ECE311

Homework 3 - Solutions

Problem 1:

Determine the region of values for the parameter k so that the systems with the following characteristic equations are stable.

For each case, compute the critical frequency of oscillation ω_c:

a. $s^4 + 7s^3 + 15s^2 + (25+k)s + 2k = 0$

b. $s^3 + 3ks^2 + (k+2)s + 4 = 0$

Solution

a. The characteristic equation is

$$s^4 + 7s^3 + 15s^2 + (25+k)s + 2k = 0 \qquad \text{(P5.2.1)}$$

Routh's tabulation is

$$\begin{array}{c|ccc} s^4 & 1 & 15 & 2k \\ s^3 & 7 & 25+k & \\ s^2 & \dfrac{80-k}{7} & 2k & \\ s^1 & \dfrac{(80-k)(25+k)-98k}{80-k} & & \\ s^0 & 2k & & \end{array}$$

For the system to be stable, it must hold that

1. $\dfrac{80-k}{7} > 0 \Rightarrow k < 80$ (P5.2.2)

2. $\dfrac{(80-k)(25+k)-98k}{80-k} > 0 \Rightarrow -71.1 < k < 28.1$ (P5.2.3)

3. $2k > 0 \Rightarrow k > 0$ (P5.2.4)

By combining inequalities (P5.2.2), (P5.2.3), and (P5.2.4), we get

$$0 < k < 28.1 \qquad \text{(P5.2.5)}$$

For $k = 28.1 = k_c$, the characteristic equation has a couple of imaginary roots, while for $k = 0$ there is no response ($y(t) = 0$). The angular frequency of oscillation if $k = 28.1 = k_c$ is

$$\frac{80-k_c}{7}s^2 + 2k_c = 0 \Rightarrow s^2 = -7.58 \Rightarrow s = \pm j2.75 \Rightarrow \omega_c = 2.75 \text{ rad/s} \qquad \text{(P5.2.6)}$$

b. The characteristic equation is

$$s^3 + 3ks^2 + (k+2)s + 4 = 0 \tag{P5.2.12}$$

Routh's tabulation is

$$\begin{array}{c|cc} s^3 & 1 & k+2 \\ s^2 & 3k & 4 \\ s^1 & \dfrac{3k^2+6k-4}{3k} & \\ s^0 & 4 & \end{array}$$

The conditions for the stability of the system are

1. $3k > 0 \Rightarrow k > 0$ (P5.2.13)

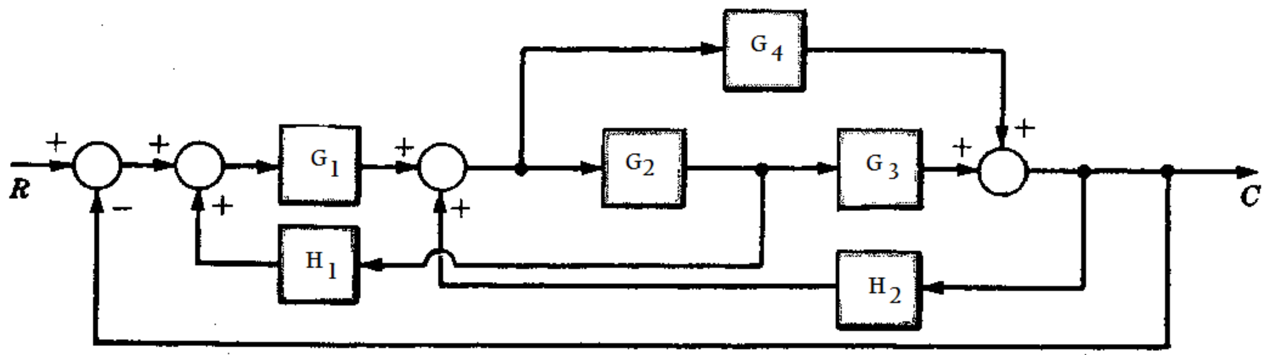
2. $\dfrac{3k^2+6k-4}{3k} > 0 \Rightarrow 3k^2+6k-4>0 \Rightarrow \begin{cases} k < -2.528 \\ \text{or} \\ k > 0.528 \end{cases}$ (P5.2.14)

$$\text{(P5.2.13)},\text{(P5.2.14)} \Rightarrow k > 0.528 \tag{P5.2.15}$$

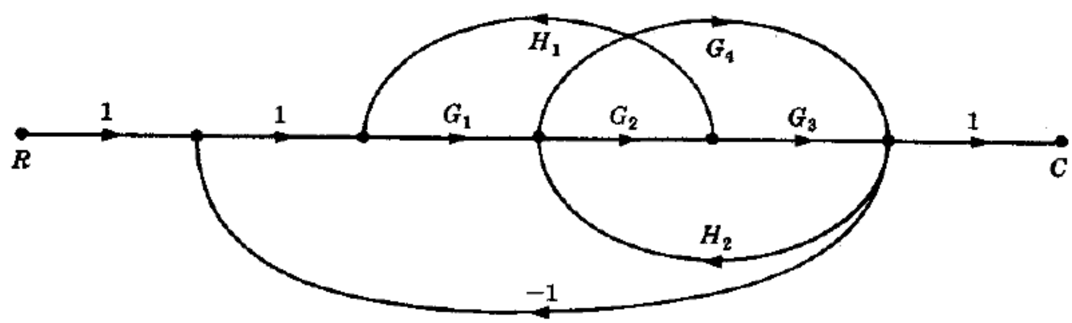
From the auxiliary equation of row s^2, we get

$$3k_c s^2 + 4 = 0 \Rightarrow s^2 = -2.525 \Rightarrow s = \pm j1.59 \Rightarrow \omega_c = 1.59 \text{ rad/s} \tag{P5.2.16}$$

Problem 2:



Signal Flow Graph:



The two forward path gains are $P_1 = G_1G_2G_3$ and $P_2 = G_1G_4$. The five feedback loop gains are $P_{11} = G_1G_2H_1$, $P_{21} = G_2G_3H_2$, $P_{31} = -G_1G_2G_3$, $P_{41} = G_4H_2$, and $P_{51} = -G_1G_4$. Hence

$$\Delta = 1 - (P_{11} + P_{21} + P_{31} + P_{41} + P_{51}) = 1 + G_1G_2G_3 - G_1G_2H_1 - G_2G_3H_2 - G_4H_2 + G_1G_4$$

and $\Delta_1 = \Delta_2 = 1$. Finally,

$$\frac{C}{R} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_1G_2G_3 + G_1G_4}{1 + G_1G_2G_3 - G_1G_2H_1 - G_2G_3H_2 - G_4H_2 + G_1G_4}$$