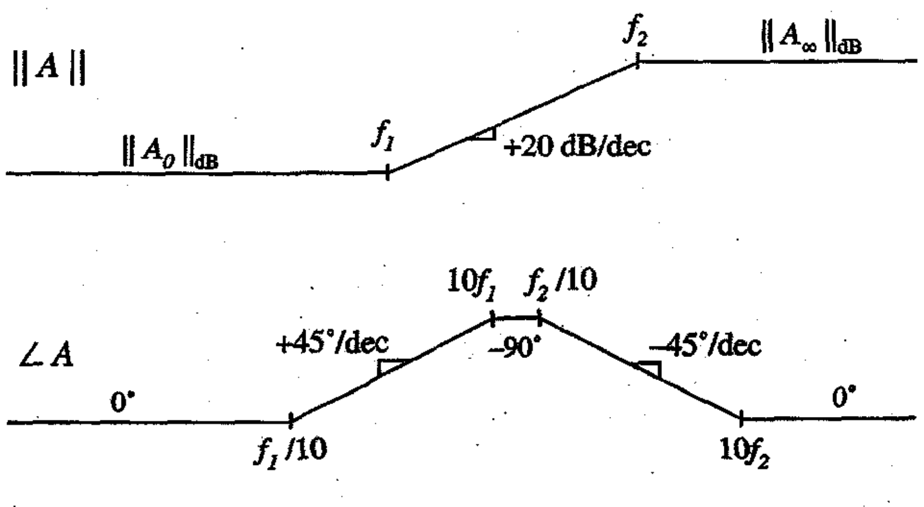
Example 1: $G(s) = \dfrac{G_0}{\left(1 + \frac{s}{\omega_1}\right)\left(1 + \frac{s}{\omega_2}\right)}$

with $G_0 = 40 \Rightarrow 32$ dB, $f_1 = \omega_1/2\pi = 100$ Hz, $f_2 = \omega_2/2\pi = 2$ kHz



Example 2

Determine the transfer function *A(s)* corresponding to the following asymptotes:



Example 2, continued

One solution:

$$A(s) = A_0 \frac{\left(1 + \frac{s}{\omega_1}\right)}{\left(1 + \frac{s}{\omega_2}\right)}$$

Analytical expressions for asymptotes:

For $f < f_1$

$$\left\| A_0 \frac{\left(1 + \cancel{\frac{s}{\omega_1}}\right)}{\left(1 + \cancel{\frac{s}{\omega_2}}\right)} \right\|_{s=j\omega} = A_0 \frac{1}{1} = A_0$$

For $f_1 < f < f_2$

$$\left\| A_0 \frac{\left(\cancel{1} + \frac{s}{\omega_1}\right)}{\left(1 + \cancel{\frac{s}{\omega_2}}\right)} \right\|_{s=j\omega} = A_0 \frac{\left\| \frac{s}{\omega_1} \right\|_{s=j\omega}}{1} = A_0 \frac{\omega}{\omega_1} = A_0 \frac{f}{f_1}$$

Example 2, continued

For $f > f_2$

$$\left\| A_0 \frac{\left(\cancel{1} + \frac{s}{\omega_1}\right)}{\left(\cancel{1} + \frac{s}{\omega_2}\right)} \right\|_{s=j\omega} = A_0 \frac{\left\| \frac{s}{\omega_1} \right\|_{s=j\omega}}{\left\| \frac{s}{\omega_2} \right\|_{s=j\omega}} = A_0 \frac{\omega_2}{\omega_1} = A_0 \frac{f_2}{f_1}$$

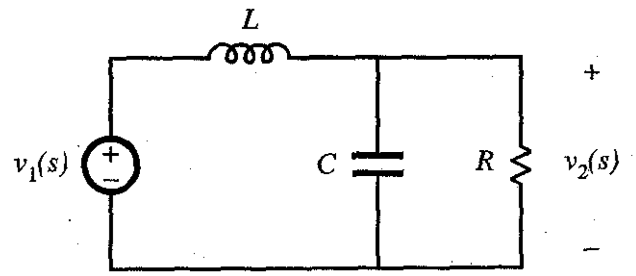
So the high-frequency asymptote is

$$A_\infty = A_0 \frac{f_2}{f_1}$$

Quadratic pole response: resonance

Example

$$G(s) = \frac{v_2(s)}{v_1(s)} = \frac{1}{1 + s\frac{L}{R} + s^2 LC}$$



Two-pole low-pass filter example

Second-order denominator, of the form

$$G(s) = \frac{1}{1 + a_1 s + a_2 s^2}$$

with $a_1 = L/R$ and $a_2 = LC$

How should we construct the Bode diagram?

Approach 1: factor denominator

$$G(s) = \frac{1}{1 + a_1 s + a_2 s^2}$$

We might factor the denominator using the quadratic formula, then construct Bode diagram as the combination of two real poles:

$$G(s) = \frac{1}{\left(1 - \frac{s}{s_1}\right)\left(1 - \frac{s}{s_2}\right)} \quad \text{with} \quad s_1 = -\frac{a_1}{2a_2}\left[1 - \sqrt{1 - \frac{4a_2}{a_1^2}}\right]$$

$$s_2 = -\frac{a_1}{2a_2}\left[1 + \sqrt{1 - \frac{4a_2}{a_1^2}}\right]$$

- If $4a_2 \le a_1^2$, then the roots s_1 and s_2 are real. We can construct Bode diagram as the combination of two real poles.
- If $4a_2 > a_1^2$, then the roots are complex. In a previous section, the assumption was made that ω_0 is real; hence, the results of that section cannot be applied and we need to do some additional work.

Approach 2: Define a standard normalized form for the quadratic case

$$G(s) = \frac{1}{1 + 2\zeta \frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2} \qquad \text{or} \qquad G(s) = \frac{1}{1 + \frac{s}{Q\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

- When the coefficients of s are real and positive, then the parameters ζ, ω_0, and Q are also real and positive
- The parameters ζ, ω_0, and Q are found by equating the coefficients of s
- The parameter ω_0 is the angular corner frequency, and we can define $f_0 = \omega_0/2\pi$
- The parameter ζ is called the *damping factor.* ζ controls the shape of the exact curve in the vicinity of $f = f_0$. The roots are complex when $\zeta < 1$.
- In the alternative form, the parameter Q is called the *quality factor.* Q also controls the shape of the exact curve in the vicinity of $f = f_0$. The roots are complex when $Q > 0.5$.

The Q-factor

In a second-order system, ζ and Q are related according to

$$Q = \frac{1}{2\zeta}$$

Q is a measure of the dissipation in the system. A more general definition of Q, for sinusoidal excitation of a passive element or system is

$$Q = 2\pi \frac{\text{(peak stored energy)}}{\text{(energy dissipated per cycle)}}$$

For a second-order passive system, the two equations above are equivalent. We will see that Q has a simple interpretation in the Bode diagrams of second-order transfer functions.

Analytical expressions for f_0 and Q

Two-pole low-pass filter example: we found that

$$G(s) = \frac{v_2(s)}{v_1(s)} = \frac{1}{1 + s\frac{L}{R} + s^2 LC}$$

Equate coefficients of like powers of s with the standard form

$$G(s) = \frac{1}{1 + \frac{s}{Q\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

Result:

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$

$$Q = R\sqrt{\frac{C}{L}}$$

Magnitude asymptotes, quadratic form

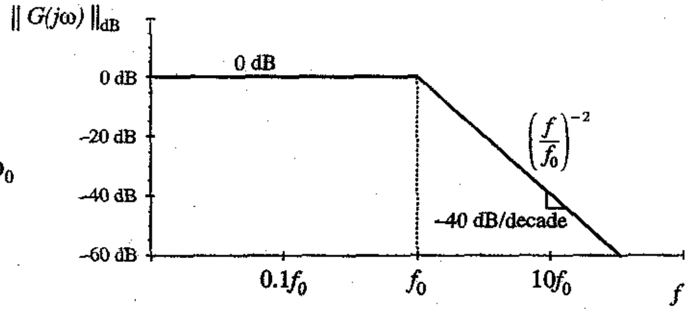
In the form $G(s) = \frac{1}{1 + \frac{s}{Q\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$

let $s = j\omega$ and find magnitude: $\left\| G(j\omega) \right\| = \frac{1}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2}}$

Asymptotes are

$\left\| G \right\| \to 1$ for $\omega \ll \omega_0$

$\left\| G \right\| \to \left(\frac{f}{f_0}\right)^{-2}$ for $\omega \gg \omega_0$



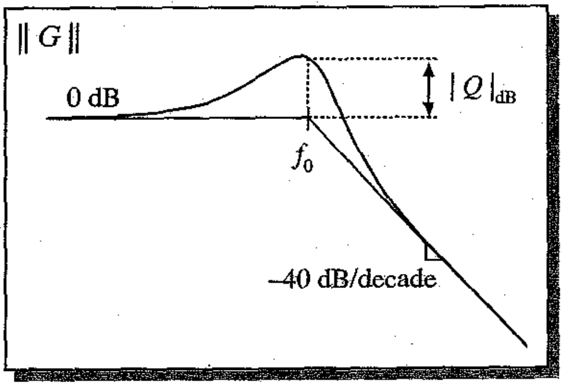
Deviation of exact curve from magnitude asymptotes

$$\left\| G(j\omega) \right\| = \frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2}}$$

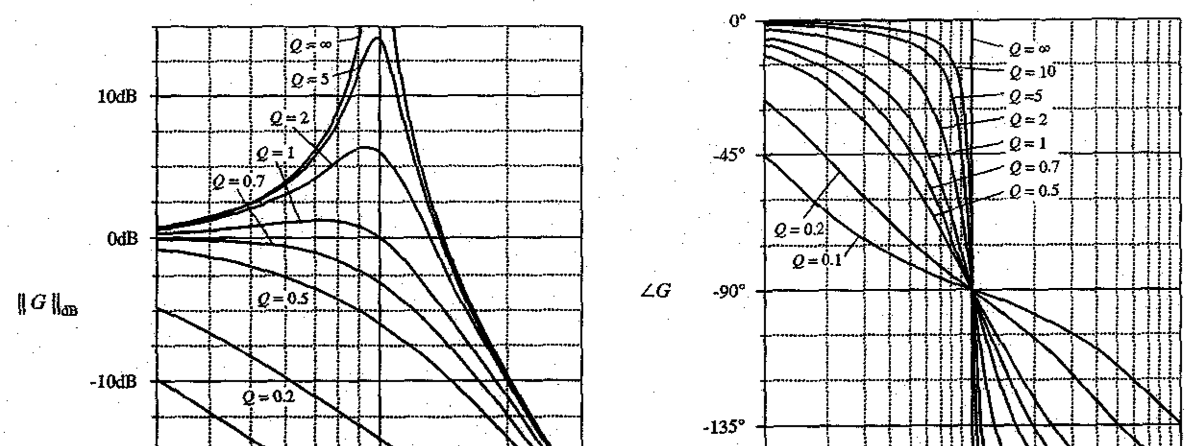
At $\omega = \omega_0$, the exact magnitude is

$$\left\| G(j\omega_0) \right\| = Q \qquad \text{or, in dB:} \qquad \left\| G(j\omega_0) \right\|_{dB} = \left| Q \right|_{dB}$$

The exact curve has magnitude Q at $f = f_0$. The deviation of the exact curve from the asymptotes is $| Q |_{dB}$



Two-pole response: exact curves



Stability

Even though the original open-loop system is stable, the closed-loop transfer functions can be unstable and contain right half-plane poles. Even when the closed-loop system is stable, the transient response can exhibit undesirable ringing and overshoot, due to the high Q -factor of the closed-loop poles in the vicinity of the crossover frequency.

When feedback destabilizes the system, the denominator $(1+T(s))$ terms in the closed-loop transfer functions contain roots in the right half-plane (i.e., with positive real parts). If $T(s)$ is a rational fraction of the form $N(s) / D(s)$, where $N(s)$ and $D(s)$ are polynomials, then we can write

$$\frac{T(s)}{1+T(s)} = \frac{\frac{N(s)}{D(s)}}{1+\frac{N(s)}{D(s)}} = \frac{N(s)}{N(s)+D(s)}$$

$$\frac{1}{1+T(s)} = \frac{1}{1+\frac{N(s)}{D(s)}} = \frac{D(s)}{N(s)+D(s)}$$

- Could evaluate stability by evaluating $N(s) + D(s)$, then factoring to evaluate roots. This is a lot of work, and is not very illuminating.

Determination of stability directly from $T(s)$

- Nyquist stability theorem: general result.
- A special case of the Nyquist stability theorem: the phase margin test

  Allows determination of closed-loop stability (i.e., whether $1/(1+T(s))$ contains RHP poles) directly from the magnitude and phase of $T(s)$.

  A good design tool: yields insight into how $T(s)$ should be shaped, to obtain good performance in transfer functions containing $1/(1+T(s))$ terms.

The phase margin test

A test on $T(s)$, to determine whether $1/(1+T(s))$ contains RHP poles.

The crossover frequency f_c is defined as the frequency where

$$\| T(j2\pi f_c) \| = 1 \Rightarrow 0\text{dB}$$

The phase margin φ_m is determined from the phase of $T(s)$ at f_c , as follows:

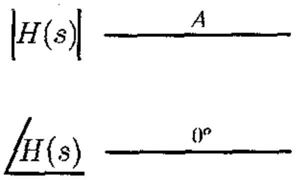
$$\varphi_m = 180^\circ + \angle T(j2\pi f_c)$$

If there is exactly one crossover frequency, and if $T(s)$ contains no RHP poles, then

the quantities $T(s)/(1+T(s))$ and $1/(1+T(s))$ contain no RHP poles whenever the phase margin φ_m is positive.

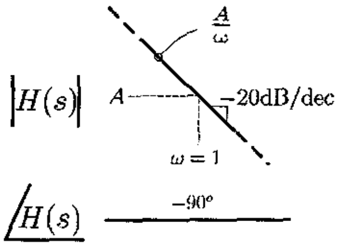
$H(s) = A$

Simple Gain



$H(s) = \frac{A}{s}$

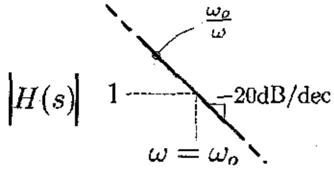
Pole at Zero



Or:

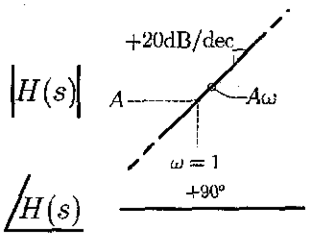
$H(s) = \frac{\omega_o}{s}$

$\omega_o = A$



$H(s) = As$

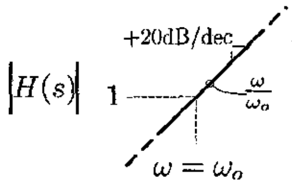
Zero at Zero



Or:

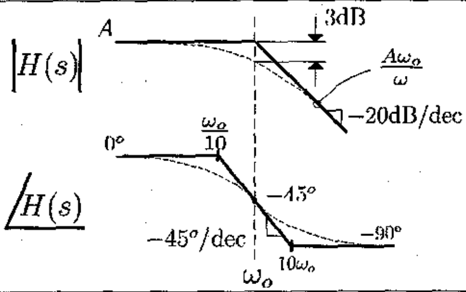
$H(s) = \frac{s}{\omega_o}$

$\omega_o = \frac{1}{A}$



$H(s) = \frac{A}{1+\frac{s}{\omega_o}}$

Pole at ω_o



$\left(\text{If } \omega \gg \omega_o,\ |H(s)| = \frac{A}{(\omega/\omega_o)} = \frac{A\omega_o}{\omega}\right)$

Maximum Error @ ω_o = 3dB

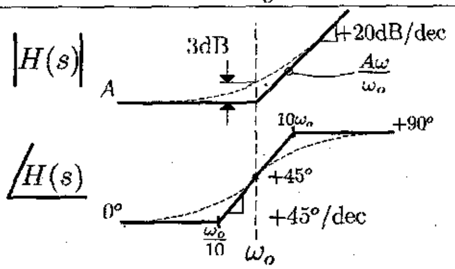
Maximum Error @ $\frac{\omega_o}{10}$ & $10\omega_o = 5.7^o$

Exact Phase: $-tan^{-1}\left(\frac{\omega}{\omega_o}\right), \forall\omega$

Approx. Phase: $-45^o log_{10}\left(\frac{10\omega}{\omega_o}\right)$, $\frac{\omega_o}{10} \le \omega \le 10\omega_o$

$H(s) = A\left(1+\frac{s}{\omega_o}\right)$

Zero at ω_o



Maximum Error @ ω_o = 3dB

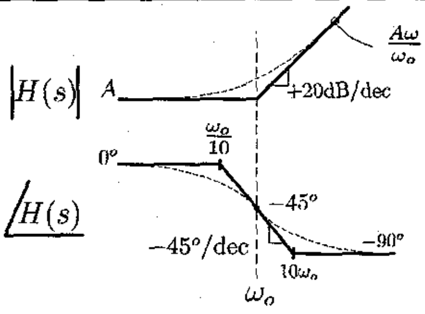
Maximum Error @ $\frac{\omega_o}{10}$ & $10\omega_o = 5.7^o$

Exact Phase: $tan^{-1}\left(\frac{\omega}{\omega_o}\right), \forall\omega$

Approx. Phase: $45^o log_{10}\left(\frac{10\omega}{\omega_o}\right)$, $\frac{\omega_o}{10} \le \omega \le 10\omega_o$

$H(s) = A\left(1-\frac{s}{\omega_o}\right)$

Right Half Plane Zero at ω_o



Maximum Error @ ω_o = 3dB

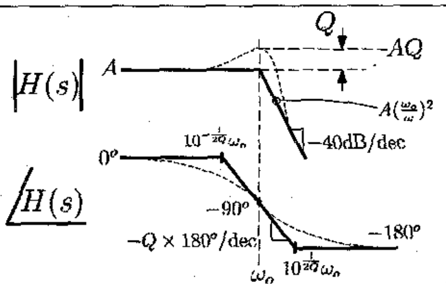
Maximum Error @ $\frac{\omega_o}{10}$ & $10\omega_o = 5.7^o$

Exact Phase: $-tan^{-1}\left(\frac{\omega}{\omega_o}\right), \forall\omega$

Approx. Phase: $-45^o log_{10}\left(\frac{10\omega}{\omega_o}\right)$, $\frac{\omega_o}{10} \le \omega \le 10\omega_o$

$H(s) = \frac{A}{1+\frac{s}{Q\omega_o}+\left(\frac{s}{\omega_o}\right)^2}$

Second Order Complex Pole



ω_o=Corner Frequency

$Q > \frac{1}{2} \implies$ Complex Roots

Q=Quality Factor: Exact Gain @ ω_o
Approximate Maximum Value

Exact Phase: $-tan^{-1}\left[\frac{\frac{1}{Q}\frac{\omega}{\omega_o}}{1-(\frac{\omega}{\omega_o})^2}\right], \forall\omega$