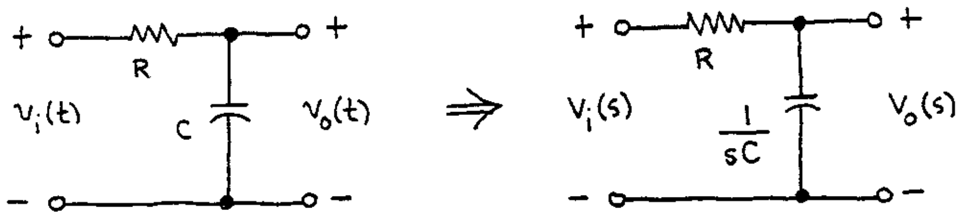
Bode Plots

The response curves of the amplitude and phase angle of $F(j\omega)$ are often approximated with a series of straight lines. Such straight-line approximations of amplitude and phase angle versus frequency, where frequency is plotted on a logarithmic scale, are called <u>idealized Bode diagrams</u>.

Consider the following circuit.



The Transfer function is

$$F(s) = \frac{V_o(s)}{V_i(s)} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} = \frac{1}{sRC + 1}$$

Letting $s \rightarrow j\omega$,

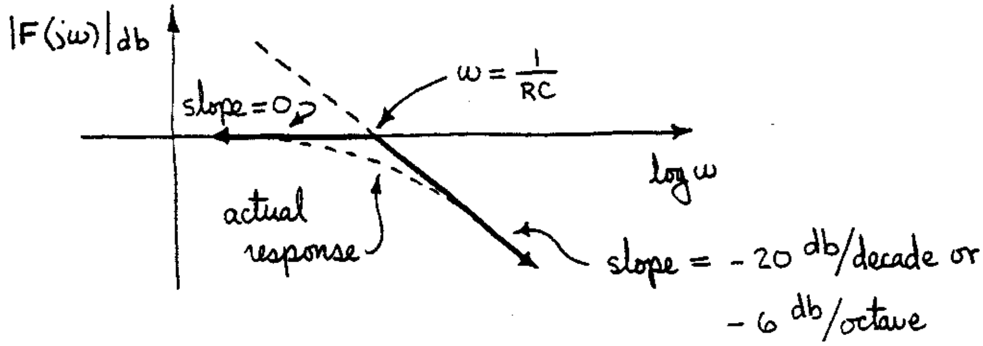
$$F(j\omega) = \frac{1}{j\omega RC + 1} = \frac{1}{1 + j\frac{\omega}{\frac{1}{RC}}} = \frac{1\angle 0^\circ}{\sqrt{1+\left(\frac{\omega}{\frac{1}{RC}}\right)^2}\ \angle \tan^{-1}\frac{\omega}{\frac{1}{RC}}}$$

The magnitude in decibels is

$$|F(j\omega)|_{db} = \overbrace{20\log_{10} 1}^{0} - 20\log_{10}\sqrt{1+\left(\frac{\omega}{\frac{1}{RC}}\right)^2}$$

$$= \begin{cases} 0 & \text{for } \omega \ll \frac{1}{RC} \\ -20\log_{10}\frac{\omega}{\frac{1}{RC}} & \text{for } \omega \gg \frac{1}{RC} \end{cases}$$

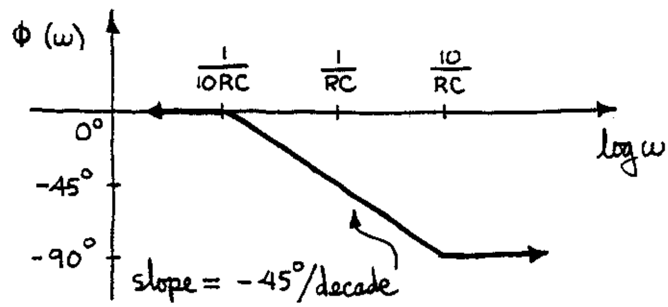
These two limiting functions are asymptotes of the original voltage transfer function in decibels. When plotted, they form an idealized Bode diagram. Note that they intersect when $\omega = \frac{1}{RC}$.



The phase angle is

$$\Phi(\omega) = \begin{cases} 0^\circ & \text{for } \omega << \frac{1}{RC} \\ -90^\circ & \text{for } \omega >> \frac{1}{RC} \end{cases}$$

The plot is



In general,

$$F(j\omega) = \frac{K\left(1+j\frac{\omega}{z_1}\right)\left(1+j\frac{\omega}{z_2}\right)\left(1+j\frac{\omega}{z_3}\right)\cdots}{\left(1+j\frac{\omega}{P_1}\right)\left(1+j\frac{\omega}{P_2}\right)\left(1+j\frac{\omega}{P_3}\right)\cdots}$$

where $z_1, z_2, z_3, \ldots, P_1, P_2, P_3, \ldots$ are called <u>break or corner frequencies</u>. The numerator corner frequencies cause the Bode amplitude plot to <u>increase</u> by 20 db/decade. The denominator corner frequencies cause the Bode amplitude plot to <u>decrease</u> by 20 db/decade (or −20 db/decade).

Example:

Consider the following transfer function.

$$F(s) = \frac{10^8 s^2 (s+100)}{(s+10)^2 (s+1{,}000)(s+10{,}000)}$$

Letting $s \to j\omega$,

$$F(j\omega) = \frac{10^8 (j\omega)^2 (j\omega+100)}{(j\omega+10)^2 (j\omega+1{,}000)(j\omega+10{,}000)}$$

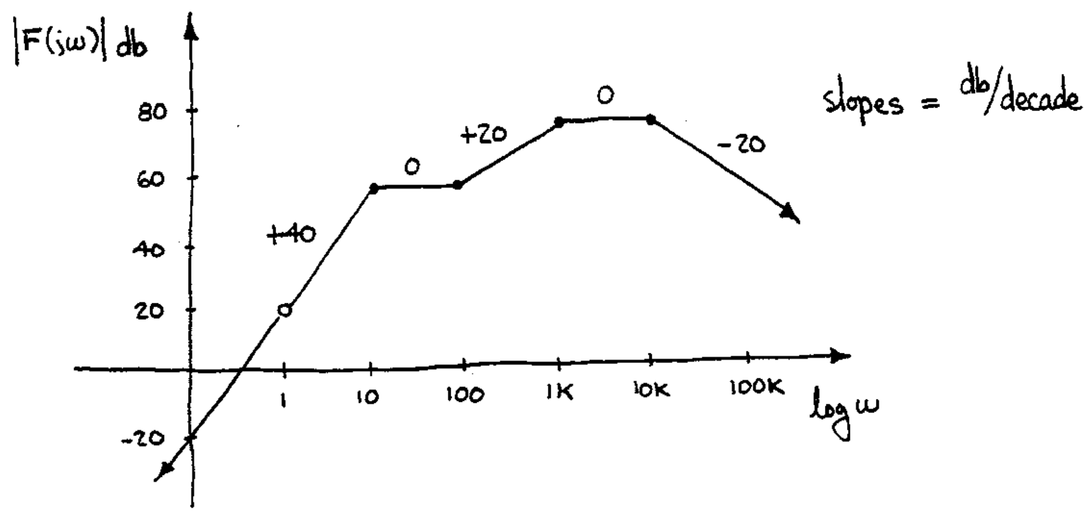
$$= \frac{-10^8 \omega^2 \left(1+j\frac{\omega}{100}\right)\left(\frac{100}{10\cdot 10\cdot 1{,}000\cdot 10{,}000}\right)}{\left(1+j\frac{\omega}{10}\right)\left(1+j\frac{\omega}{10}\right)\left(1+j\frac{\omega}{1{,}000}\right)\left(1+j\frac{\omega}{10{,}000}\right)}$$

$$= \frac{-10\omega^2 \left(1+j\frac{\omega}{100}\right)}{\left(1+j\frac{\omega}{10}\right)^2 \left(1+j\frac{\omega}{1{,}000}\right)\left(1+j\frac{\omega}{10{,}000}\right)}$$

At $\omega = 1$,

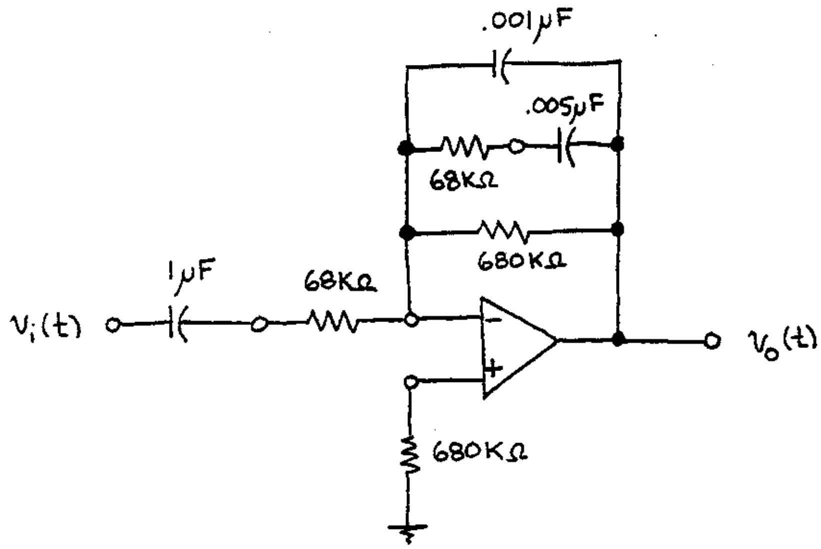
$$|F(j\omega)|_{db} = 20 \log_{10} 10 = 20 \text{ db}$$

The idealized Bode plot is



Example:

Plot the idealized Bode diagram $|F(j\omega)|_{db}$ for the following circuit from f = 0.1 Hz to 10 KHz.



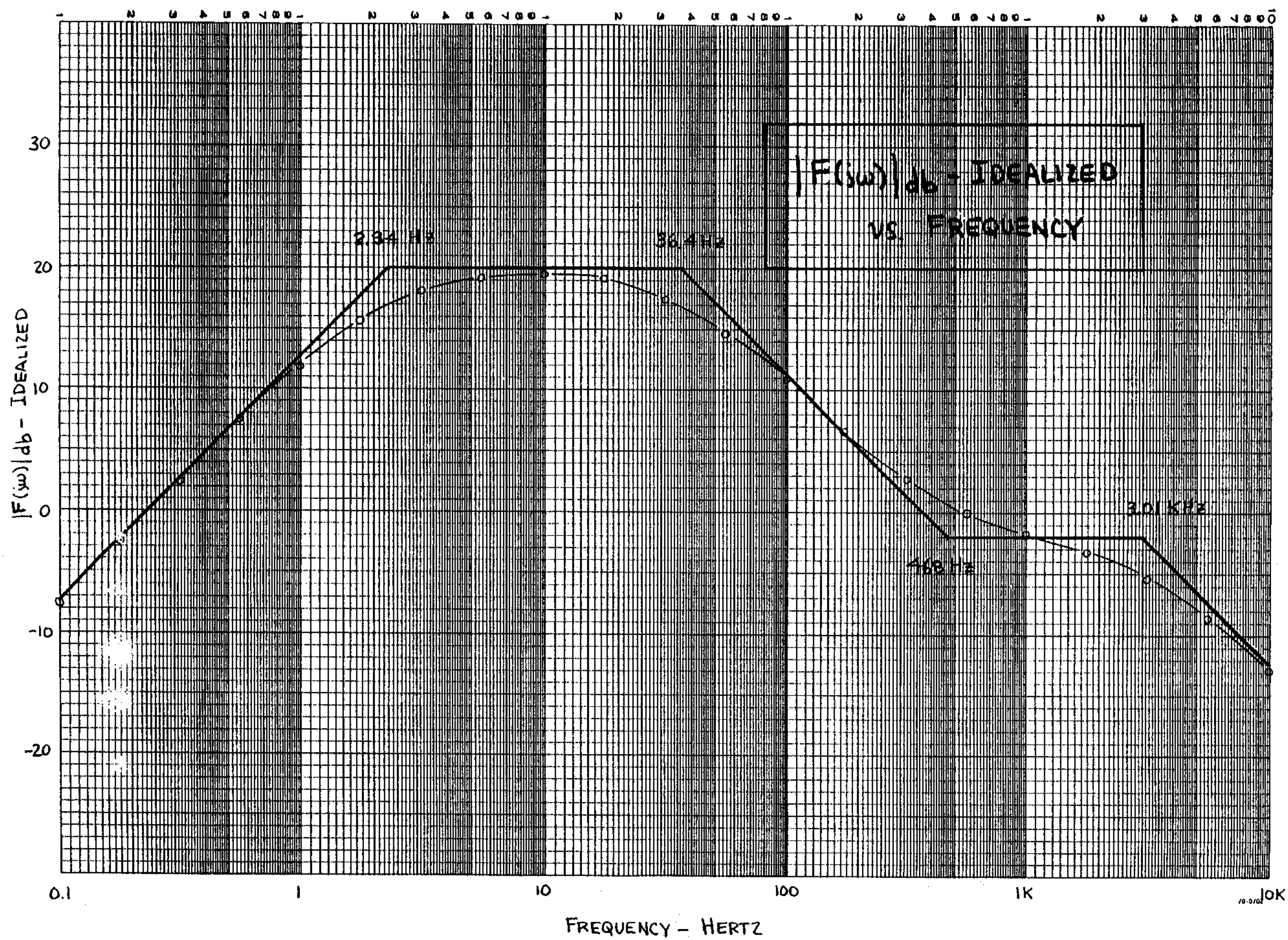
Obtaining the "factored form" of the transfer function,

$$F(s) = \frac{-1.47\times10^4 s\,(s+2.94\times10^3)}{(s+1.47\times10^1)(s+2.29\times10^2)(s+1.89\times10^4)}$$

Letting $s \to j\omega$,

$$F(j\omega) = \frac{-j\,0.68\,\omega\left(1+j\frac{\omega}{2.94\times10^3}\right)}{\left(1+j\frac{\omega}{1.47\times10^1}\right)\left(1+j\frac{\omega}{2.29\times10^2}\right)\left(1+j\frac{\omega}{1.89\times10^4}\right)}$$

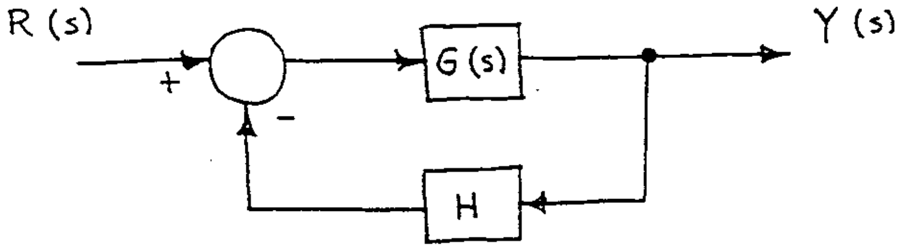
$$= \frac{-j\,4.27 f\left(1+j\frac{f}{468}\right)}{\left(1+j\frac{f}{2.34}\right)\left(1+j\frac{f}{36.4}\right)\left(1+j\frac{f}{3.01\times10^3}\right)}$$



|F(jω)|db - IDEALIZED
VS. FREQUENCY
2.34 Hz
36.4 Hz
468 Hz
3.01 KHz
30
20
10
0
-10
-20
|F(jω)|db - IDEALIZED
0.1
1
10
100
1K
10K
FREQUENCY - HERTZ

Gain and Phase Margin

Consider a simple feedback system with constant H



where

$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + HG(s)}$$

Letting $s = j2\pi f$,

$$T(f) = \frac{G(f)}{1 + HG(f)}$$

Now suppose that as $f \to f_1$,

$$\lim_{f \to f_1} HG(f) = -1$$

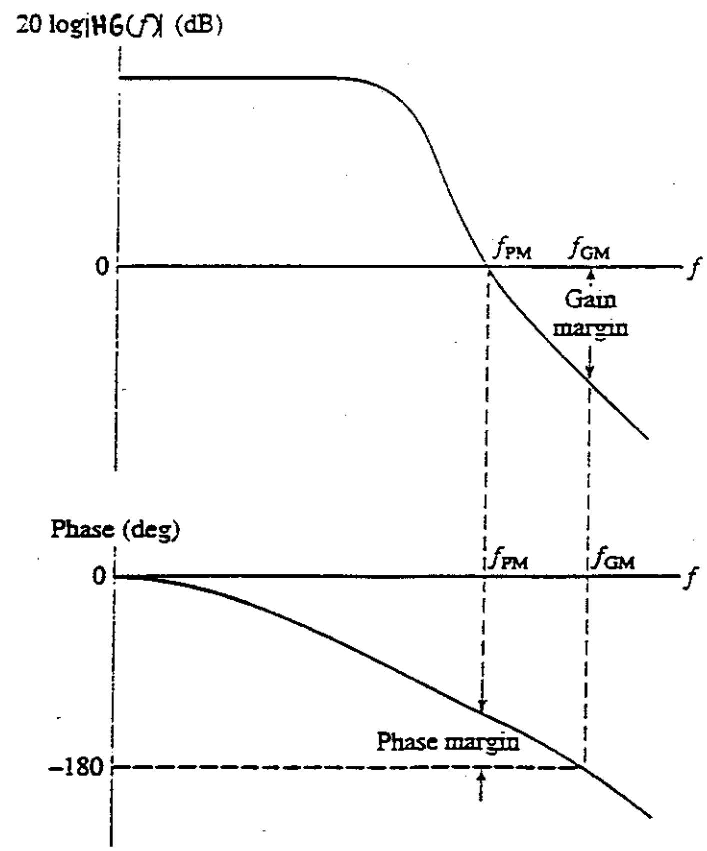
As a result,

$$\lim_{f \to f_1} T(f) = \lim_{f \to f_1} \frac{G(f)}{1 + HG(f)} = \infty$$

which corresponds to a pole on the $j\omega$-axis at $s = j2\pi f_1$. The resulting transient response would contain a constant-amplitude sinusoid.

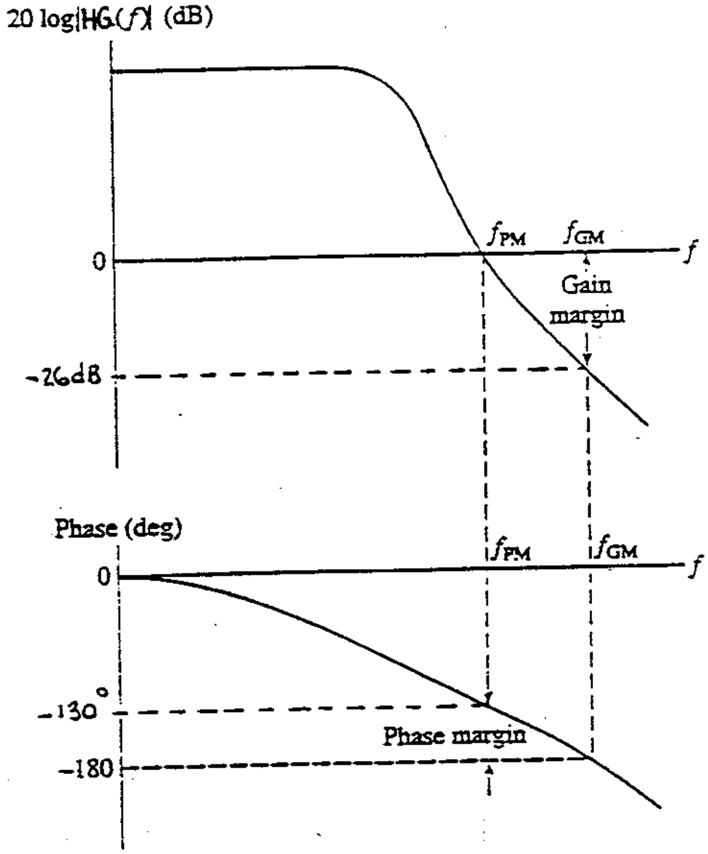
The Bode plot of the loop gain HG(f) can be used to determine the stability of a system. First we examine the Bode plot of the phase shift of HG(f) to determine the frequency f_{GM} for which the phase shift is 180°. If the magnitude of the loop gain is less than unity at this frequency, the system is stable. The amount that the gain magnitude is less than unity (or 0 db) is called the <u>gain margin</u>.

Another measure of stability that can be obtained from the Bode plots is the <u>phase margin</u>. Phase margin is determined at the frequency f_{PM} for which the loop gain $HG(f_{PM})$ is unity in magnitude (or 0 db). The phase margin is the difference between the actual phase and 180°.



Example:

Determine the gain and phase margin of a system with the following Bode plots.



By definition,

Gain Margin = 0 − (−26) = 26 db ←

Phase Margin = −130° − (−180°) = 50° ←

A generally accepted rule-of-thumb is to design for a minimum gain margin of 10 db and a minimum phase margin of 45°.

Review of Bode plots

Decibels

$$\|G\|_{dB} = 20 \log_{10}\left(\|G\|\right)$$

Decibels of quantities having units (impedance example): normalize before taking log

$$\|Z\|_{dB} = 20 \log_{10}\left(\frac{\|Z\|}{R_{base}}\right)$$

Expressing magnitudes in decibels

Actual magnitude	*Magnitude in dB*
1/2	– 6dB
1	0 dB
2	6 dB
5 = 10/2	20 dB – 6 dB = 14 dB
10	20dB
1000 = 10^3	3 · 20dB = 60 dB

5Ω is equivalent to 14dB with respect to a base impedance of R_{base} = 1Ω, also known as 14dBΩ.

60dBμA is a current 60dB greater than a base current of 1μA, or 1mA.

Bode plot of f^n

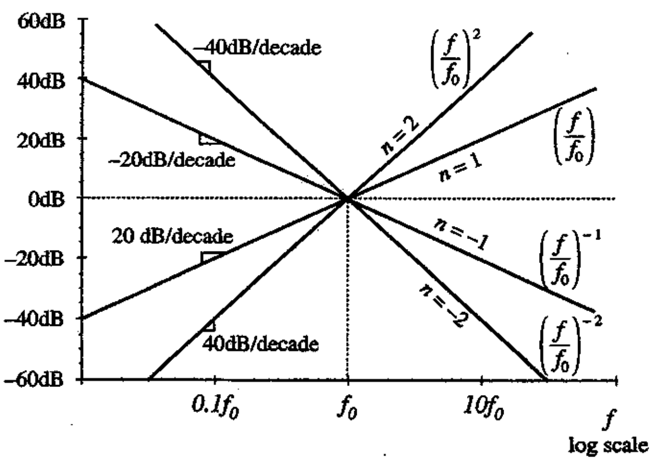
Bode plots are effectively log-log plots, which cause functions which vary as f^n to become linear plots. Given:

$$\|G\| = \left(\frac{f}{f_0}\right)^n$$

Magnitude in dB is

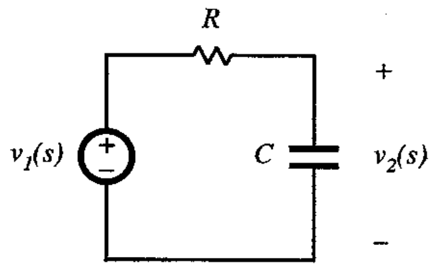
$$\|G\|_{dB} = 20 \log_{10}\left(\frac{f}{f_0}\right)^n = 20n \log_{10}\left(\frac{f}{f_0}\right)$$

- Slope is $20n$ dB/decade
- Magnitude is 1, or 0dB, at frequency $f = f_0$



Single pole response

Simple R-C example



Transfer function is

$$G(s) = \frac{v_2(s)}{v_1(s)} = \frac{\frac{1}{sC}}{\frac{1}{sC} + R}$$

Express as rational fraction:

$$G(s) = \frac{1}{1 + sRC}$$

This coincides with the normalized form

$$G(s) = \frac{1}{\left(1 + \frac{s}{\omega_0}\right)}$$

with $\omega_0 = \frac{1}{RC}$

$G(j\omega)$ and $\| G(j\omega) \|$

Let $s = j\omega$:

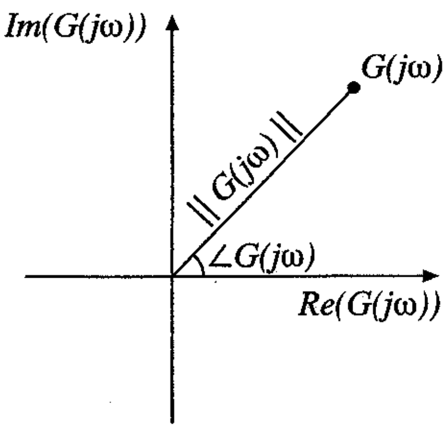
$$G(j\omega) = \frac{1}{\left(1 + j\frac{\omega}{\omega_0}\right)} = \frac{1 - j\frac{\omega}{\omega_0}}{1 + \left(\frac{\omega}{\omega_0}\right)^2}$$

Magnitude is

$$\left| G(j\omega) \right| = \sqrt{\left[\mathrm{Re}\,(G(j\omega))\right]^2 + \left[\mathrm{Im}\,(G(j\omega))\right]^2}$$
$$= \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}}$$

Magnitude in dB:

$$\left| G(j\omega) \right|_{dB} = -20 \log_{10}\left(\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}\right) \text{ dB}$$

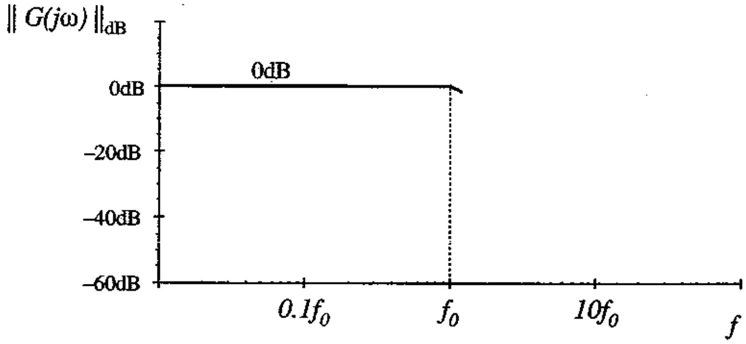


Asymptotic behavior: low frequency

For small frequency,
$\omega << \omega_0$ and $f << f_0$:

$$\left(\frac{\omega}{\omega_0}\right) << 1$$

$$\left\| G(j\omega) \right\| = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}}$$



Then $\| G(j\omega) \|$
becomes

$$\left| G(j\omega) \right| \approx \frac{1}{\sqrt{1}} = 1$$

Or, in dB,

$$\left| G(j\omega) \right|_{dB} \approx 0\text{dB}$$

This is the low-frequency
asymptote of $\| G(j\omega) \|$

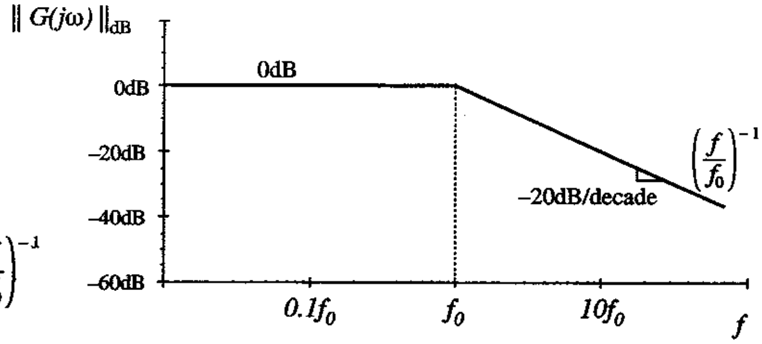
Asymptotic behavior: high frequency

For high frequency,
$\omega >> \omega_0$ and $f >> f_0$:

$$\left(\frac{\omega}{\omega_0}\right) >> 1$$

$$1 + \left(\frac{\omega}{\omega_0}\right)^2 \approx \left(\frac{\omega}{\omega_0}\right)^2$$

$$\left\| G(j\omega) \right\| = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}}$$



Then $\| G(j\omega) \|$
becomes

$$\left| G(j\omega) \right| \approx \frac{1}{\sqrt{\left(\frac{\omega}{\omega_0}\right)^2}} = \left(\frac{f}{f_0}\right)^{-1}$$

The high-frequency asymptote of $\| G(j\omega) \|$ varies as f^{-1}.
Hence, $n = -1$, and a straight-line asymptote having a slope of -20dB/decade is obtained. The asymptote has a value of 1 at $f = f_0$.

Deviation of exact curve near $f = f_0$

Evaluate exact magnitude:

at $f = f_0$:

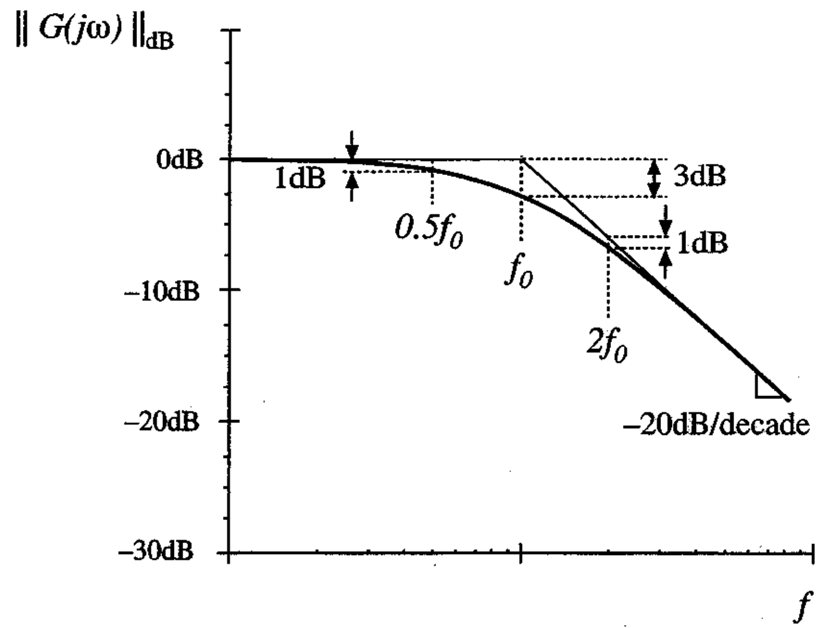
$$\left| G(j\omega_0) \right| = \frac{1}{\sqrt{1 + \left(\frac{\omega_0}{\omega_0}\right)^2}} = \frac{1}{\sqrt{2}}$$

$$\left| G(j\omega_0) \right|_{dB} = -20 \log_{10}\left(\sqrt{1 + \left(\frac{\omega_0}{\omega_0}\right)^2}\right) \approx -3 \text{ dB}$$

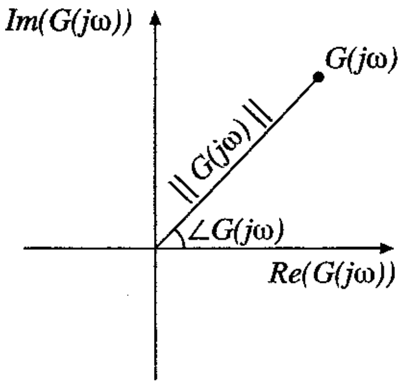
at $f = 0.5f_0$ *and* $2f_0$:

Similar arguments show that the exact curve lies 1dB below the asymptotes.

Summary: magnitude



Phase of $G(j\omega)$

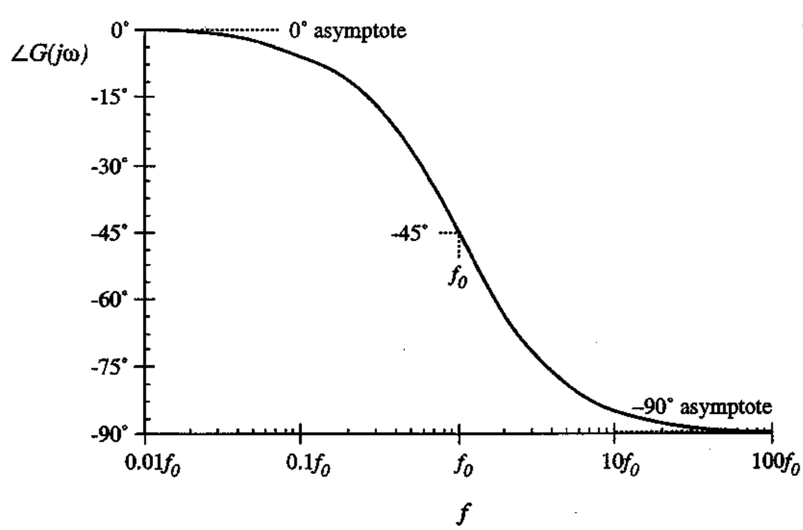


$$\angle G(j\omega) = \tan^{-1}\left(\frac{\mathrm{Im}\left(G(j\omega)\right)}{\mathrm{Re}\left(G(j\omega)\right)}\right)$$

$$G(j\omega) = \frac{1}{\left(1 + j\frac{\omega}{\omega_0}\right)} = \frac{1 - j\frac{\omega}{\omega_0}}{1 + \left(\frac{\omega}{\omega_0}\right)^2}$$

$$\angle G(j\omega) = -\tan^{-1}\left(\frac{\omega}{\omega_0}\right)$$

Phase of $G(j\omega)$



$$\angle G(j\omega) = -\tan^{-1}\left(\frac{\omega}{\omega_0}\right)$$

ω	$\angle G(j\omega)$
0	0°
ω_0	–45°
∞	–90°

Phase asymptotes

Low frequency: 0˚

High frequency: –90˚

Low- and high-frequency asymptotes do not intersect

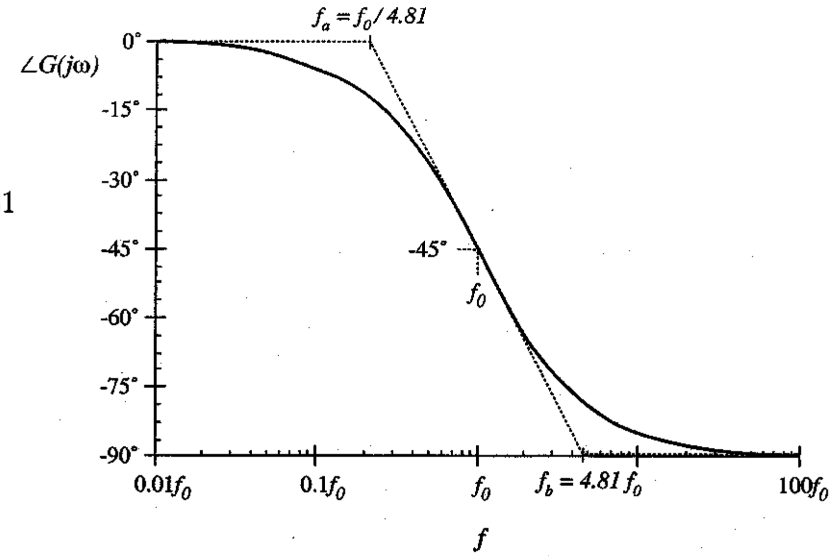
Hence, need a midfrequency asymptote

Try a midfrequency asymptote having slope identical to actual slope at the corner frequency f_0. One can show that the asymptotes then intersect at the break frequencies

$$f_a = f_0 \, e^{-\pi/2} \approx f_0 / 4.81$$
$$f_b = f_0 \, e^{\pi/2} \approx 4.81 \, f_0$$

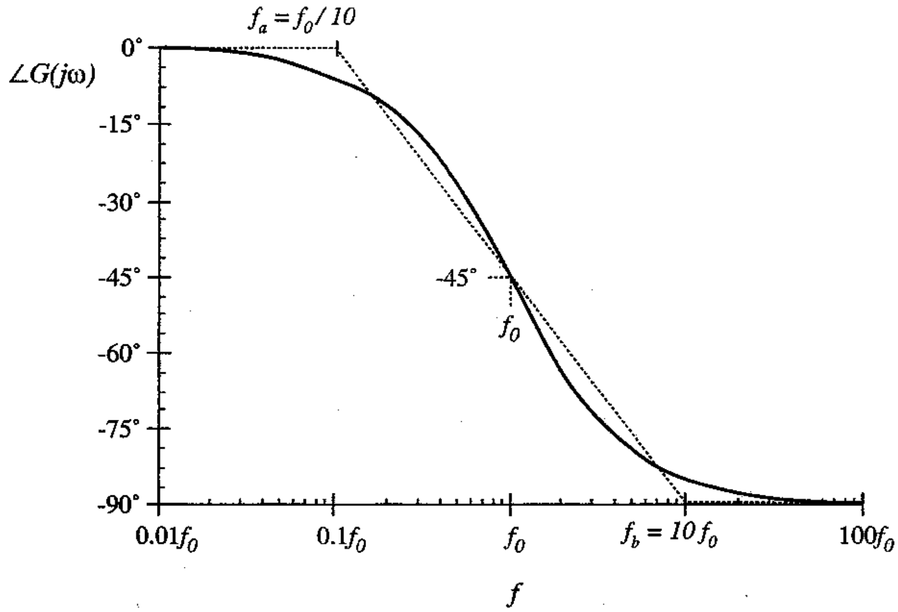
Phase asymptotes

$$f_a = f_0 \, e^{-\pi/2} \approx f_0 / 4.81$$
$$f_b = f_0 \, e^{\pi/2} \approx 4.81 \, f_0$$



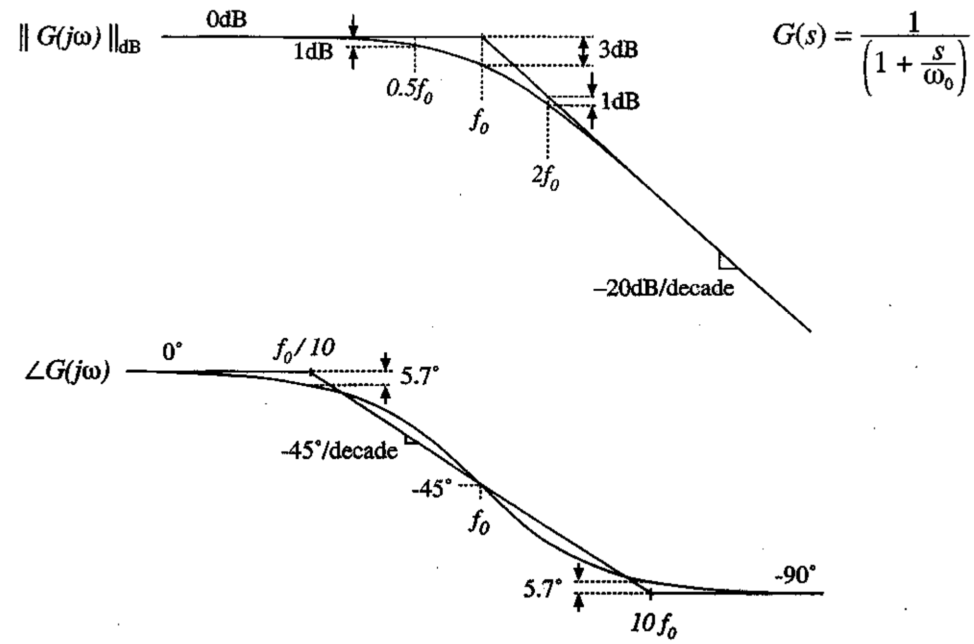
Phase asymptotes: a simpler choice

$f_a = f_0 / 10$
$f_b = 10\, f_0$



Summary: Bode plot of real pole

$$G(s) = \frac{1}{\left(1 + \frac{s}{\omega_0}\right)}$$



Single zero response

Normalized form:

$$G(s) = \left(1 + \frac{s}{\omega_0}\right)$$

Magnitude:

$$\left\| G(j\omega) \right\| = \sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}$$

Use arguments similar to those used for the simple pole, to derive asymptotes:

0dB at low frequency, $\omega << \omega_0$

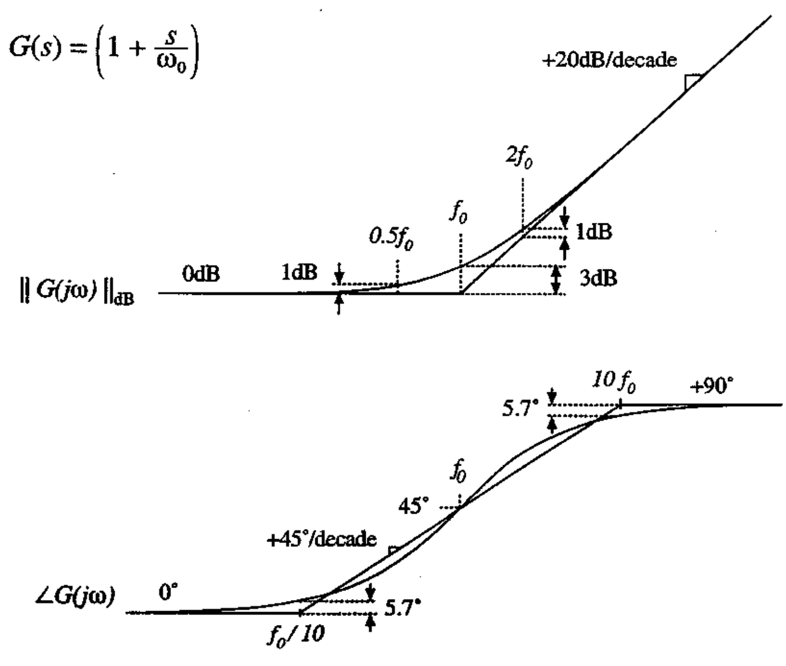
+20dB/decade slope at high frequency, $\omega >> \omega_0$

Phase:

$$\angle G(j\omega) = \tan^{-1}\left(\frac{\omega}{\omega_0}\right)$$

—with the exception of a missing minus sign, same as simple pole

Summary: Bode plot, real zero



Combinations

Suppose that we have constructed the Bode diagrams of two complex-values functions of frequency, $G_1(\omega)$ and $G_2(\omega)$. It is desired to construct the Bode diagram of the product, $G_3(\omega) = G_1(\omega)\, G_2(\omega)$.

Express the complex-valued functions in polar form:

$$G_1(\omega) = R_1(\omega)\, e^{j\theta_1(\omega)}$$
$$G_2(\omega) = R_2(\omega)\, e^{j\theta_2(\omega)}$$
$$G_3(\omega) = R_3(\omega)\, e^{j\theta_3(\omega)}$$

The product $G_3(\omega)$ can then be written

$$G_3(\omega) = G_1(\omega)\, G_2(\omega) = R_1(\omega)\, e^{j\theta_1(\omega)}\, R_2(\omega)\, e^{j\theta_2(\omega)}$$

$$G_3(\omega) = \left(R_1(\omega)\, R_2(\omega)\right) e^{j(\theta_1(\omega) + \theta_2(\omega))}$$

Combinations

$$G_3(\omega) = \left(R_1(\omega)\, R_2(\omega)\right) e^{j(\theta_1(\omega) + \theta_2(\omega))}$$

The composite phase is

$$\theta_3(\omega) = \theta_1(\omega) + \theta_2(\omega)$$

The composite magnitude is

$$R_3(\omega) = R_1(\omega)\, R_2(\omega)$$

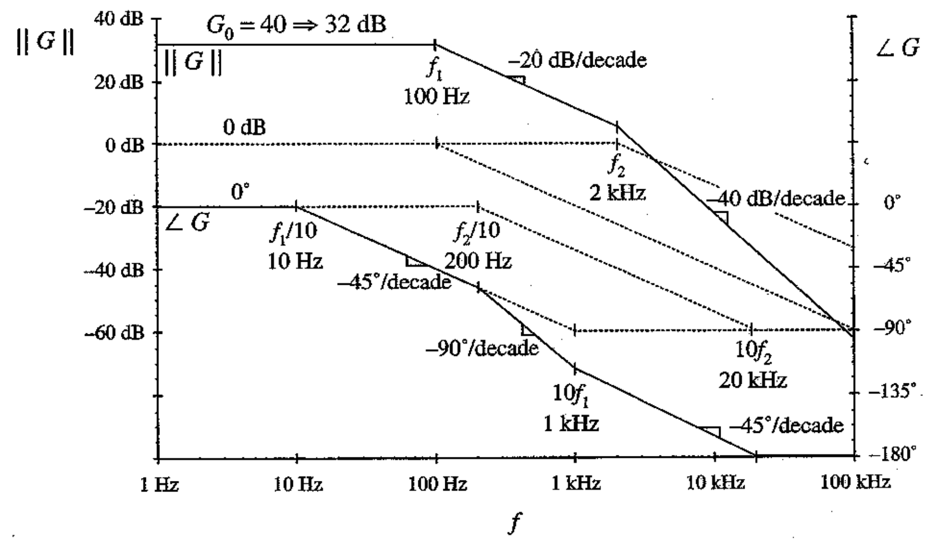
$$\left| R_3(\omega) \right|_{\mathrm{dB}} = \left| R_1(\omega) \right|_{\mathrm{dB}} + \left| R_2(\omega) \right|_{\mathrm{dB}}$$

Composite phase is sum of individual phases.

Composite magnitude, when expressed in dB, is sum of individual magnitudes.

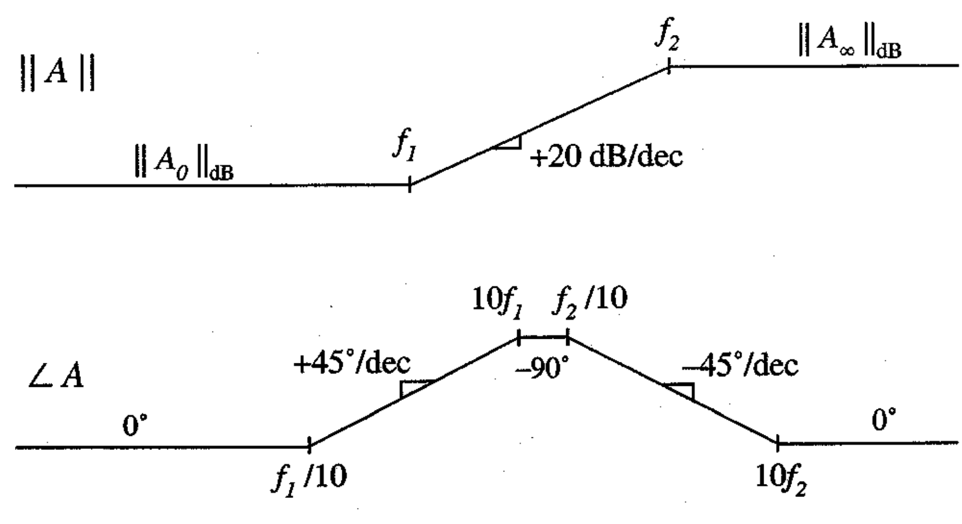
Example 1: $G(s) = \dfrac{G_0}{\left(1 + \frac{s}{\omega_1}\right)\left(1 + \frac{s}{\omega_2}\right)}$

with $G_0 = 40 \Rightarrow 32\text{ dB}, f_1 = \omega_1/2\pi = 100\text{ Hz}, f_2 = \omega_2/2\pi = 2\text{ kHz}$



Example 2

Determine the transfer function *A(s)* corresponding to the following asymptotes:



Example 2, continued

One solution:

$$A(s) = A_0 \frac{\left(1 + \frac{s}{\omega_1}\right)}{\left(1 + \frac{s}{\omega_2}\right)}$$

Analytical expressions for asymptotes:

For $f < f_1$

$$\left\| A_0 \frac{\left(1 + \cancel{\frac{s}{\omega_1}}\right)}{\left(1 + \cancel{\frac{s}{\omega_2}}\right)} \right\|_{s = j\omega} = A_0 \frac{1}{1} = A_0$$

For $f_1 < f < f_2$

$$\left\| A_0 \frac{\left(\cancel{1} + \frac{s}{\omega_1}\right)}{\left(1 + \cancel{\frac{s}{\omega_2}}\right)} \right\|_{s = j\omega} = A_0 \frac{\left\| \frac{s}{\omega_1} \right\|_{s = j\omega}}{1} = A_0 \frac{\omega}{\omega_1} = A_0 \frac{f}{f_1}$$

Example 2, continued

For $f > f_2$

$$\left\| A_0 \frac{\left(\cancel{1} + \frac{s}{\omega_1}\right)}{\left(\cancel{1} + \frac{s}{\omega_2}\right)} \right\|_{s = j\omega} = A_0 \frac{\left\| \frac{s}{\omega_1} \right\|_{s = j\omega}}{\left\| \frac{s}{\omega_2} \right\|_{s = j\omega}} = A_0 \frac{\omega_2}{\omega_1} = A_0 \frac{f_2}{f_1}$$

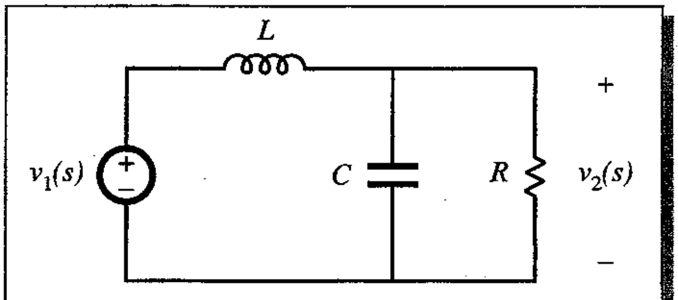
So the high-frequency asymptote is

$$A_\infty = A_0 \frac{f_2}{f_1}$$

Quadratic pole response: resonance

Example

$$G(s) = \frac{v_2(s)}{v_1(s)} = \frac{1}{1 + s\frac{L}{R} + s^2LC}$$



Two-pole low-pass filter example

Second-order denominator, of the form

$$G(s) = \frac{1}{1 + a_1 s + a_2 s^2}$$

with $a_1 = L/R$ and $a_2 = LC$

How should we construct the Bode diagram?

Approach 1: factor denominator

$$G(s) = \frac{1}{1 + a_1 s + a_2 s^2}$$

We might factor the denominator using the quadratic formula, then construct Bode diagram as the combination of two real poles:

$$G(s) = \frac{1}{\left(1 - \frac{s}{s_1}\right)\left(1 - \frac{s}{s_2}\right)} \quad \text{with} \quad s_1 = -\frac{a_1}{2a_2}\left[1 - \sqrt{1 - \frac{4a_2}{a_1^2}}\right]$$

$$s_2 = -\frac{a_1}{2a_2}\left[1 + \sqrt{1 - \frac{4a_2}{a_1^2}}\right]$$

- If $4a_2 \le a_1^2$, then the roots s_1 and s_2 are real. We can construct Bode diagram as the combination of two real poles.
- If $4a_2 > a_1^2$, then the roots are complex. In a previous section, the assumption was made that ω_0 is real; hence, the results of that section cannot be applied and we need to do some additional work.

Approach 2: Define a standard normalized form for the quadratic case

$$G(s) = \frac{1}{1 + 2\zeta \frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2} \qquad \text{or} \qquad G(s) = \frac{1}{1 + \frac{s}{Q\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

- When the coefficients of s are real and positive, then the parameters ζ, ω_0, and Q are also real and positive
- The parameters ζ, ω_0, and Q are found by equating the coefficients of s
- The parameter ω_0 is the angular corner frequency, and we can define $f_0 = \omega_0/2\pi$
- The parameter ζ is called the *damping factor.* ζ controls the shape of the exact curve in the vicinity of $f = f_0$. The roots are complex when $\zeta < 1$.
- In the alternative form, the parameter Q is called the *quality factor.* Q also controls the shape of the exact curve in the vicinity of $f = f_0$. The roots are complex when $Q > 0.5$.

The Q-factor

In a second-order system, ζ and Q are related according to

$$Q = \frac{1}{2\zeta}$$

Q is a measure of the dissipation in the system. A more general definition of Q, for sinusoidal excitation of a passive element or system is

$$Q = 2\pi \frac{\text{(peak stored energy)}}{\text{(energy dissipated per cycle)}}$$

For a second-order passive system, the two equations above are equivalent. We will see that Q has a simple interpretation in the Bode diagrams of second-order transfer functions.

Analytical expressions for f_0 and Q

Two-pole low-pass filter example: we found that

$$G(s) = \frac{v_2(s)}{v_1(s)} = \frac{1}{1 + s\frac{L}{R} + s^2 LC}$$

Equate coefficients of like powers of s with the standard form

$$G(s) = \frac{1}{1 + \frac{s}{Q\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$

Result:

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$

$$Q = R\sqrt{\frac{C}{L}}$$

Magnitude asymptotes, quadratic form

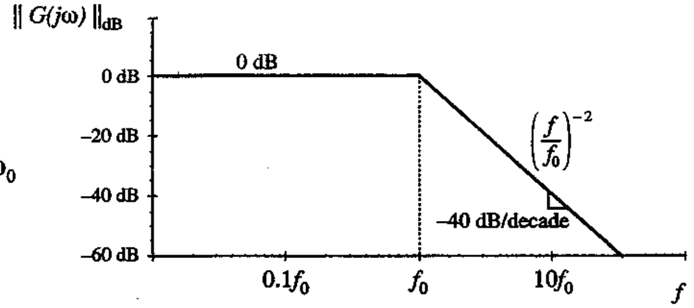
In the form $G(s) = \frac{1}{1 + \frac{s}{Q\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$

let $s = j\omega$ and find magnitude: $\left\| G(j\omega) \right\| = \frac{1}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2}}$

Asymptotes are

$\left\| G \right\| \to 1$ for $\omega << \omega_0$

$\left\| G \right\| \to \left(\frac{f}{f_0}\right)^{-2}$ for $\omega >> \omega_0$



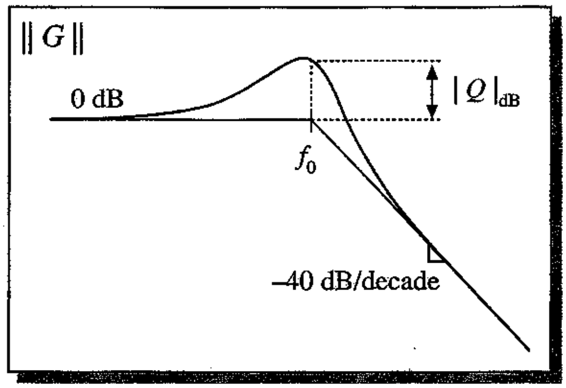
Deviation of exact curve from magnitude asymptotes

$$\left\| G(j\omega) \right\| = \frac{1}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2}}$$

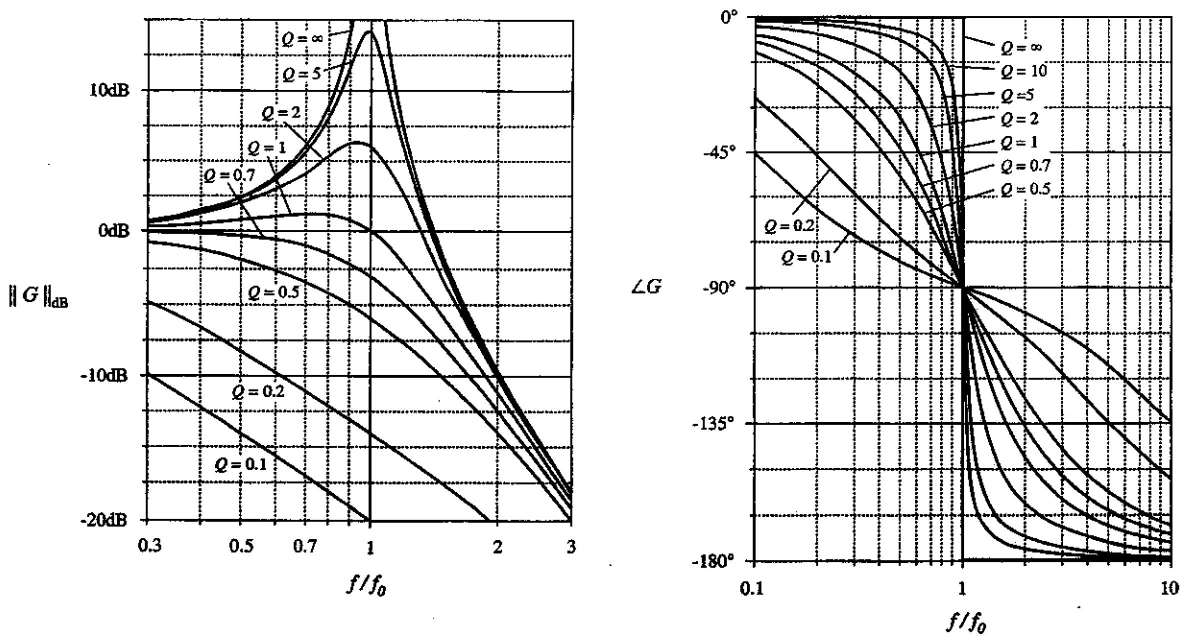
At $\omega = \omega_0$, the exact magnitude is

$\left\| G(j\omega_0) \right\| = Q$ or, in dB: $\left\| G(j\omega_0) \right\|_{dB} = \left| Q \right|_{dB}$

The exact curve has magnitude Q at $f = f_0$. The deviation of the exact curve from the asymptotes is $| Q |_{dB}$



Two-pole response: exact curves



Stability

Even though the original open-loop system is stable, the closed-loop transfer functions can be unstable and contain right half-plane poles. Even when the closed-loop system is stable, the transient response can exhibit undesirable ringing and overshoot, due to the high Q -factor of the closed-loop poles in the vicinity of the crossover frequency.

When feedback destabilizes the system, the denominator $(1+T(s))$ terms in the closed-loop transfer functions contain roots in the right half-plane (i.e., with positive real parts). If $T(s)$ is a rational fraction of the form $N(s) / D(s)$, where $N(s)$ and $D(s)$ are polynomials, then we can write

$$\frac{T(s)}{1+T(s)} = \frac{\frac{N(s)}{D(s)}}{1+\frac{N(s)}{D(s)}} = \frac{N(s)}{N(s)+D(s)}$$

$$\frac{1}{1+T(s)} = \frac{1}{1+\frac{N(s)}{D(s)}} = \frac{D(s)}{N(s)+D(s)}$$

- Could evaluate stability by evaluating $N(s) + D(s)$, then factoring to evaluate roots. This is a lot of work, and is not very illuminating.

Determination of stability directly from $T(s)$

- Nyquist stability theorem: general result.
- A special case of the Nyquist stability theorem: the phase margin test

  Allows determination of closed-loop stability (i.e., whether $1/(1+T(s))$ contains RHP poles) directly from the magnitude and phase of $T(s)$.

  A good design tool: yields insight into how $T(s)$ should be shaped, to obtain good performance in transfer functions containing $1/(1+T(s))$ terms.

The phase margin test

A test on $T(s)$, to determine whether $1/(1+T(s))$ contains RHP poles.

The crossover frequency f_c is defined as the frequency where

$$\| T(j2\pi f_c) \| = 1 \Rightarrow 0\text{dB}$$

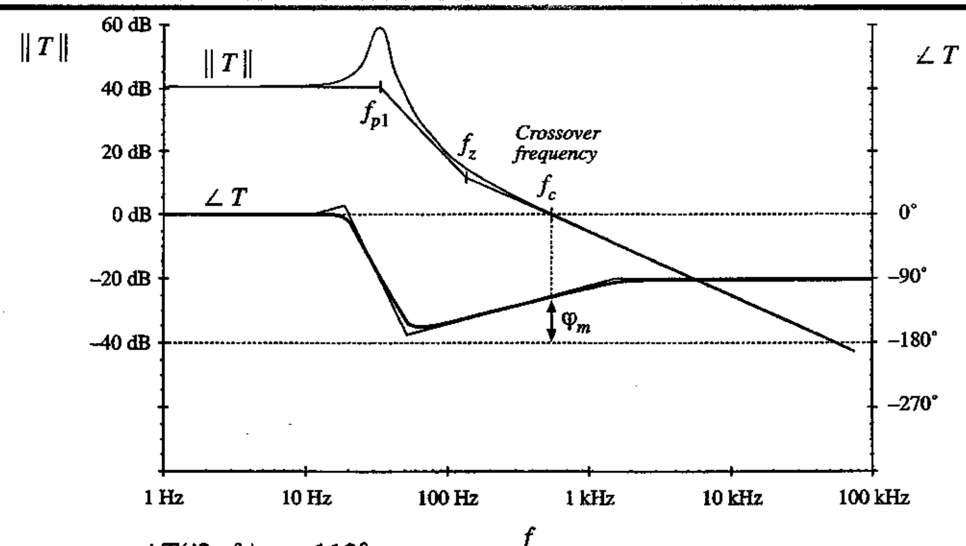
The phase margin φ_m is determined from the phase of $T(s)$ at f_c , as follows:

$$\varphi_m = 180^\circ + \angle T(j2\pi f_c)$$

If there is exactly one crossover frequency, and if $T(s)$ contains no RHP poles, then

> the quantities $T(s)/(1+T(s))$ and $1/(1+T(s))$ contain no RHP poles whenever the phase margin φ_m is positive.

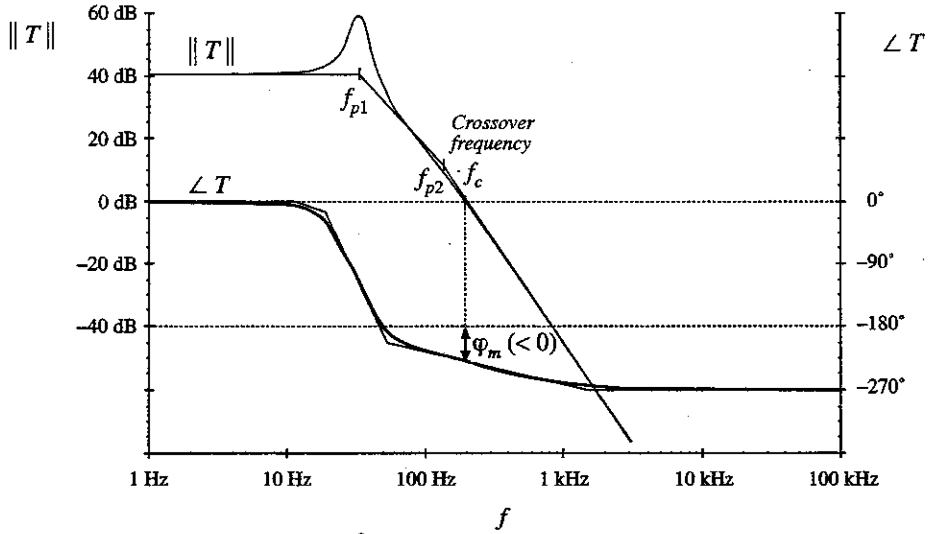
Example: a loop gain leading to a stable closed-loop system



$$\angle T(j2\pi f_c) = -\,112^\circ$$

$$\varphi_m = 180^\circ - 112^\circ = +\,68^\circ$$

Example: a loop gain leading to an unstable closed-loop system



$\angle T(j2\pi f_c) = -230^\circ$

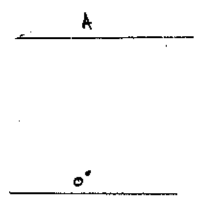
$\varphi_m = 180^\circ - 230^\circ = -50^\circ$

BODE PLOT SUMMARY SHEET

Straight gain

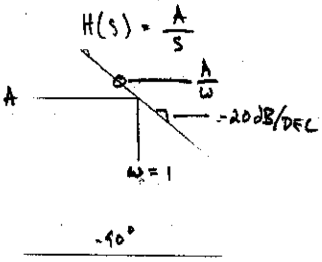
TRANSFER FUNCTION: $H(s) = A$

MAGNITUDE RESPONSE / PHASE RESPONSE



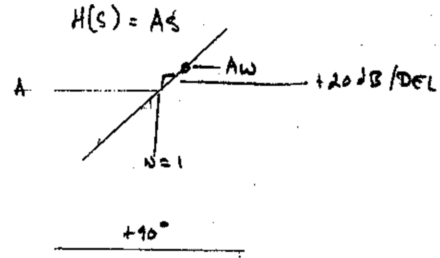
Pole (at zero)

TRANSFER FUNCTION: $H(s) = \frac{A}{s}$



Zero (at zero)

TRANSFER FUNCTION: $H(s) = As$



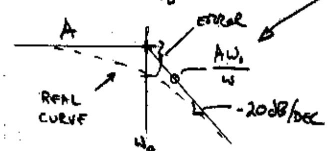
Pole

TRANSFER FUNCTION: $H(s) = \frac{A}{1 + s/\omega_0}$

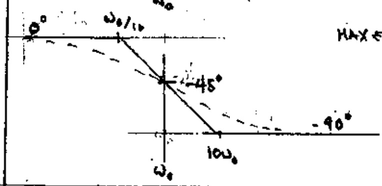
IF $\omega >> \omega_0$, $H(s) = \frac{A}{\omega/\omega_0} = \frac{A\omega_0}{\omega}$

MAXIMUM ERROR @ ω_0 = 3dB

MAGNITUDE RESPONSE



PHASE RESPONSE



MAX ERROR @ $\omega_0/10$ & $10\omega_0$ = ±5.7°

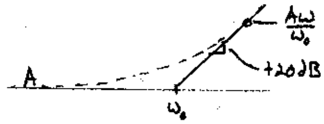
EXACT VALUE: $-\tan^{-1}\left(\frac{\omega}{\omega_0}\right)$

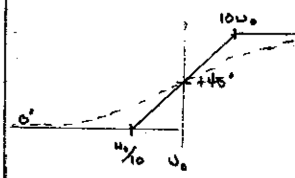
APPROX VALUE: $-45\log_{10}\left(\frac{10\omega}{\omega_0}\right)$

Zero

TRANSFER FUNCTION: $H(s) = A\left(1 + \frac{s}{\omega_0}\right)$

MAXIMUM ERROR @ ω_0 = 3dB





MAX ERROR @ $\omega_0/10$ & $10\omega_0$ = 5.7°

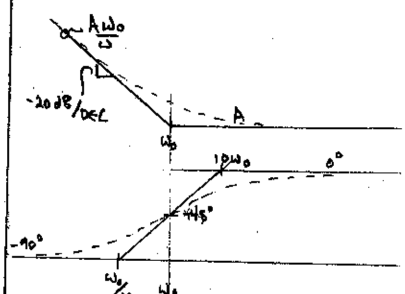
EXACT VALUE: $\tan^{-1}\left(\frac{\omega}{\omega_0}\right)$

APPROX VALUE: $45\log_{10}\left(\frac{10\omega}{\omega_0}\right)$

Inverted zero

TRANSFER FUNCTION: $H(s) = A\left(1 + \frac{\omega_0}{s}\right)$ (INVERTED ZERO)

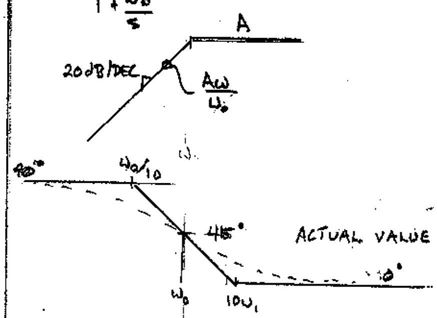
MAGNITUDE RESPONSE / PHASE RESPONSE



ACTUAL VALUE = $\tan^{-1}\left(\frac{\omega_0}{\omega}\right)$

Inverted pole

TRANSFER FUNCTION: $H(s) = \frac{A}{1 + \frac{\omega_0}{s}}$ (INVERTED POLE)

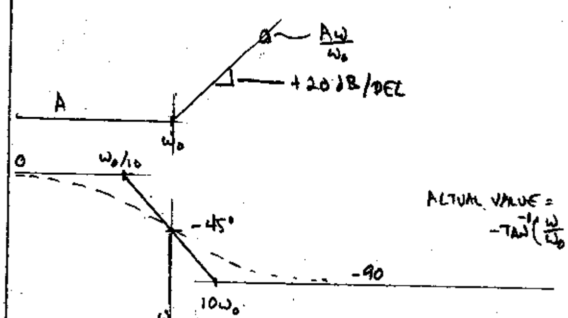


ACTUAL VALUE = $\tan^{-1}\left(\frac{\omega_0}{\omega}\right)$

Right-hand plane zero

TRANSFER FUNCTION: $H(s) = A\left(1 - \frac{s}{\omega_0}\right)$ (RIGHT-HAND PLANE ZERO)

MAGNITUDE RESPONSE / PHASE RESPONSE



ACTUAL VALUE = $-\tan^{-1}\left(\frac{\omega}{\omega_0}\right)$

Second order complex pole

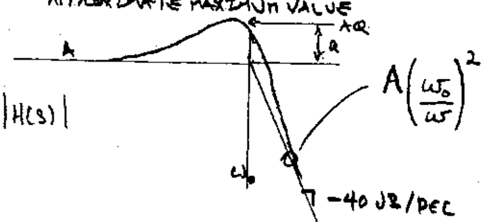
$H(s) = \frac{A}{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \frac{s^2}{\omega_0^2}}$ SECOND ORDER COMPLEX POLE

ω_0 = CORNER FREQUENCY $Q > \frac{1}{2} \Rightarrow$ COMPLEX ROOTS

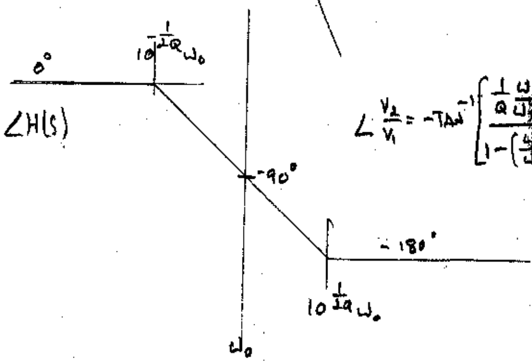
Q = QUALITY FACTOR: EXACT GAIN @ ω_0

APPROXIMATE MAXIMUM VALUE

$|H(s)|$



$\angle H(s)$



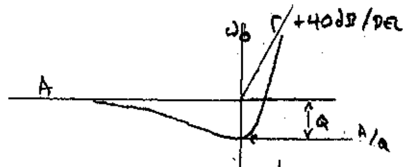
$\angle\frac{V_2}{V_1} = -\tan^{-1}\left[\frac{\frac{1}{Q}\frac{\omega}{\omega_0}}{1-\left(\frac{\omega}{\omega_0}\right)^2}\right]$

Second order complex zero

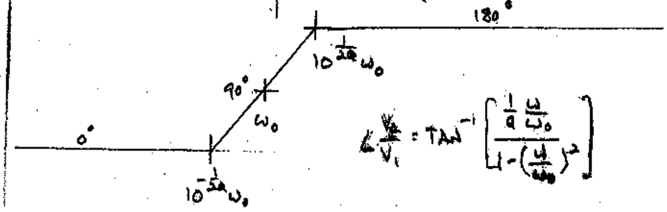
TRANSFER FUNCTION: $H(s) = A\left(1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \frac{s^2}{\omega_0^2}\right)$ SECOND ORDER COMPLEX ZERO

$Q > \frac{1}{2} \Rightarrow$ COMPLEX ROOTS

MAGNITUDE RESPONSE



PHASE RESPONSE



$\angle\frac{V_2}{V_1} = \tan^{-1}\left[\frac{\frac{1}{Q}\frac{\omega}{\omega_0}}{1-\left(\frac{\omega}{\omega_0}\right)^2}\right]$

Example: Consider the system

$$A(s) = \frac{A_o}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)\left(1+\frac{s}{\omega_3}\right)} \qquad k(s) = k_o \text{ (constant)}$$

with

$A_o = 500 \Rightarrow 54\,db$

$k_o = 0.5 \Rightarrow -6\,db$

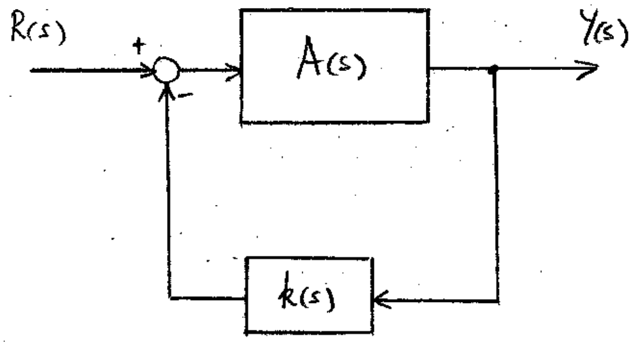
$f_1 = \omega_1/2\pi = 100\,Hz$

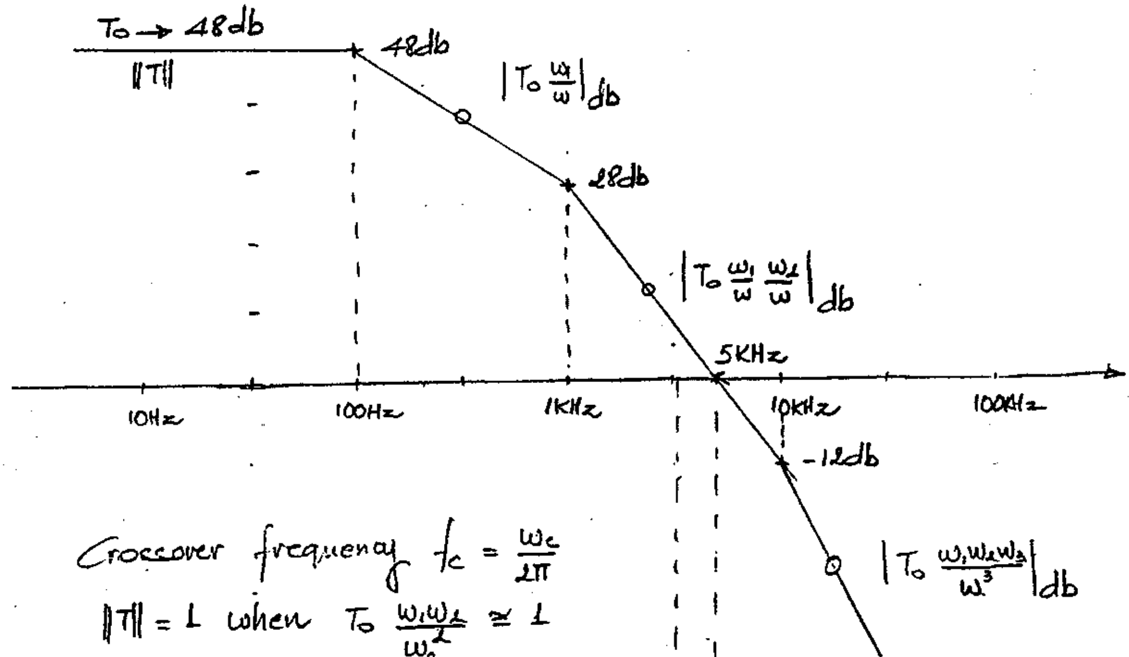
$f_2 = \omega_2/2\pi = 1\,KHz$

$f_3 = \omega_3/2\pi = 10\,KHz$

$$\Rightarrow T(s) = A(s)\,k(s) = \frac{A_o k_o}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)\left(1+\frac{s}{\omega_3}\right)}$$

$$A_o k_o = T_o = 250 \Rightarrow 48\,db$$



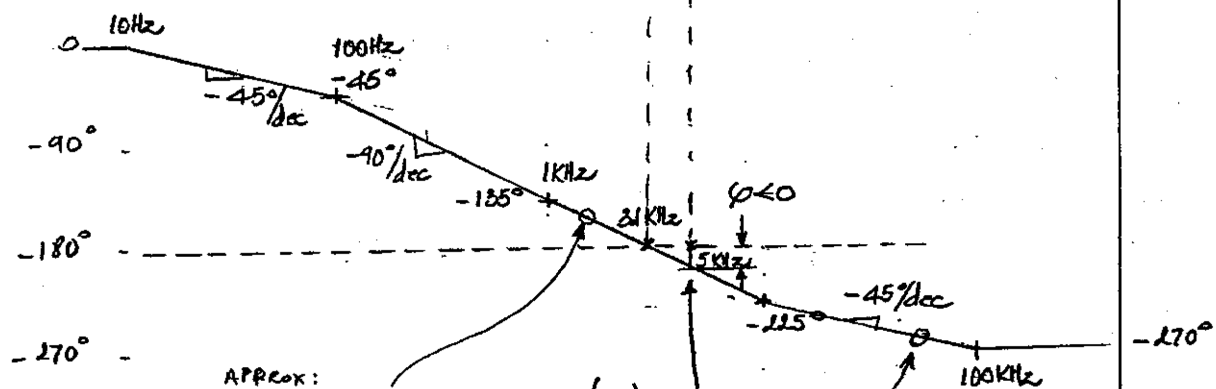


Crossover frequency $f_c = \frac{\omega_c}{2\pi}$

$\|T\| = 1$ when $T_o \frac{\omega_1 \omega_2}{\omega_c^2} \cong 1$

$\Rightarrow \omega_c^2 = T_o \omega_1 \omega_2$

$$f_c = \frac{\sqrt{T_o \omega_1 \omega_2}}{2\pi} = 5KHz$$



APPROX: $\angle T = -135° - 90° \log_{10}\left(\frac{f}{1KHz}\right)$

EXACT: $\angle T = -\tan^{-1}\left(\frac{f}{100}\right) - \tan^{-1}\left(\frac{f}{1{,}000}\right) - \tan^{-1}\left(\frac{f}{10{,}000}\right)$

At 5KHz, $\angle T \cong -198°$

At 5KHz, $\angle T = -194°$

phase margin $= 180° - 194° = -14° < 0$

$\Rightarrow$ unstable

Phase crossover frequency and gain margin?

Problem:

Given the following transmittance:

$$T(s) = \frac{As}{\left(\frac{s^2}{w_o^2} + \frac{s}{Qw_o} + 1\right)\left(1 + \frac{s}{w_p}\right)}$$

where

$A = 40$
$w_o = 15$
$Q = 3$
$w_p = 0.2$

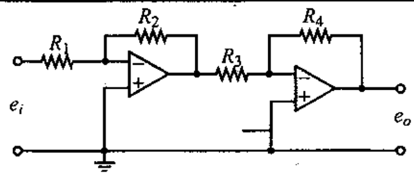
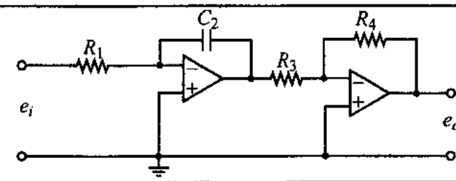
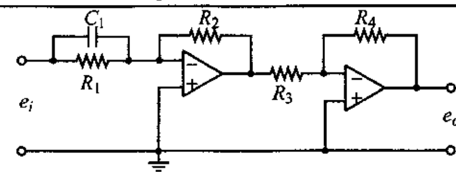
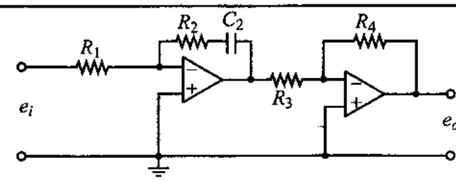
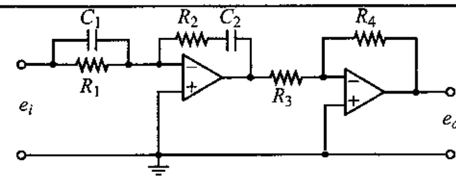
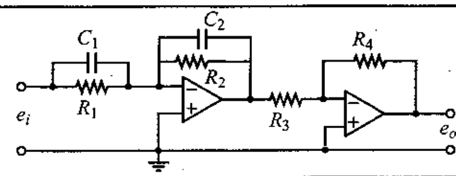
Using an approach that uses asymptotic approximations,

a). Sketch $|T(s)|$, the magnitude characteristic of T(s). Be sure to label the break frequencies, the slopes of sloping lines and gains of constant gain lines.

b). Determine the maximum gain of T(s) (expressed as an absolute value). At what frequency, w, does this maximum occur?

c). Determine the range of frequencies, $w_{low} \le w \le w_{high}$ for which $|T(jw_{low})| \le 2 \le |T(jw_{high})|$. That is, find the range of frequencies where the gain is greater than or equal to 2.

Table 3–1 Operational-Amplifier Circuits That May Be Used as Compensators

	Control Action	$G(s) = \frac{E_o(s)}{E_i(s)}$	Operational Amplifier Circuits
1	P	$\frac{R_4}{R_3}\frac{R_2}{R_1}$	
2	I	$\frac{R_4}{R_3}\frac{1}{R_1C_2s}$	
3	PD	$\frac{R_4}{R_3}\frac{R_2}{R_1}(R_1C_1s+1)$	
4	PI	$\frac{R_4}{R_3}\frac{R_2}{R_1}\frac{R_2C_2s+1}{R_2C_2s}$	
5	PID	$\frac{R_4}{R_3}\frac{R_2}{R_1}\frac{(R_1C_1s+1)(R_2C_2s+1)}{R_2C_2s}$	
6	Lead or lag	$\frac{R_4}{R_3}\frac{R_2}{R_1}\frac{R_1C_1s+1}{R_2C_2s+1}$	
7	Lag–lead	$\frac{R_6}{R_5}\frac{R_4}{R_3}\frac{[(R_1+R_3)C_1s+1](R_2C_2s+1)}{(R_1C_1s+1)[(R_2+R_4)C_2s+1]}$