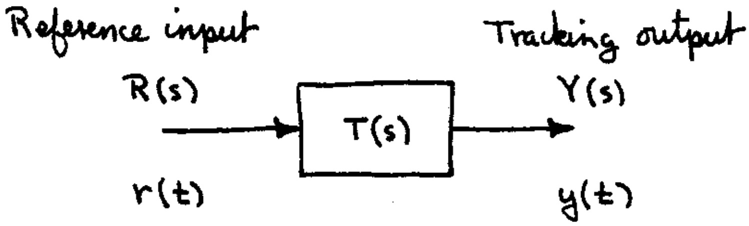
Tracking Systems

The performance of control systems is often characterized by two criteria: the transient performance and the steady state performance. Transient performance is determined by the natural response of a system and defines the stability of the system. Steady state performance refers to the ability of a system to track (follow) a desired command.

A tracking system is a control system that produces an output $y(t)$ which tracks a reference input $r(t)$ within a given level of tolerance.



In many cases, $r(t)$ can be approximated by the sum of steps, ramps, etc. For example, using a Taylor's series expansion of a reference input $r(t)$ near $t = a$,

$$r(t) = r(a) + \left.\frac{dr}{dt}\right|_{t=a}(t-a) + \frac{\left.\frac{d^2r}{dt^2}\right|_{t=a}}{2!}(t-a)^2 + \cdots$$

$$= A_0 + A_1 t + A_2 t^2 + \cdots$$

The analysis and design of tracking systems can be separated into parts:

1. Locating the characteristic roots (poles) of the transfer function, which determine the character of the system's natural response component.

2. Tracking of the reference input by the system's forced response component for the types of commands the system will encounter.

Consider a second-order, underdamped system with the following characteristic roots

$$s_1, s_2 = -a \pm j\omega$$

The natural response component is of the form

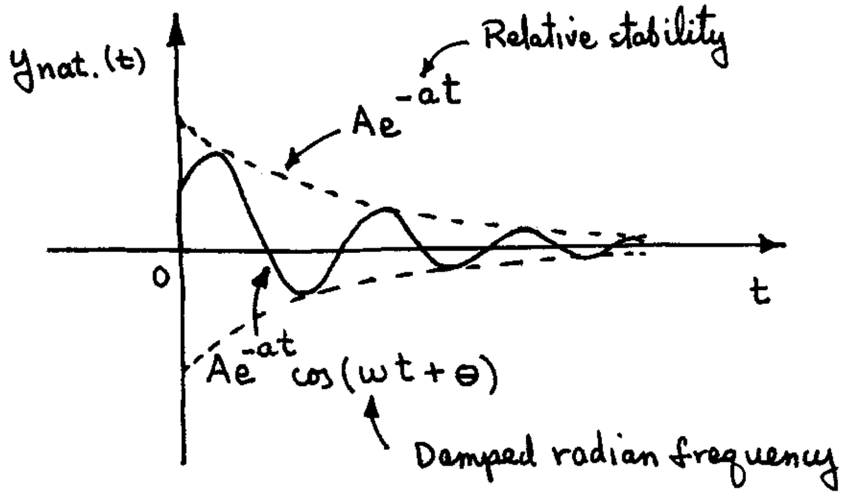
$$Y_{nat.}(s) = \frac{\text{num. poly.}}{s^2 + a_1 s + a_0} = \frac{\text{num. poly.}}{(s+a-j\omega)(s+a+j\omega)}$$

$$= \frac{\text{num. poly.}}{(s+a)^2 + \omega^2}$$

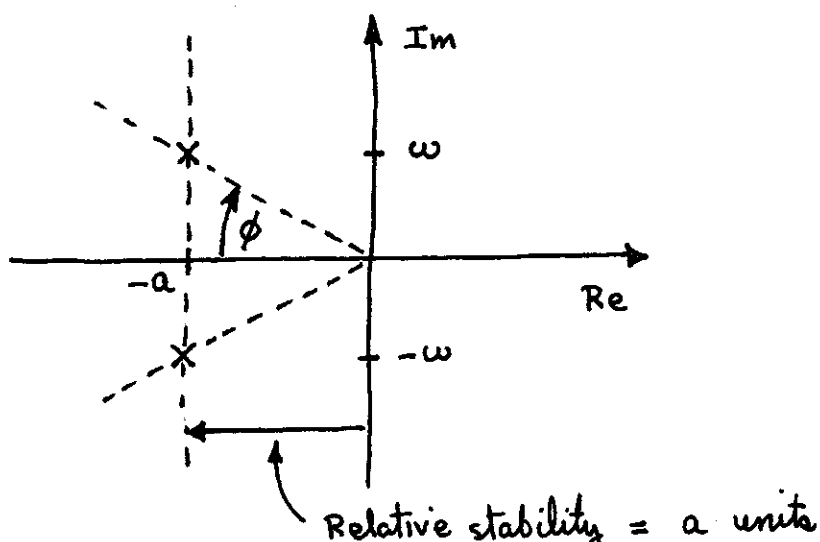
$$y_{nat.}(t) = \left[A e^{-at} \cos(\omega t + \theta)\right] u(t)$$

where the constants A and θ depend on the initial conditions.

Graphing,



The character of the natural response component of a system can be determined by graphing the characteristic roots of its transfer function.

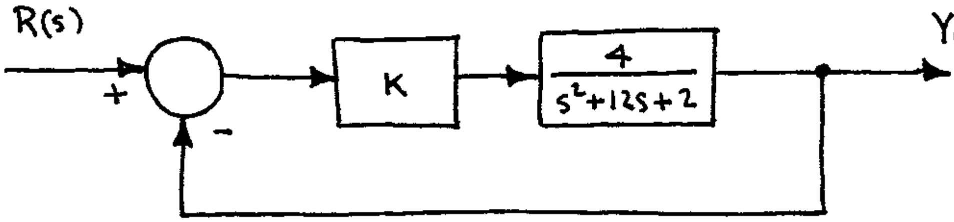


The damping ratio ζ is a function of the damping angle ϕ.

$$\zeta = \cos\phi$$

Example :

Consider the following control system and graph the characteristic roots for an underdamped system as a function of K. Select K for a damping ratio of $\zeta = \frac{1}{\sqrt{2}}$.



The Transfer function is

$$T(s) = \frac{\dfrac{4K}{s^2+12s+2}}{1+\dfrac{4K}{s^2+12s+2}}$$

$$= \frac{4K}{s^2+12s+4K+2}$$

The characteristic roots are

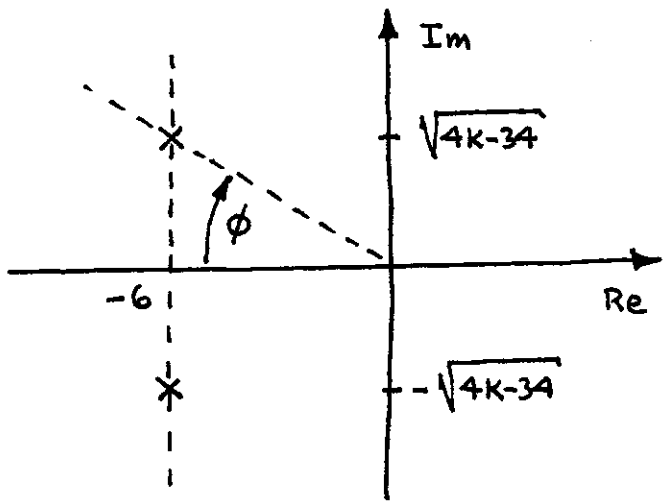
$$s = \frac{-12 \pm \sqrt{144-4(4K+2)}}{2}$$

$$= \frac{-12 \pm \sqrt{136-16K}}{2}$$

For the underdamped case, $K > 8.5$ and

$$s = -6 \pm j\sqrt{4K-34} \,, \quad K > 8.5$$

Graphing,



The damping angle is

$$\phi = \cos^{-1}(\zeta) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45°$$

Therefore,

$$\tan 45° = 1 = \frac{\sqrt{4K-34}}{6}$$

$$4K-34 = 36$$

$$K = \underline{\underline{17.5}} \leftarrow$$

As a check for $K = 17.5$,

$$p(s) = s^2 + 12s + 72$$

and

$$\omega_n = \sqrt{72} = \underline{8.49 \text{ rad./sec.}}$$

Therefore,

$$2\zeta\omega_n = 12$$

$$\zeta = \frac{12}{2\omega_n} = \frac{6}{8.49}$$

$$= \underline{\underline{0.71}} \leftarrow$$

Also,

$$\omega = \omega_n\sqrt{1-\zeta^2}$$

$$= 8.49\sqrt{1-(0.71)^2}$$

$$= \underline{\underline{6.00 \text{ rad./sec.}}} \leftarrow$$

Steady State Error

The error between the output and reference input of a tracking system is given by

$$E(s) = R(s) - Y(s) = R(s) - T(s)R(s)$$

$$= \left[1 - T(s)\right] R(s)$$

$$= T_E(s)\, R(s)$$

The transmittance function which relates the transforms of the input and error signals is

$$T_E(s) = 1 - T(s) = \left.\frac{E(s)}{R(s)}\right|_{\text{zero initial conditions}}$$

It is important to realize that the characteristic polynomial for $T_E(s)$ is the same as that for $T(s)$.

The error signal is generally composed of natural and forced parts. The natural response component decays to zero as a function of the relative stability of the system σ. The forced part of the error signal is

$$e_{forced}(t) = r(t) - y_{forced}(t)$$

For perfect tracking, $e_{forced}(t) = 0$.

Many reference inputs $r(t)$ are either power-of-time signals or can be represented as a sum of power-of-time signals. The Laplace Transform of power-of-time reference inputs is given by

$$\mathcal{L}\{r(t)\} = \mathcal{L}\left\{\frac{t^i u(t)}{i!}\right\} = \frac{1}{s^{i+1}} = R_i(s)$$

The Transform of the error signal is

$$E_i(s) = T_E(s)\, R_i(s) = T_E(s)\left(\frac{1}{s^{i+1}}\right)$$

For a stable system driven by a power-of-time input, the natural response component of the error will decay to zero and the forced response component of the error will do one of three things:

1. The forced error will become zero.
2. The forced error will become constant.
3. The forced error will increase without bound.

The final value theorem can be applied to $E(s)$ to determine which of the three conditions occurs.

Example:

A tracking system has the following transfer function.

$$T(s) = \frac{-2s^2+5}{s^3+3s^2+2s+5}$$

Find the steady state error with unit step, ramp, and parabolic inputs.

The error transmittance is

$$T_E(s) = 1 - T(s) = 1 - \frac{-2s^2+5}{s^3+3s^2+2s+5}$$

$$= \frac{s^3+5s^2+2s}{s^3+3s^2+2s+5}$$

Testing for stability,

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 3 & 5 \\ s^1 & \frac{1}{3} & 0 \\ s^0 & 5 & 0 \end{array}$$

For a unit step input,

$$E_o(s) = T_E(s)\left(\frac{1}{s^1}\right) = \frac{s^2+5s+2}{s^3+3s^2+2s+5}$$

Applying the final value theorem,

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE_0(s) = \lim_{s\to 0} \frac{s^3+5s^2+2s}{s^3+3s^2+2s+5} = \underline{\underline{0}} \leftarrow$$

For a unit ramp input,

$$E_1(s) = T_E(s)\left(\frac{1}{s^2}\right) = \frac{s^2+5s+2}{s(s^3+3s^2+2s+5)}$$

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE_1(s) = \lim_{s\to 0} \frac{s^2+5s+2}{s^3+3s^2+2s+5} = \underline{\underline{\frac{2}{5}}} \leftarrow$$

For a unit parabolic input,

$$E_2(s) = T_E(s)\left(\frac{2}{s^3}\right) = \frac{2s^2+10s+4}{s^2(s^3+3s^2+2s+5)}$$

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE_2(s) = \lim_{s\to 0} \frac{2s^2+10s+4}{s^4+3s^3+2s^2+5s} = \underline{\underline{\infty}} \leftarrow$$

The steady state error of a control system to power-of-time inputs is closely related to its <u>system type</u>. Consider an error transmittance $T_E(s)$ of the form

$$T_E(s) = \frac{s^j n(s)}{d(s)}$$

where $n(s)$ and $d(s)$ are polynomials with constant terms. The power of s in the numerator j determines the <u>system type number</u>. If there are no factors of s in the numerator, $j = 0$ and the type number of the system is zero.

Now consider a power-of-time input transform

$$R(s) = \frac{A}{s^i}$$

where the error is

$$E(s) = T_E(s)\frac{A}{s^i} = \frac{A s^j n(s)}{s^i d(s)} = A s^{j-i} \frac{n(s)}{d(s)}$$

The steady state error for a stable system then becomes

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} A s^{j-i+1} \frac{n(s)}{d(s)}$$

If $i > j+1$,

$$\lim_{t \to \infty} e(t) = \infty$$

If $i = j+1$,

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$

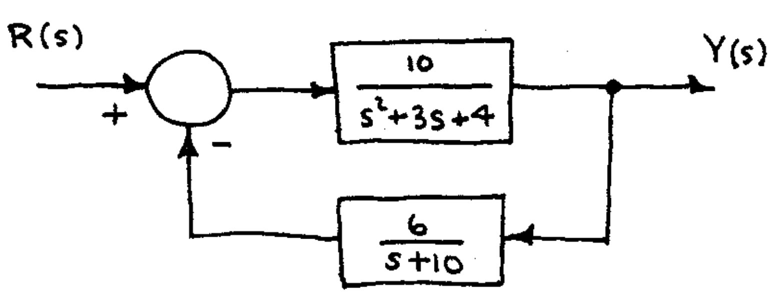
$$= A \lim_{s \to 0} \left[\frac{T_E(s)}{s^j} \right]$$

And if $i \leq j$,

$$\lim_{t \to \infty} e(t) = 0$$

<u>Example</u>:

Determine the type number of the following control system and the steady state error to a unit step and a unit ramp input.



The transfer function is

$$T(s) = \frac{\dfrac{10}{s^2+3s+4}}{1 + \left(\dfrac{6}{s+10}\right)\left(\dfrac{10}{s^2+3s+4}\right)}$$

$$= \frac{10s + 100}{s^3 + 13s^2 + 34s + 100}$$

The error transmittance is

$$T_E(s) = 1 - T(s) = 1 - \frac{10s+100}{s^3+13s^2+34s+100}$$

$$= \frac{s^1(s^2+13s+24)}{s^3+13s^2+34s+100}$$

and the system type number is <u>one</u>, or $j = 1$.

For a unit step input, $i = 1$ and therefore $i = j$. The steady state error is

$$\lim_{t\to\infty} e(t) = \underline{\underline{0}} \leftarrow$$

For a unit ramp input, $i = 2$ and $i = j+1$. Therefore,

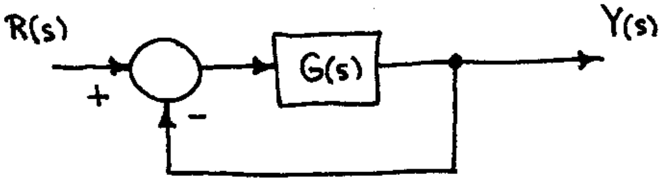
$$\lim_{t\to\infty} e(t) = \lim_{s\to 0}\left[\frac{T_E(s)}{s}\right] = \lim_{s\to 0}\left[\frac{s^2+13s+24}{s^3+13s^2+34s+100}\right]$$

$$= \frac{24}{100} = \underline{\underline{\frac{6}{25}}} \leftarrow$$

Table 3.2 Steady State Errors

System Type (Number of $s = 0$ numerator roots of the error transmittance T_E)	**Steady State Error to Step Input $r(t) = Au(t)$ $R(s) = A/s$**	**Steady State Error to Ramp Input $r(t) = Atu(t)$ $R(s) = A/s^2$**	**Steady State Error to Parabolic Input $r(t) = \frac{1}{2}At^2u(t)$ $R(s) = A/s^3$**	**Steady State Error to Input $r(t) = \frac{1}{6}At^3u(t)$ $R(s) = A/s^4$**
0	$AT_E(0)$	∞	∞	∞
1	0	$A \lim_{s \to 0} \left[\frac{T_E(s)}{s}\right]$	∞	∞
2	0	0	$A \lim_{s \to 0} \left[\frac{T_E(s)}{s^2}\right]$	∞
3	0	0	0	$A \lim_{s \to 0} \left[\frac{T_E(s)}{s^3}\right]$
⋮				

When a control system has unity feedback,



the input r and output y are compared directly, and the error transmittance is

$$T_E(s) = 1 - T(s) = 1 - \frac{G(s)}{1+G(s)} = \frac{1}{1+G(s)}$$

If the transmittance G(s) is of the form

$$G(s) = \frac{p(s)}{s^j q(s)}$$

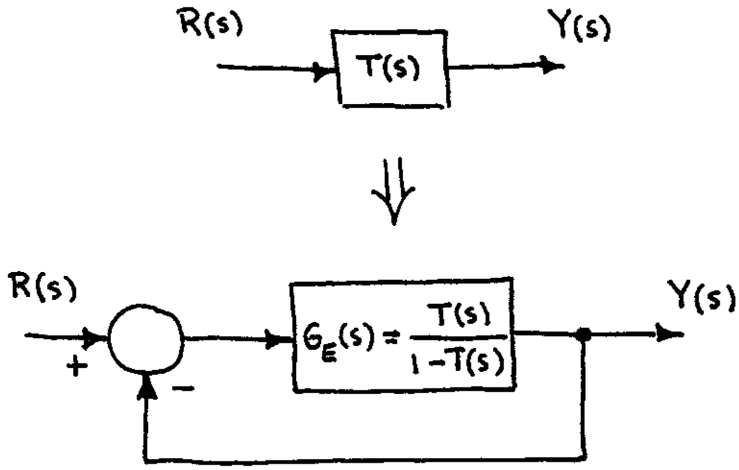
where p(s) and q(s) are polynomials with constant terms, then

$$T_E(s) = \frac{1}{1+G(s)} = \frac{1}{1 + \dfrac{p(s)}{s^j q(s)}}$$

$$= \frac{s^j q(s)}{s^j q(s) + p(s)} = \frac{s^j q(s)}{p'(s)}$$

Therefore, for unity feedback systems, the system type can be determined directly from the factors of s in the <u>denominator</u> of the transmittance G(s).

Systems that do not have unity feedback can be converted by first finding their transfer function $T(s)$ and then using the following rule.



However, it may be easier to find the steady state error of a non-unity feedback system using the $T_E(s)$ approach.

We now define the <u>steady state error coefficient</u> of a unity feedback system as

$$K_m = \lim_{s \to 0} s^m G(s)$$

For $m=0$,

$$K_0 = \lim_{s \to 0} G(s)$$

for $m=1$,

$$K_1 = \lim_{s \to 0} s\,G(s)$$

and so forth.

Since the error transmittance is

$$T_E(s) = \frac{1}{1+G(s)}$$

the error to a power-of-time input transform

$$R(s) = \frac{A}{s^i}$$

is

$$E(s) = T_E(s)\,R(s) = \frac{A}{s^i\left[1+G(s)\right]}$$

When the limits exist and are finite,

$$e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{A}{s^{i-1}\left[1+G(s)\right]}$$

For a step input, $i=1$ and

$$e(t) = \lim_{s\to 0} \frac{A}{1+G(s)} = \frac{A}{1+K_0}$$

For higher power-of-time inputs, $i = 2, 3, 4, \ldots,$

$$e(t) = \lim_{s\to 0} \frac{A}{s^{i-1}\left[1+G(s)\right]}$$

$$= \lim_{s\to 0} \frac{A}{s^{i-1}\,G(s)}$$

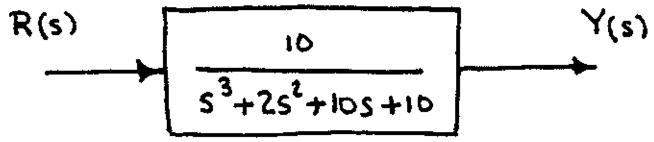
$$= \frac{A}{K_{i-1}}, \quad i = 2, 3, 4, \ldots$$

Table 3.3 Steady State Error of Unity Feedback Systems in Terms of Error Coefficients

System Type	Steady State Error to Step Input $R(s) = A/s$	Steady State Error to Ramp Input $R(s) = A/s^2$	Steady State Error to Parabolic Input $R(s) = A/s^3$	Steady State Error to Input $R(s) = A/s^4$
0	$\frac{A}{1+\kappa_0}$	∞	∞	∞
1	0	$\frac{A}{\kappa_1}$	∞	∞
2	0	0	$\frac{A}{\kappa_2}$	∞
3	0	0	0	$\frac{A}{\kappa_3}$
⋮				

Example:

Find the type number of the system, the steady state error coefficients K_0, K_1, and K_2, and the steady state errors to unit step, ramp, and parabolic inputs.



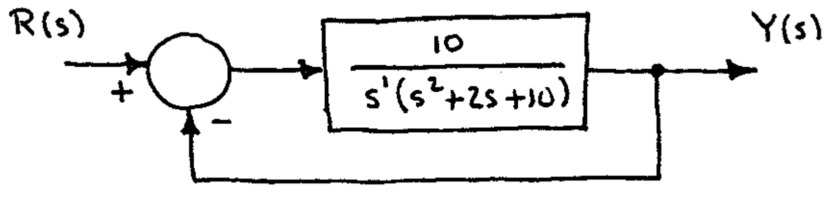
Converting to a unity feedback system,

$$G_E(s) = \frac{T(s)}{1 - T(s)}$$

$$= \frac{\dfrac{10}{s^3+2s^2+10s+10}}{1 - \dfrac{10}{s^3+2s^2+10s+10}}$$

$$= \frac{10}{s^1(s^2+2s+10)}$$

Therefore,



and the type number of the system is one.

The steady state error coefficients,

$$K_0 = \lim_{s \to 0} G_E(s) = \underline{\underline{\infty}}$$

$$K_1 = \lim_{s \to 0} s G_E(s)$$

$$= \lim_{s \to 0} \frac{10}{s^2+2s+10} = \underline{\underline{1}}$$

$$K_2 = \lim_{s \to 0} s^2 G_E(s) = \underline{\underline{0}}$$

From the Table,

$$e_{step}(t) = \underline{\underline{0}}$$

$$e_{ramp}(t) = \frac{A}{K_1} = \frac{1}{1} = \underline{\underline{1}}$$

$$e_{para}(t) = \underline{\underline{\infty}}$$

As a check,

$$T_E(s) = 1 - T(s) = 1 - \frac{10}{s^3+2s^2+10s+10}$$

$$= \frac{s^1(s^2+2s+10)}{s^3+2s^2+10s+10}$$

and the type number of the system is $\underline{\underline{\text{one}}}$.

From the table,

$$e_{step}(t) = \underline{\underline{0}} \leftarrow$$

$$e_{ramp}(t) = A \lim_{s \to 0} \left[\frac{T_E(s)}{s} \right]$$

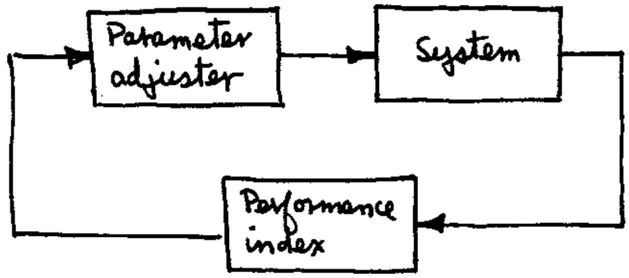
$$= \lim_{s \to 0} \frac{s^2 + 2s + 10}{s^3 + 2s^2 + 10s + 10}$$

$$= \underline{\underline{1}} \leftarrow$$

$$e_{para}(t) = \underline{\underline{\infty}} \leftarrow$$

Optimizing Control Systems

If a measure or index of performance can be expressed mathematically, a control system can often be optimized by adjusting parameters until the best performance of the system is realized.



The selection of an appropriate performance index is part of the design process. Some of the more common ones are:

Integral square error to a step signal - I_S

$$I_S = \int_0^{\infty} e^2_{step}(t)\,dt$$

Integral magnitude error - I_M

$$I_M = \int_0^{\infty} |e(t)|\,dt$$

Integral time-weighted square error - I_{TS}

$$I_{TS} = \int_0^{\infty} t\,e^2(t)\,dt$$

Integral time-weighted magnitude error – I_{TM}

$$I_{TM} = \int_0^{\infty} t\,|e(t)|\,dt$$

Example:

Select a parameter K to give minimum integral square error to a step input for a system with an error transmittance

$$T_E(s) = \frac{Ks^2 + (1-K)s}{s^2+3s+2}$$

The error to a step input is

$$E(s) = T_E(s)\left(\frac{A}{s}\right) = \frac{A\left[Ks+(1-K)\right]}{s^2+3s+2} = A\left(\frac{-2K+1}{s+1} + \frac{3K-1}{s+2}\right)$$

Therefore,

$$e(t) = \mathcal{L}^{-1}\{E(s)\} = A\left[(-2K+1)e^{-t} + (3K-1)e^{-2t}\right]u(t)$$

and

$$e^2(t) = A^2\left[(-2K+1)^2 e^{-2t} + 2(-2K+1)(3K-1)e^{-3t} + (3K-1)^2 e^{-4t}\right]u(t)$$

The integral square error is

$$I_s(K) = \int_0^\infty e^2(t)\,dt$$

$$= A^2\left[(4K^2-4K+1)\frac{e^{-2t}}{-2} + 2(-6K^2+5K-1)\frac{e^{-3t}}{-3} + (9K^2-6K+1)\frac{e^{-4t}}{-4}\right]_0^\infty$$

$$= A^2\left[(4K^2-4K+1)\left(\frac{1}{2}\right) + 2(-6K^2+5K-1)\left(\frac{1}{3}\right) + (9K^2-6K+1)\left(\frac{1}{4}\right)\right]$$

$$= \frac{A^2}{12}(3K^2-2K+1)$$

Minimizing,

$$\frac{dI_s}{dK} = \frac{A^2}{12}(6K-2) = 0$$

$$K = \underline{\underline{\frac{1}{3}}} \leftarrow$$

The system type number is one, and from the table the steady state error is zero. The actual error signal is

$$e(t) = \frac{A}{3}e^{-t}u(t)$$

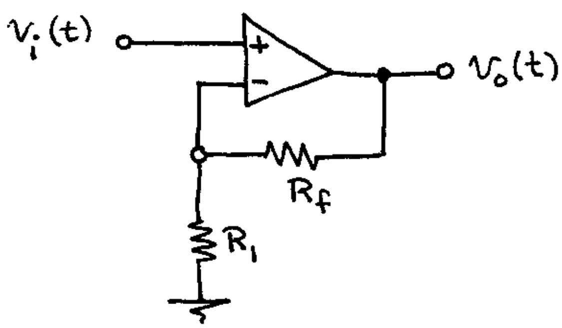
Control System Sensitivity

The <u>sensitivity</u> of a single-input, single-output transfer function $T(s)$ to changes in a specific parameter a is defined as

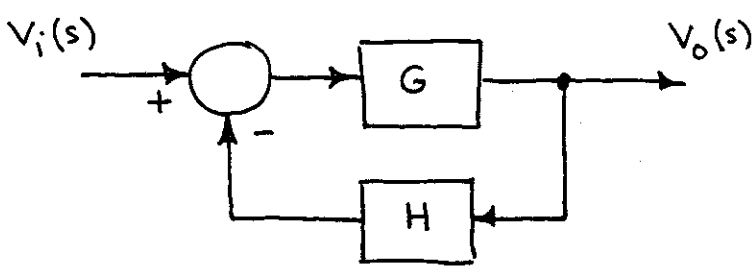
$$S_a = \lim_{\Delta a \to 0} \frac{\frac{\Delta T}{T}}{\frac{\Delta a}{a}} = \lim_{\Delta a \to 0} \frac{a}{T}\frac{\Delta T}{\Delta a} = \frac{a}{T}\frac{\partial T}{\partial a}$$

and is the limiting ratio of the fractional change in the transfer function to the fractional change in the parameter.

Consider a noninverting amplifier circuit where the operational amplifier has a finite open loop gain.



The control system block diagram is



The feedback transmittance is determined by the resistors

$$H = \frac{R_1}{R_1 + R_f}$$

and the transfer function is

$$T = \frac{G}{1+GH} = \frac{G}{1 + G\left(\frac{R_1}{R_1+R_f}\right)}$$

$$= \frac{G(R_1+R_f)}{(G+1)R_1 + R_f}$$

$$\cong \frac{G}{G+1}\,\frac{R_1+R_f}{R_1} \quad , \quad (G+1)R_1 \gg R_f$$

For large G,

$$T \cong \frac{R_1+R_f}{R_1} = 1 + \frac{R_f}{R_1}$$

The sensitivity of T to changes in G is

$$S_G = \frac{G}{T}\frac{\partial T}{\partial G} = \frac{\cancel{G}}{\frac{\cancel{G}}{1 \cancel{+ GH}}}\;\frac{1 + \cancel{GH} - \cancel{GH}}{(1+GH)^2} = \frac{1}{1+GH}$$

and

$$S_G \cong \frac{1}{GH} = \frac{R_1+R_f}{GR_1} \quad , \quad GH \gg 1$$

The sensitivity of T to changes in H is

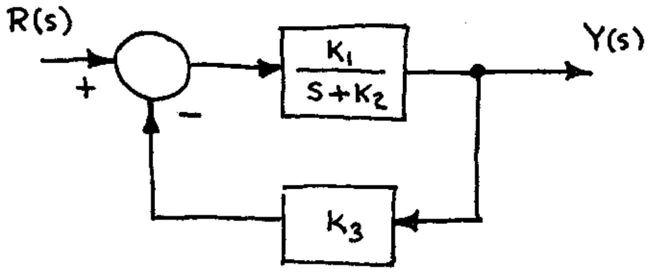
$$S_H = \frac{H}{T}\frac{\partial T}{\partial H} = \frac{H}{\frac{\cancel{G}}{\cancel{1+GH}}}\;\frac{-G^{\cancel{2}}}{(1+GH)^{\cancel{2}}} = \frac{-GH}{1+GH}$$

and

$$S_H \cong -1 \,, \quad GH >> 1$$

Example:

Find the sensitivities of the transfer function to small changes in K_1, K_2, and K_3 about the nominal values of $K_1 = 1$, $K_2 = 2$, and $K_3 = 3$.



The transfer function is

$$T = \frac{\frac{K_1}{S+K_2}}{1+\frac{K_1K_3}{S+K_2}} = \frac{K_1}{S+K_2+K_1K_3}$$

The sensitivities are

$$S_{K_1} = \frac{K_1}{T}\frac{\partial T}{\partial K_1} = \frac{\cancel{K_1}}{\frac{\cancel{K_1}}{\cancel{S+K_2+K_1K_3}}} \; \frac{S+K_2+\cancel{K_1K_3}-\cancel{K_1K_3}}{(S+K_2+K_1K_3)^{\cancel{2}}}$$

$$= \frac{S+K_2}{S+K_2+K_1K_3} = \frac{S+2}{S+2+(1)(3)} = \underline{\underline{\frac{S+2}{S+5}}} \leftarrow$$

$$S_{K_2} = \frac{K_2}{T}\frac{\partial T}{\partial K_2} = \frac{K_2}{\frac{\cancel{K_1}}{\cancel{S+K_2+K_1K_3}}} \; \frac{-\cancel{K_1}}{(S+K_2+K_1K_3)^{\cancel{2}}}$$

$$= \frac{-K_2}{S+K_2+K_1K_3} = \frac{-2}{S+2+(1)(3)} = \underline{\underline{\frac{-2}{S+5}}} \leftarrow$$

$$S_{K_3} = \frac{K_3}{T}\frac{\partial T}{\partial K_3} = \frac{K_3}{\frac{\cancel{K_1}}{\cancel{S+K_2+K_1K_3}}} \; \frac{-K_1^{\cancel{2}}}{(S+K_2+K_1K_3)^{\cancel{2}}}$$

$$= \frac{-K_1K_3}{S+K_2+K_1K_3} = \frac{-(1)(3)}{S+2+(1)(3)} = \underline{\underline{\frac{-3}{S+5}}} \leftarrow$$