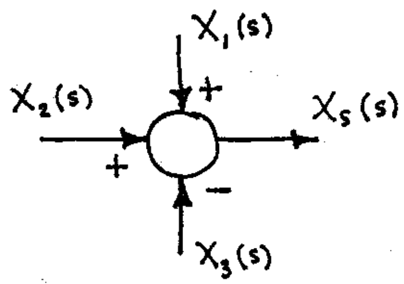
Block Diagrams

Block diagrams are often used to describe systems in the frequency domain. The basic elements are the <u>block</u>, <u>summer</u>, and the <u>junction</u>. The block uses a <u>transmittance</u> function to describe the relationship between two Laplace-transformed signals.

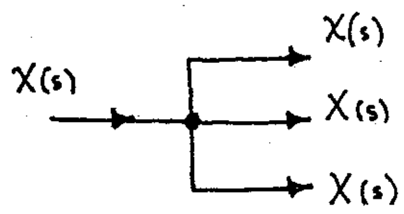
$$G(s) = \frac{X_2(s)}{X_1(s)}$$

A summer permits the addition and subtraction of signals.



$$X_S(s) = X_1(s) + X_2(s) - X_3(s)$$

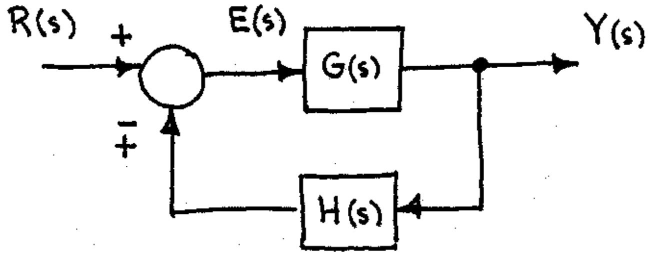
A junction (or pickoff point) is used to distribute a signal to multiple paths.



Block Diagram Algebra

Block diagram algebra is used to reduce block diagrams to a single block containing a transfer function which relates the input and output signals of the block diagram.

Consider the following basic feedback configuration.



This system contains a forward transmittance block G(s), a feedback transmittance block H(s), a summer and a junction. The block diagram representing the system can be reduced by observing

$$\begin{cases} Y(s) = G(s)\,E(s) \\ E(s) = R(s) \mp H(s)\,Y(s) \end{cases}$$

and solving for

$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)\,H(s)}$$

Therefore,

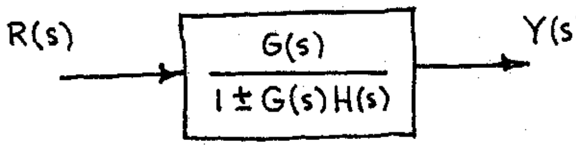
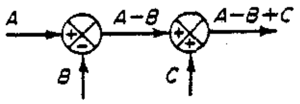
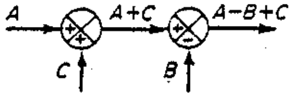
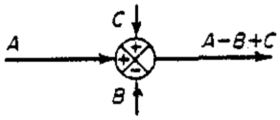
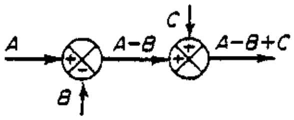
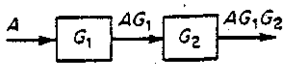
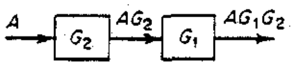
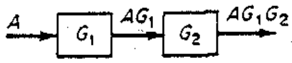
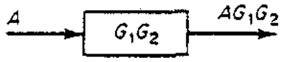
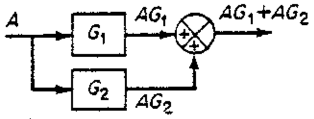
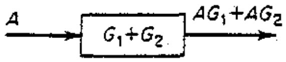
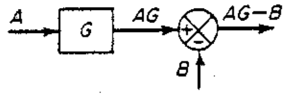
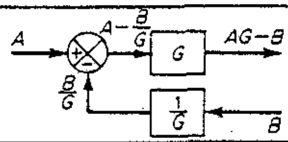
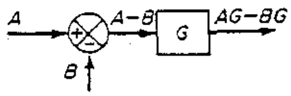
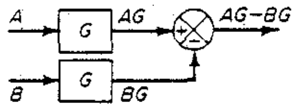
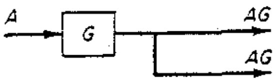
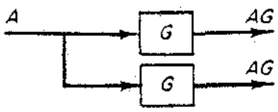
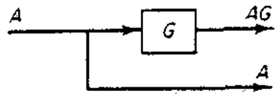
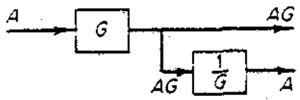
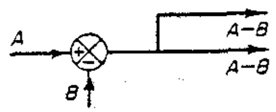
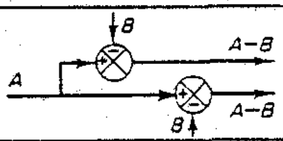
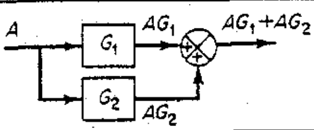
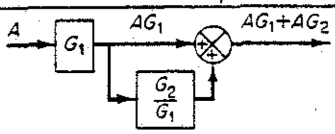
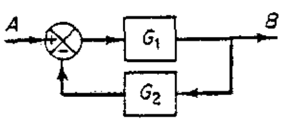
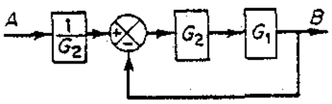
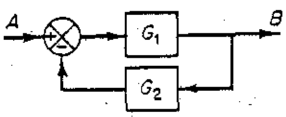
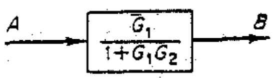


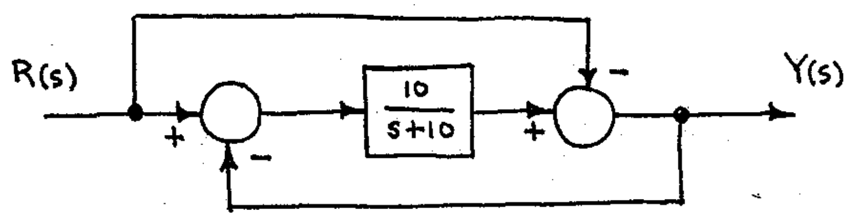
Table 1 Rules of Block Diagram Algebra

	Original Block Diagrams	Equivalent Block Diagrams
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		
11		
12		
13		

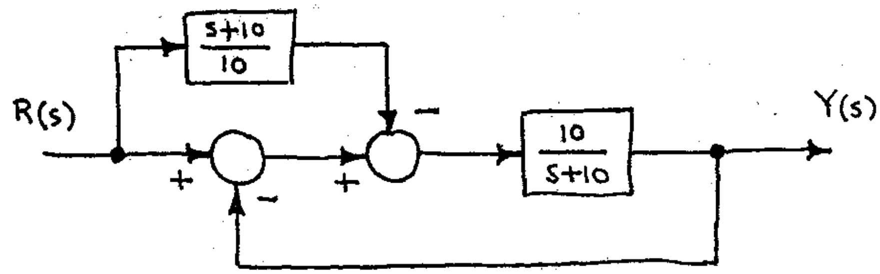
A list of rules for block diagram algebra can be found in Table 1. For example, the reduction for the basic feedback configuration becomes Rule 13 in this table.

<u>Example</u>:

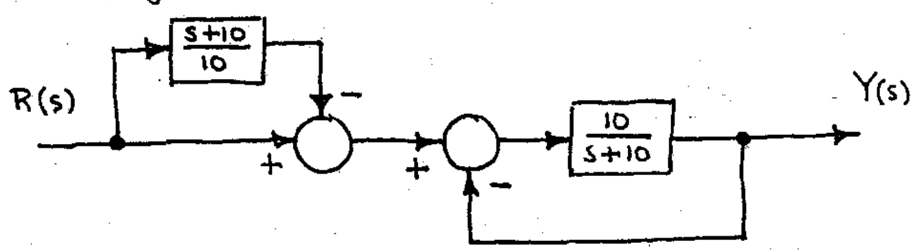
Consider the following block diagram.



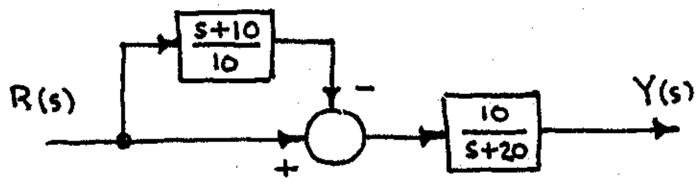
The reduction using block diagram algebra is as follows. Beginning with Rule 6,



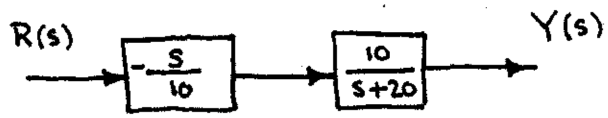
Then using Rule 1,



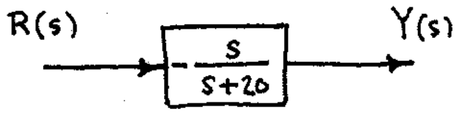
Using Rule 13,



Rule 5 yields

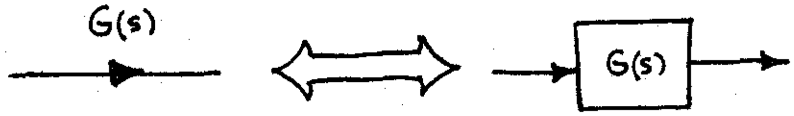


Finally, Rule 4 achieves reduction.



Signal Flow Graphs

Signal flow graphs are also used to describe systems in the frequency domain. The two basic elements are the branch and the node. The branch is equivalent to a block (in block diagrams).



The node is equivalent to a summer, with all plus signs, followed by a junction.

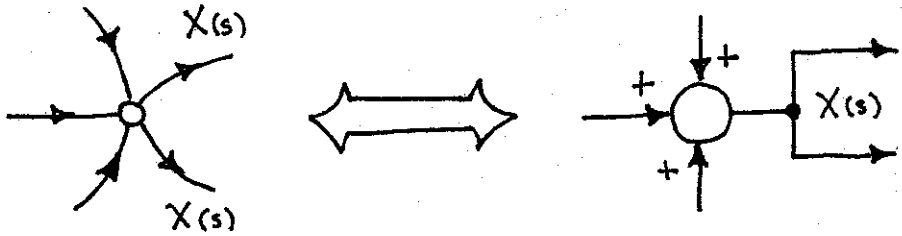


Table 1.10 SIGNAL FLOW GRAPH DEFINITIONS

Path:	A succession of branches, from input to output, in the direction of the arrows, which does not pass any node more than once.
Path gain:	Product of the transmittances of the branches of the path. For the ith path, the path gain is denoted by P_i.
Loop:	A closed succession of branches, in the direction of the arrows, which does not pass any node more than once.
Loop gain:	Product of the transmittances of the branches of the loop.
Touching:	Loops with one or more nodes in common are termed touching. A loop and a path are touching if they have a common node.
Determinant:	The determinant of a signal flow graph is $\Delta = 1 -$ (sum of all loop gains) + (sum of products of gains of all combinations of 2 nontouching loops) $-$ (sum of products of gains of all combinations of 3 nontouching loops) $+ \cdots$
Cofactor:	The cofactor of the ith path, denoted by Δ_i, is the determinant of the signal flow graph formed by deleting all loops touching path i.
Mason's gain rule:	$T(s) = \dfrac{P_1\Delta_1 + P_2\Delta_2 + \cdots}{\Delta}$

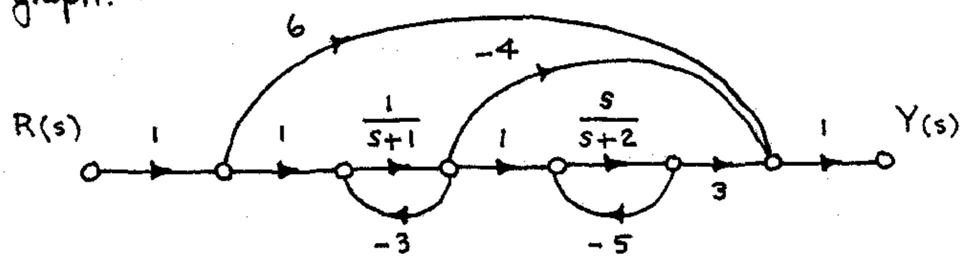
Mason's Gain Rule

Mason's gain rule is a straightforward procedure for determining the transfer function of a system from its signal flow graph. Using definitions from Table 1.10 to find a system's <u>path gains P_i</u>, <u>determinant Δ</u>, and <u>cofactors Δ_i</u>, Mason's gain rule for a system with n paths is

$$T(s) = \frac{\sum_{i=1}^{n} P_i \Delta_i}{\Delta}$$

<u>Example</u>:

consider the following single-input, single-output signal flow graph.



The $n=3$ path gains are

$$P_1 = (1)(6)(1) = 6$$

$$P_2 = (1)(1)\left(\frac{1}{s+1}\right)(-4)(1) = -\frac{4}{s+1}$$

$$P_3 = (1)(1)\left(\frac{1}{s+1}\right)(1)\left(\frac{s}{s+2}\right)(3)(1) = \frac{3s}{(s+1)(s+2)}$$

The two loop gains are

$$L_1 = \left(\frac{1}{s+1}\right)(-3) = -\frac{3}{s+1}$$

$$L_2 = \left(\frac{s}{s+2}\right)(-5) = -\frac{5s}{s+2}$$

and are nontouching.

The determinant of the signal flow graph is

$$\Delta = 1 - (L_1 + L_2) + (L_1 L_2)$$

$$= 1 - \left[\left(-\frac{3}{s+1}\right) + \left(-\frac{5s}{s+2}\right)\right] + \left(-\frac{3}{s+1}\right)\left(-\frac{5s}{s+2}\right)$$

$$= 1 + \frac{3}{s+1} + \frac{5s}{s+2} + \frac{15s}{(s+1)(s+2)}$$

which is also the cofactor Δ_1.

The other two cofactors are

$$\Delta_2 = 1 - L_2$$

$$= 1 - \left(-\frac{5s}{s+2}\right) = \frac{6s+2}{s+2}$$

$$\Delta_3 = 1$$

Applying Mason's gain rule,

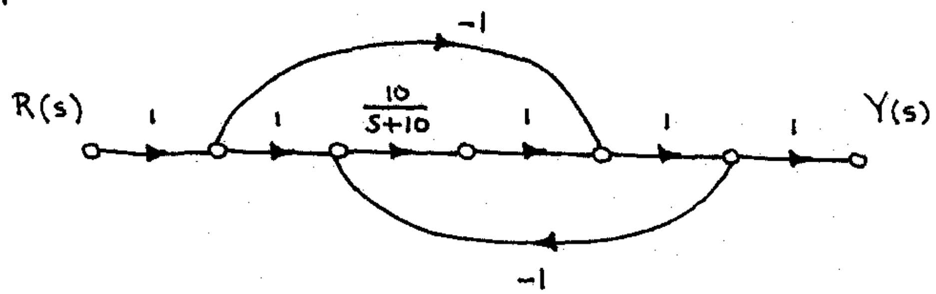
$$T(s) = \frac{\sum_{i=1}^{3} P_i \Delta_i}{\Delta} = \frac{P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3}{\Delta}$$

$$= \frac{6\left[1+\frac{3}{s+1}+\frac{5s}{s+2}+\frac{15s}{(s+1)(s+2)}\right]+\left(-\frac{4}{s+1}\right)\left(\frac{6s+2}{s+2}\right)+\frac{3s}{(s+1)(s+2)}}{1+\frac{3}{s+1}+\frac{5s}{s+2}+\frac{15s}{(s+1)(s+2)}}$$

$$= \underline{\underline{\frac{36s^2+135s+40}{6s^2+26s+8}}} \leftarrow$$

Example:

Consider the earlier block diagram example. The signal flow graph is.



The n = 2 path gains are

$$P_1 = -1 \qquad P_2 = \frac{10}{s+10}$$

The loop gain is

$$L_1 = -\frac{10}{s+10}$$

and the determinant is

$$\Delta = 1 - L_1 = 1 + \frac{10}{s+10} = \frac{s+20}{s+10}$$

The cofactors are

$$\Delta_1 = \Delta_2 = 1$$

Using Mason's gain rule,

$$T(s) = \frac{\sum_{i=1}^{2} P_i \Delta_i}{\Delta} = \frac{P_1 \Delta_1 + P_2 \Delta_2}{\Delta}$$

$$= \frac{-1 + \frac{10}{s+10}}{\frac{s+20}{s+10}} = \underline{\underline{-\frac{s}{s+20}}} \leftarrow$$

Response of a First-Order System

Consider a first-order system, with input $r(t)$ and output $y(t)$, that is represented by a differential equation of the form

$$\frac{dy}{dt} + a_0 y = b_1 \frac{dr}{dt} + b_0 r$$

The Laplace-transformed equation is

$$sY(s) - y(0^-) + a_0 Y(s) = b_1 s R(s) - b_1 r(0^-) + b_0 R(s)$$

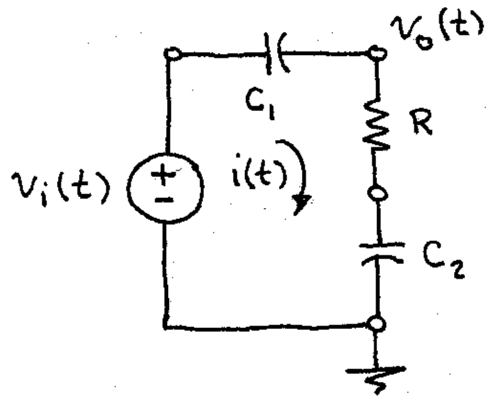
$$Y(s) = \frac{b_1 s + b_0}{s + a_0} R(s) + \frac{y(0^-) - b_1 r(0^-)}{s + a_0}$$

$$= T(s) R(s) + \frac{y(0^-) - b_1 r(0^-)}{s + a_0}$$

This system is stable if and only if $a_0 > 0$.

Example:

Consider the following first-order circuit.



Using KVL,

$$\begin{cases} v_i = \frac{1}{C_1}\int_{-\infty}^{t} i\,d\tau + iR + \frac{1}{C_2}\int_{-\infty}^{t} i\,d\tau \\ v_o = iR + \frac{1}{C_2}\int_{-\infty}^{t} i\,d\tau \end{cases}$$

Solving these simultaneous equations,

$$\frac{dv_o}{dt} + \left(\frac{C_1 + C_2}{RC_1C_2}\right)v_o = \frac{dv_i}{dt} + \left(\frac{1}{RC_2}\right)v_i$$

Letting $R = \frac{1}{10}\Omega$, $C_1 = C_2 = 1F$, and $v_i(t) = u(t)$,

$$\frac{dv_o}{dt} + 20v_o = \frac{dv_i}{dt} + 10v_i$$

$$= \delta(t) + 10u(t)$$

By inspection, $a_o = 20$, $b_1 = 1$, and $b_o = 10$.
Therefore,

$$V_o(s) = \frac{b_1 s + b_o}{s + a_o}R(s) + \frac{v_o(0^-) - b_1 v_i(0^-)}{s + a_o}$$

$$= \frac{s+10}{s+20}\left(\frac{1}{s}\right)$$

$$= \frac{\frac{1}{2}}{s} + \frac{\frac{1}{2}}{s+20}$$

And,

$$v_o(t) = \underline{\underline{\left(\tfrac{1}{2} + \tfrac{1}{2}e^{-20t}\right)u(t)\ \text{V}}} \leftarrow$$

As a check,

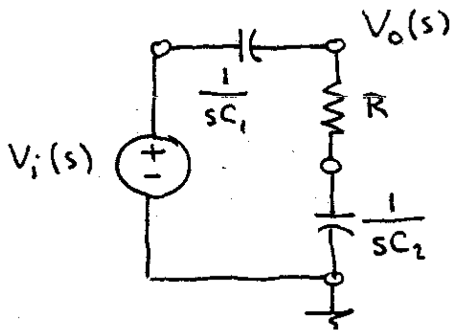
$$\frac{dv_o}{dt} + 20v_o = \delta(t) + 10u(t)$$

$$\left(\tfrac{1}{2} + \tfrac{1}{2}e^{-20t}\right)\delta(t) - 10e^{-20t}u(t) + \left(10 + 10e^{-20t}\right)u(t)$$

$$= \delta(t) - \cancel{10e^{-20t}u(t)} + 10u(t) + \cancel{10e^{-20t}u(t)}$$

$$= \underline{\underline{\delta(t) + 10u(t)}} \leftarrow$$

In the frequency domain,



Using the voltage divider theorem,

$$T(s) = \frac{V_o(s)}{V_i(s)} = \frac{R + \frac{1}{sC_2}}{R + \frac{1}{sC_1} + \frac{1}{sC_2}} = \frac{s + \frac{1}{RC_2}}{s + \frac{C_1 + C_2}{RC_1C_2}}$$

Inserting values,

$$T(s) = \frac{s+10}{s+20}$$

and

$$V_o(s) = T(s)\,R(s)$$

$$= \frac{s+10}{s+20}\left(\frac{1}{s}\right)$$

$$= \frac{\frac{1}{2}}{s} + \frac{\frac{1}{2}}{s+20}$$

$$v_o(t) = \underline{\underline{\left(\tfrac{1}{2} + \tfrac{1}{2}e^{-20t}\right)u(t)\ V}} \leftarrow$$

Now consider a first-order system described by

$$\frac{dy}{dt} + a_o y = b_o r$$

For a step function input signal,

$$r(t) = A\,u(t) \Rightarrow R(s) = \frac{A}{s}$$

and

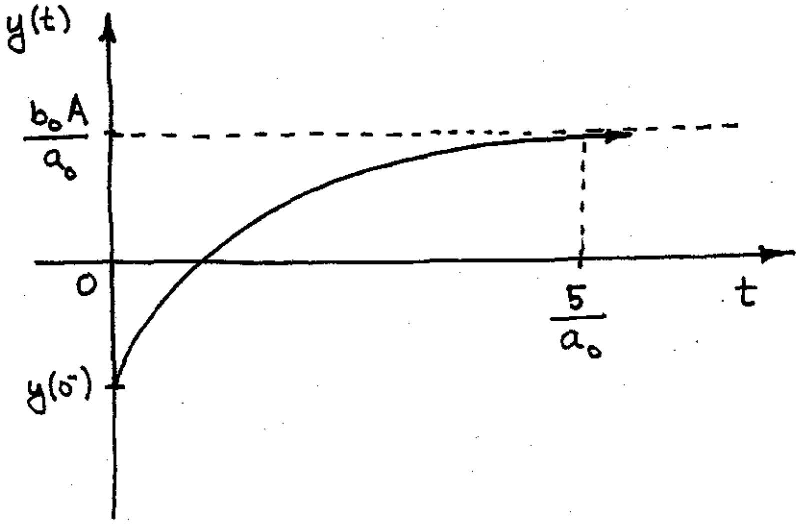
$$Y(s) = \frac{b_o A}{s(s+a_o)} + \frac{y(0^-)}{s+a_o}$$

$$= \frac{\frac{b_o A}{a_o}}{s} + \frac{y(0^-) - \frac{b_o A}{a_o}}{s+a_o}$$

The time-domain response is

$$y(t) = \left\{ \frac{b_0 A}{a_0} + \left[y(0^-) - \frac{b_0 A}{a_0} \right] e^{-a_0 t} \right\} u(t)$$

and the time constant $\tau = \frac{1}{a_0}$. Graphing,



<u>Example</u>:

Find $y(t)$ for a system described by: $T(s) = \frac{3}{s+3}$, $r(t) = 6u(t)$, and $y(0^-) = 10$.

Solving for $Y(s)$,

$$Y(s) = T(s)\,R(s) + \frac{y(0^-) - b_1 r(0^-)}{s + a_0}$$

$$= \frac{3}{s+3}\left(\frac{6}{s}\right) + \frac{10}{s+3}$$

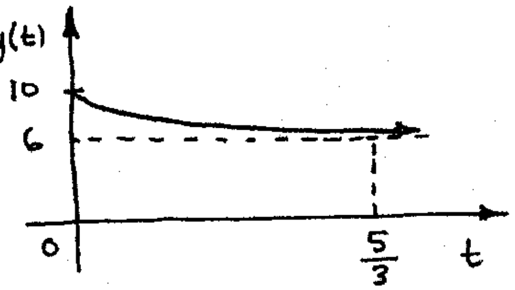
Expanding,

$$Y(s) = \frac{6}{s} + \frac{4}{s+3}$$

Therefore,

$$y(t) = \underline{\underline{(6+4e^{-3t})u(t)}} \leftarrow$$

Graphing,



<u>Example</u>:

Find $y(t)$ for a system described by: $T(s) = \frac{20s}{s+300}$, $r(t) = 8 \sin 100t \, u(t)$, and $y(0^-) = -10$.

Solving for $Y(s)$,

$$Y(s) = T(s)R(s) + \frac{y(0^-) - b_1 r(0^-)}{s+a_0}$$

$$= \frac{20s}{s+300}\left(\frac{800}{s^2+100^2}\right) + \frac{-10}{s+300}$$

$$= \frac{-10s^2 + 1.60 \times 10^4 s - 10^5}{(s+300)(s^2+100^2)}$$

Expanding,

$$Y(s) = -\frac{58}{s+300} + \frac{25.30\angle -18.43^\circ}{s-j100} + \frac{25.30\angle 18.43^\circ}{s+j100}$$

Therefore,

$$y(t) = \underline{\underline{\left[-58e^{-300t} + 50.60\cos(100t - 18.43^\circ)\right]u(t)}} \leftarrow$$

Response of Second-Order Systems

Consider a second-order system, with input $r(t)$ and output $y(t)$, that is represented by a differential equation of the form

$$\frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = b_2 \frac{d^2r}{dt^2} + b_1 \frac{dr}{dt} + b_0 r$$

The Laplace-transformed equation is

$$s^2Y(s) - sy(0^-) - y'(0^-) + a_1 s Y(s) - a_1 y(0^-) + a_0 Y(s) =$$

$$b_2 s^2 R(s) - b_2 s r(0^-) - b_2 r'(0^-) + b_1 s R(s) - b_1 r(0^-) + b_0 R(s)$$

$$Y(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^2 + a_1 s + a_0} R(s) + \frac{\text{initial condition terms}}{s^2 + a_1 s + a_0}$$

$$= T(s) R(s) + \frac{\text{initial condition terms}}{s^2 + a_1 s + a_0}$$

This system is stable if and only if a_0 and $a_1 > 0$.

The characteristic polynomial of a second-order system is

$$s^2 + a_1 s + a_0 = (s - s_1)(s - s_2)$$

where the roots s_1 and s_2 are determined by the quadratic formula

$$s_1, s_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0}}{2}$$

Now consider a second-order system that is represented by the following differential equation.

$$\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0 y = b_1\frac{dr}{dt} + b_0 r$$

Solving for Y(s),

$$Y(s) = \underbrace{T(s)\,R(s)}_{\text{zero-state component}} + \underbrace{\frac{s\,y(0^-) + y'(0^-) + a_1\,y(0^-) - b_1\,r(0^-)}{s^2 + a_1 s + a_0}}_{\text{zero-input component}}$$

Overdamped Response

If the characteristic roots s_1 and s_2 are real and distinct, the natural response of the system is

$$Y_{nat.}(s) = \frac{K_1}{s-s_1} + \frac{K_2}{s-s_2}$$

$$y_{nat.}(t) = (K_1 e^{s_1 t} + K_2 e^{s_2 t})\,u(t)$$

Critically Damped Response

If the characteristic roots are equal,

$$Y_{nat.}(s) = \frac{K_1}{s-s_1} + \frac{K_2}{(s-s_1)^2}$$

$$y_{nat.}(t) = (K_1 e^{s_1 t} + K_2 t e^{s_1 t})\,u(t)$$

Underdamped Response

If the characteristic roots are complex numbers, they are complex conjugates of one another,

$$s_1, s_2 = -a \pm j\omega$$

and the natural response of the system is

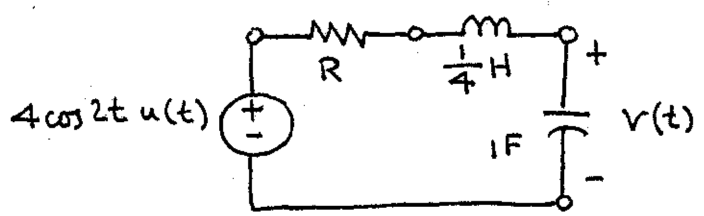
$$Y_{nat.}(s) = \frac{\text{num. poly.}}{s^2 + a_1 s + a_0} = \frac{\text{num. poly.}}{(s+a-j\omega)(s+a+j\omega)}$$

$$= \frac{\text{num. poly.}}{(s+a)^2 + \omega^2}$$

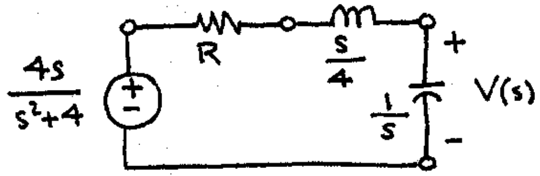
$$y_{nat.}(t) = \left[A e^{-at} \cos(\omega t + \theta) \right] u(t)$$

Example:

Find the zero-state response $v(t)$ of the following second-order circuit.



The s-domain circuit is



Using the voltage divider theorem,

$$V(s) = \frac{\frac{1}{s}}{\frac{s}{4}+R+\frac{1}{s}} \times \frac{4s}{s^2+4} = \frac{4}{s^2+4Rs+4} \times \frac{4s}{s^2+4}$$

---

when $R = \frac{5}{4}\Omega$, the <u>overdamped response</u> is

$$V(s) = \frac{4}{s^2+5s+4} \times \frac{4s}{s^2+4} = -\frac{\frac{16}{15}}{s+1} + \frac{\frac{16}{15}}{s+4} + \frac{\frac{4}{5}\angle{-90^\circ}}{s-j2} + \frac{\frac{4}{5}\angle 90^\circ}{s+j2}$$

$$v(t) = \Big[\underbrace{-\tfrac{16}{15}e^{-t} + \tfrac{16}{15}e^{-4t}}_{\text{natural response}} + \underbrace{\tfrac{8}{5}\cos(2t-90^\circ)}_{\text{forced response}}\Big]u(t)$$

---

when $R = 1\Omega$, the <u>critically damped response</u> is

$$V(s) = \frac{4}{s^2+4s+4} \times \frac{4s}{s^2+4} = -\frac{4}{(s+2)^2} + \frac{1\angle{-90^\circ}}{s-j2} + \frac{1\angle 90^\circ}{s+j2}$$

$$v(t) = \Big[\underbrace{-4te^{-2t}}_{\text{natural response}} + \underbrace{2\cos(2t-90^\circ)}_{\text{forced response}}\Big]u(t)$$

When $R = \frac{3}{4}\,\Omega$, the underdamped response is

$$V(s) = \frac{4}{s^2 + 3s + 4} \times \frac{4s}{s^2+4}$$

$$= \frac{\sqrt{\frac{256}{63}}\angle 90^\circ}{s + \frac{3}{2} - j\frac{\sqrt{7}}{2}} + \frac{\sqrt{\frac{256}{63}}\angle -90^\circ}{s + \frac{3}{2} - j\frac{\sqrt{7}}{2}} + \frac{\frac{4}{3}\angle -90^\circ}{s - j2} + \frac{\frac{4}{3}\angle 90^\circ}{s + j2}$$

$$v(t) = \Bigg[\underbrace{2\sqrt{\frac{256}{63}}\, e^{-\frac{3}{2}t} \cos\left(\frac{\sqrt{7}}{2}t + 90^\circ\right)}_{\text{natural response}} + \underbrace{\frac{8}{3}\cos\left(2t - 90^\circ\right)}_{\text{forced response}}\Bigg] u(t)$$

Example :

Find $y(t)$ for a system described by : $T(s) = \dfrac{4s-20}{s^2+4s+29}$,

$r(t) = 10\delta(t)$, $y(0^-) = 0$, and $y'(0^-) = 6$.

Solving for $Y(s)$,

$$Y(s) = T(s)\,R(s) + \frac{sy(0^-) + y'(0^-) + a_1 y(0^-) - b_1 r(0^-)}{s^2 + a_1 s + a_0}$$

$$= \frac{4s-20}{s^2+4s+29}(10) + \frac{6}{s^2+4s+29}$$

$$= \frac{40s - 194}{s^2+4s+29}$$

Expanding,

$$Y(s) = \frac{33.92\angle 53.87^\circ}{s+2-j5} + \frac{33.92\angle -53.87^\circ}{s+2+j5}$$

Therefore,

$$y(t) = \underline{\underline{67.84 e^{-2t} \cos(5t + 53.87^\circ)\, u(t)}} \leftarrow$$

Undamped Natural Frequency and Damping Ratio

Consider the following characteristic polynomial which determines the response of a second-order underdamped system.

$$s^2 + a_1 s + a_0 = (s+\sigma)^2 + \omega^2$$

The quantity σ is an exponential constant and ω is the radian frequency of oscillation.

This system can also be described in terms of the undamped radian frequency ω_n and the damping ratio ζ.

$$s^2 + a_1 s + a_0 = s^2 + 2\zeta\omega_n s + \omega_n^2$$

Comparing the two expressions,

$$\begin{cases} s^2 + 2\sigma s + (\sigma^2 + \omega^2) \\ s^2 + 2\zeta\omega_n s + \omega_n^2 \end{cases}$$

yields the following relationships

$$\boxed{\begin{aligned} \sigma &= \zeta\omega_n \\ \omega &= \omega_n\sqrt{1-\zeta^2} \end{aligned}}$$

Solving for the roots of the characteristic polynomial,

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$s = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2}$$

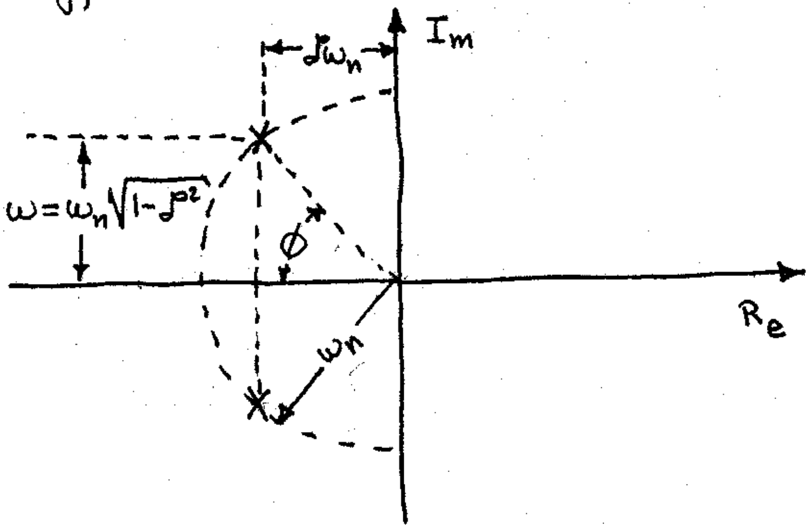
$$= -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}, \quad \zeta < 1$$

The magnitude of s is

$$|s| = \sqrt{(\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}$$

$$= \omega_n$$

Graphing,



The <u>damping angle</u> ϕ is

$$\cos\phi = \frac{\zeta\omega_n}{\omega_n} = \zeta$$

Example:

Given the following transfer function of a second-order system

$$T(s) = \frac{s^2+20}{s^2+2s+20}$$

find the undamped natural frequency ω_n, the damping ratio ζ, and the oscillation frequency ω.

Observing $T(s)$,

$$\omega_n = \sqrt{20} = \underline{\underline{4.47 \text{ rad./sec.}}} \leftarrow$$

$$2\zeta\omega_n = 2$$

$$\zeta = \frac{2}{2\omega_n} = \frac{1}{4.47} = \underline{\underline{0.224}} \leftarrow$$

$$\omega = \omega_n\sqrt{1-\zeta^2}$$

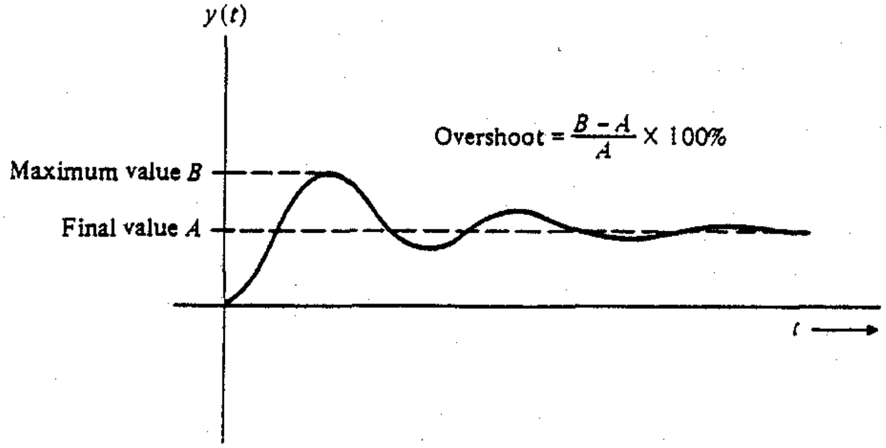
$$= 4.47\sqrt{1-(0.224)^2}$$

$$= \underline{\underline{4.36 \text{ rad./sec.}}} \leftarrow$$

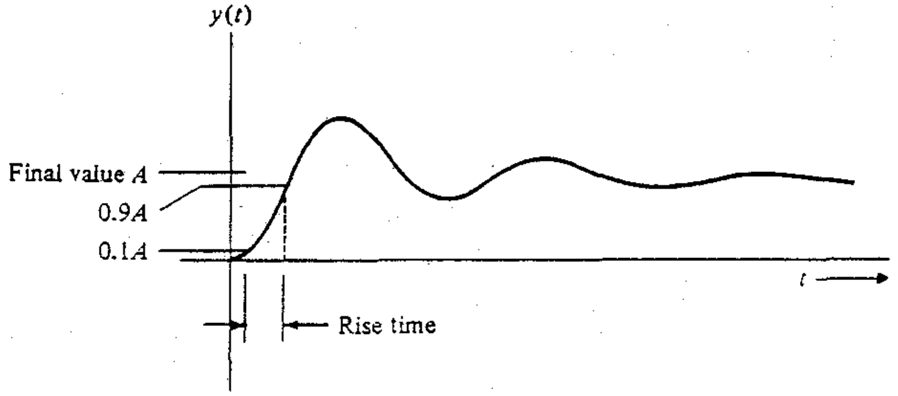
The impulse response is

$$Y(s) = \frac{s^2+20}{s^2+2s+20} = 1 + \frac{1.03\angle -167.08^\circ}{s+1-j4.36} + \frac{1.03\angle 167.08^\circ}{s+1+j4.36}$$

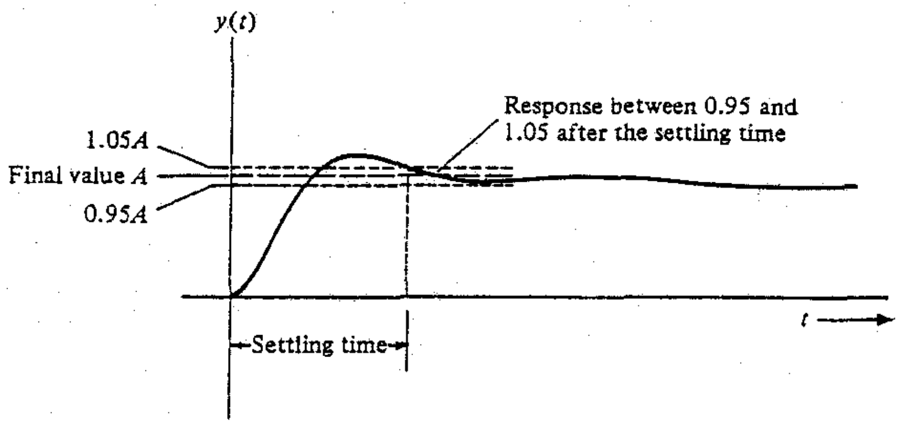
$$y(t) = \delta(t) + \left[2.06 e^{-t}\cos(\underline{\underline{4.36}}t - 167.08^\circ)\right]u(t)$$



(a)



(b)



(c)

Figure 2.9 Step response specifications. **(a)** Overshoot. **(b)** Rise time. **(c)** Settling time.

Coefficient Tests for Stability

First- and Second-Order Systems

The stability of first- and second-order systems can be determined by inspection of the coefficients of the characteristic polynomial. Both systems are stable, with all roots in the left half of the complex plane, if and only if all polynomial coefficients have the same algebraic sign.

For first-order polynomials, the proof is trivial. Now consider the following second-order polynomial with leading coefficient equal to one.

$$s^2 + a_1 s + a_0 = 0$$

Applying the quadratic formula,

$$s = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0}}{2}$$

$$= -\frac{a_1}{2} \pm \frac{\sqrt{a_1^2 - 4a_0}}{2}$$

Clearly $a_1 > 0$ is required. In addition, for real and distinct roots,

$$-a_1 + \sqrt{a_1^2 - 4a_0} < 0$$

$$\sqrt{a_1^2 - 4a_0} < a_1$$

$$a_1^2 - 4a_0 < a_1^2$$

$$-4a_0 < 0$$

$$\underline{\underline{a_0 > 0}} \leftarrow$$

Higher-Order Systems

For higher-order polynomials representing higher-order systems, the algebraic signs of the polynomial coefficients may or may not yield information as to stability. The following two conditions do result in conclusions about polynomial roots.

1. Differing algebraic signs - At least one RHP root.
2. Zero-valued coefficients - Imaginary axis roots or RHP roots or both.

Examples:

$$s^5 + 4s^4 - 3s^3 + s^2 + 7s + 10 = 0$$

roots = $0.9098 + j\,0.7826$
$0.9098 - j\,0.7826$
-1.3689
$-0.6254 + j\,0.7895$
$-0.6254 - j\,0.7895$

$$s^4 + 3s^3 + 2s + 6 = 0$$

roots = -3.0000
$0.6300 + j\,1.0911$
$0.6300 - j\,1.0911$
-1.2599

Routh – Hurwitz Test for Stability

The Routh – Hurwitz test is a numerical procedure for determining the numbers of right half-plane (RHP) and imaginary axis (IA) roots of a polynomial.

Consider the following polynomial.

$$p(s) = 4s^4 + 3s^3 + 10s^2 + 8s + 1$$

An array is now formed from the coefficients.

$$\begin{array}{c|ccc}
s^4 & 4 & 10 & 1 \\
s^3 & 3 & 8 & 0 \\
s^2 & -\frac{2}{3} & 1 & 0 \\
s^1 & \frac{25}{2} & 0 & \\
s^0 & 1 & 0 &
\end{array}$$

Elements are computed as follows.

$$-\frac{\begin{vmatrix} 4 & 10 \\ 3 & 8 \end{vmatrix}}{3} = \underline{\underline{-\frac{2}{3}}} \qquad -\frac{\begin{vmatrix} 4 & 1 \\ 3 & 0 \end{vmatrix}}{3} = \underline{\underline{1}}$$

$$-\frac{\begin{vmatrix} 3 & 8 \\ -\frac{2}{3} & 1 \end{vmatrix}}{-\frac{2}{3}} = \underline{\underline{\frac{25}{2}}} \qquad -\frac{\begin{vmatrix} -\frac{2}{3} & 1 \\ \frac{25}{2} & 0 \end{vmatrix}}{\frac{25}{2}} = \underline{\underline{1}}$$

The two sign changes in the left column indicate that $p(s)$ has two RHP roots.

Verifying,

$$4s^4 + 3s^3 + 10s^2 + 8s + 1 = 0$$

$$\text{roots} = \begin{array}{l} 0.0340 + j1.5661 \\ 0.0340 - j1.5661 \\ -0.6646 \\ -0.1533 \end{array}$$

The Routh-Hurwitz test can be used to verify conditions for stability for general polynomials. For example,

$$p(s) = s^2 + a_1 s + a_0$$

has an array of

$$\begin{array}{c|cc} s^2 & 1 & a_0 \\ s^1 & a_1 & 0 \\ s^0 & a_0 & 0 \end{array}$$

$$-\frac{\begin{vmatrix} 1 & a_0 \\ a_1 & 0 \end{vmatrix}}{a_1} = -\frac{-a_0 a_1}{a_1} = \underline{\underline{a_0}}$$

which demonstrates that $a_1 > 0$ and $a_0 > 0$ are required for stability.

Left - Column Zeroes of the Array

when left-column zeroes appear in the Routh-Hurwitz array, due to zero-valued polynomial coefficients or element calculations, the array cannot be completed in the usual way.

Consider the following polynomial.

$$p(s) = 3s^4 + 6s^3 + 2s^2 + 4s + 5$$

The array is

$$\begin{array}{c|ccc} s^4 & 3 & 2 & 5 \\ s^3 & 6 & 4 & 0 \\ s^2 & \underline{\underline{0}} & 5 & 0 \\ s^1 & & & \\ s^0 & & & \end{array}$$

Because of division by zero in the next step, the process fails.

One method to overcome this problem is to form a new polynomial of higher order by adding a known root and changing the coefficients so that a left-column zero does not occur.

For example,

$$p'(s) = (s+1)(3s^4 + 6s^3 + 2s^2 + 4s + 5)$$

$$= 3s^5 + 9s^4 + 8s^3 + 6s^2 + 9s + 5$$

Now the array becomes

$$\begin{array}{c|ccc}
s^5 & 3 & 8 & 9 \\
s^4 & 9 & 6 & 5 \\
s^3 & 6 & \frac{66}{9} & 0 \\
s^2 & -5 & 5 & 0 \\
s^1 & \frac{44}{3} & 0 & \\
s^0 & 5 & 0 &
\end{array}$$

The array now indicates the new polynomial and therefore the original polynomial has two RHP roots.

OUTDATED METHOD

Another method of solving the problem associated with left-column zeroes is to replace the zero by a small nonzero quantity ε, which will usually be positive.

The original array then becomes

$$\begin{array}{c|ccc}
s^4 & 3 & 2 & 5 \\
s^3 & 6 & 4 & 0 \\
s^2 & \varepsilon & 5 & 0 \\
s^1 & \frac{4\varepsilon-30}{\varepsilon} & 0 & \\
s^0 & 5 & 0 &
\end{array}
\longrightarrow
\begin{array}{c|c}
s^4 & 3 \\
s^3 & 6 \\
s^2 & 0^+ \\
s^1 & -\infty \\
s^0 & 5
\end{array}$$

due to the following limit.

$$\lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \frac{4\varepsilon - 30}{\varepsilon} = -\infty$$

THE FOLLOWING "ROW - SHIFTING" METHOD IS BETTER THAN THE ABOVE EPSILON METHOD.

AGAIN LET

$$p(s) = 3s^4 + 6s^3 + 2s^2 + 4s + 5$$

$$\Rightarrow \quad \begin{array}{c|ccc} s^4 & 3 & 2 & 5 \\ s^3 & 6 & 4 & \\ s^2 & \underline{\underline{0}} & 5 & \\ s^1 & & & \\ s^0 & & & \end{array}$$

s^2 ROW IS PROBLEMATIC. REPLACE IT WITH THE ORIGINAL ROW ADDED TO $(-1)^n$ TIMES THE ROW LEFT SHIFTED n TIMES UNTIL ALL ZEROS DISSAPEAR.

IN THE ABOVE EXAMPLE:

$$\begin{array}{rcc} \text{ORIGNAL ROW :} & 0 & 5 \\ (-1)^n\big|_{n=1} \times \text{ SHIFTED ROW :} & -5 & 0 \\ \hline \text{SUM :} & -5 & 5 \end{array}$$

THE ABOVE ROUTH ARRAY NOW BECOMES

$$\begin{array}{c|ccc} s^4 & 3 & 2 & 5 \\ s^3 & 6 & 4 & \\ s^2 & -5 & 5 & \\ s^1 & +10 & & \\ s^0 & 5 & & \end{array}$$

$\Rightarrow$ 2 SIGN CHANGES

Premature Termination of the Array

<u>Premature termination</u> occurs whenever a polynomial divisor of the original polynomial occurs in the Routh-Hurwitz array producing an entire row of zeroes.

Consider the following polynomial.

$$p(s) = s^5 + s^4 + 4s^3 + 24s^2 + 3s + 63$$

The Routh-Hurwitz array is

s^5	1	4	3
s^4	1	24	63
s^3	-20	-60	0
s^2	21	63	0
s^1	0	0	
s^0			

and prematurely terminates at the s^1 row.

By inspection of the s^2 row, the polynomial divisor is

$$P_{div.}(s) = 21s^2 + 63 = 21(s^2+3) = 21(s+j\sqrt{3})(s-j\sqrt{3})$$

revealing two IA roots, at $s = \pm\sqrt{3}\,j$.

Long division can be avoided by completing the array with the derivative of the divisor polynomial.

$$P_{div.}(s) = 21s^2 + 63$$

$$\frac{d\,P_{div.}(s)}{ds} = 42s$$

The array now becomes

$$\begin{array}{c|ccc} s^5 & 1 & 4 & 3 \\ s^4 & 1 & 24 & 63 \\ s^3 & -20 & -60 & 0 \\ \hline s^2 & 21 & 63 & 0 \\ s^1 & 42 & 0 & \\ s^0 & 63 & 0 & \end{array} \left.\vphantom{\begin{array}{c} s^2 \\ s^1 \\ s^0 \end{array}}\right\} \text{divisor test}$$

which indicates no RHP roots in the divisor polynomial. Therefore the divisor polynomial has two IA roots. The entire polynomial has two left-column sign changes indicating two RHP roots.

As before,

$$\text{RHP} = 2$$

$$\text{LHP} = 1$$

$$\text{IA} = 2$$

Long division now produces

$$
\begin{array}{r|l}
 & s^3 + s^2 + s + 21 \\
\hline
s^2 + 3 & s^5 + s^4 + 4s^3 + 24s^2 + 3s + 63 \\
 & s^5 \qquad + 3s^3 \\
\hline
 & \qquad s^4 + s^3 + 24s^2 + 3s + 63 \\
 & \qquad s^4 \qquad + 3s^2 \\
\hline
 & \qquad\qquad s^3 + 21s^2 + 3s + 63 \\
 & \qquad\qquad s^3 \qquad\quad + 3s \\
\hline
 & \qquad\qquad\qquad 21s^2 \qquad + 63 \\
 & \qquad\qquad\qquad 21s^2 \qquad + 63 \\
\hline
 & \qquad\qquad\qquad\qquad\qquad\quad 0
\end{array}
$$

The Routh-Hurwitz array on the resulting polynomial is

$$
\begin{array}{c|cc}
s^3 & 1 & 1 \\
s^2 & 1 & 21 \\
s^1 & -20 & 0 \\
s^0 & 21 & 0
\end{array}
$$

which indicates two RHP roots. Therefore the example polynomial has the following numbers of the various root types:

RHP = 2

LHP = 1

IA = 2

Verifying,

$$p(s) = s^5 + s^4 + 4s^3 + 24s^2 + 3s + 63$$

roots = -3.0000

$1.0000 + j\,2.4495$

$1.0000 - j\,2.4495$

$0.0000 + j\,1.7321$

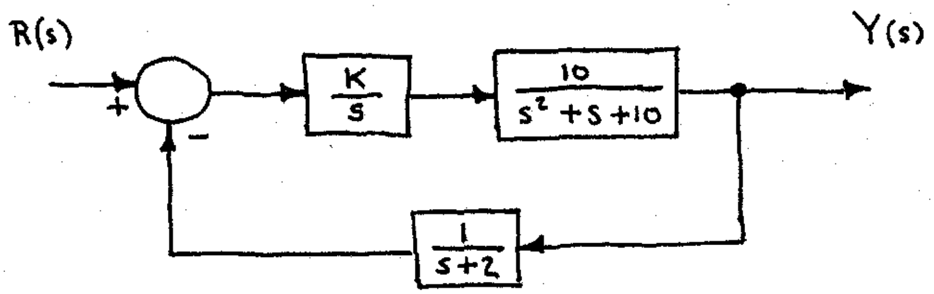
$0.0000 - j\,1.7321$

Parameter Shifting

Adjustable Systems

Many systems contain an adjustable parameter K which can be used to control stability. The Routh-Hurwitz test can be used to determine for what range of K a system is stable.

Consider the following system.



The transfer function is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{\left(\frac{K}{s}\right)\left(\frac{10}{s^2+s+10}\right)}{1 + \left(\frac{1}{s+2}\right)\left(\frac{K}{s}\right)\left(\frac{10}{s^2+s+10}\right)}$$

$$= \frac{10K(s+2)}{(s+2)(s)(s^2+s+10) + 10K}$$

$$= \frac{10Ks + 20K}{s^4 + 3s^3 + 12s^2 + 20s + 10K}$$

The Routh-Hurwitz array is

$$
\begin{array}{c|ccc}
s^4 & 1 & 12 & 10K \\
s^3 & 3 & 20 & 0 \\
s^2 & \frac{16}{3} & 10K & 0 \\
s^1 & \frac{320-90K}{16} & 0 & \\
s^0 & 10K & 0 &
\end{array}
$$

Stability requires

$$\frac{320-90K}{16} > 0$$

$$10K > 0$$

Combining,

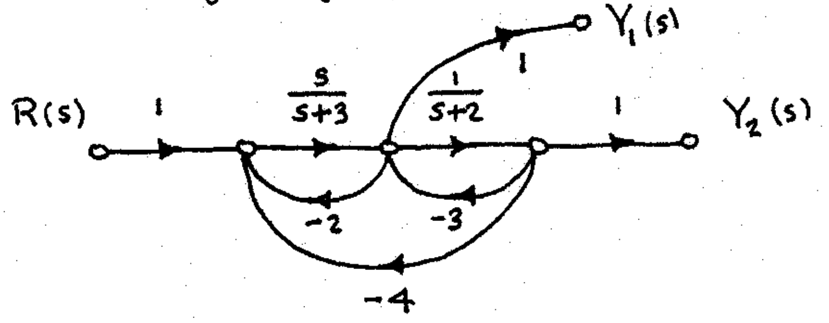
$$0 < K < \frac{32}{9} \longleftarrow$$

This range of K ensures the system will have no RHP roots and thus will be stable.

Relative Stability

The distance on the complex plane between the nearest characteristic root and the imaginary axis is termed the <u>relative stability</u> of the system. The relative stability concept is normally used only with a stable system.

Consider the following system.



The path gain for the Y_1 output is

$$P_1 = (1)\left(\frac{s}{s+3}\right)(1) = \frac{s}{s+3}$$

The three loop gains are

$$L_1 = \left(\frac{s}{s+3}\right)(-2) = \frac{-2s}{s+3}$$

$$L_2 = \left(\frac{1}{s+2}\right)(-3) = \frac{-3}{s+2}$$

$$L_3 = \left(\frac{s}{s+3}\right)\left(\frac{1}{s+2}\right)(-4) = \frac{-4s}{(s+3)(s+2)}$$

and are all touching.

The determinant of the signal flow graph is

$$\Delta = 1 - (L_1 + L_2 + L_3)$$

$$= 1 - \left\{ \left[\frac{-2s}{s+3} \right] + \left[\frac{-3}{s+2} \right] + \left[\frac{-4s}{(s+3)(s+2)} \right] \right\}$$

$$= \frac{3s^2 + 16s + 15}{(s+3)(s+2)}$$

and the cofactors are all one.

Applying Mason's gain rule,

$$T_1(s) = \frac{P_1}{\Delta} = \frac{\frac{s}{s+3}}{\frac{3s^2+16s+15}{(s+3)(s+2)}} = \frac{s(s+2)}{3s^2+16s+15}$$

For Y_2, the path gain is

$$P_2 = (1)\left(\frac{s}{s+3}\right)\left(\frac{1}{s+2}\right)(1) = \frac{s}{(s+3)(s+2)}$$

and

$$T_2(s) = \frac{P_2}{\Delta} = \frac{\frac{s}{(s+3)(s+2)}}{\frac{3s^2+16s+15}{(s+3)(s+2)}} = \frac{s}{3s^2+16s+15}$$

Therefore, the characteristic polynomial for both transfer functions is

$$p(s) = 3s^2 + 16s + 15$$

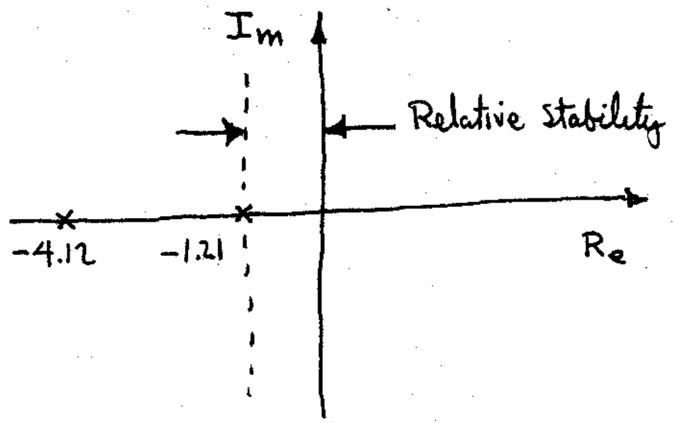
The relative stability can be determined using the quadratic formula.

$$p(s) = 3s^2 + 16s + 15$$

$$s = \frac{-16 \pm \sqrt{256 - 180}}{6}$$

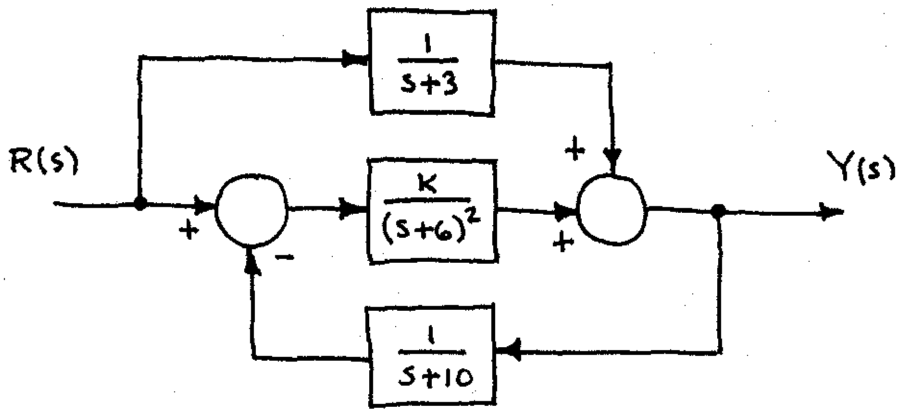
$$= \underline{\underline{-1.21}},\ -4.12$$

The relative stability of the system is 1.21.

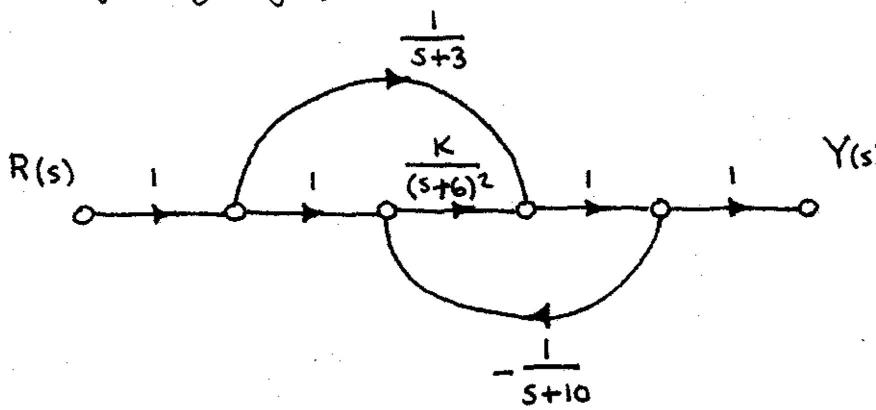


Example:

Determine the range of K that will produce a relative stability in the following system of greater than two units.



The signal flow graph is



The n = 2 path gains are

$$P_1 = \frac{K}{(s+6)^2} \qquad P_2 = \frac{1}{s+3}$$

The loop gain is

$$L_1 = \frac{-K}{(s+6)^2(s+10)}$$

and the determinant is

$$\Delta = 1 - L_1 = 1 + \frac{K}{(s+6)^2(s+10)}$$

$$= \frac{(s+6)^2(s+10)+K}{(s+6)^2(s+10)}$$

The cofactors are

$$\Delta_1 = \Delta_2 = 1$$

Using Mason's gain rule,

$$T(s) = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

$$= \frac{\dfrac{K}{(s+6)^2} + \dfrac{1}{s+3}}{\dfrac{(s+6)^2(s+10)+K}{(s+6)^2(s+10)}}$$

$$= \frac{K(s+3)(s+10) + (s+6)^2(s+10)}{(s+3)(s^3+22s^2+156s+360+K)}$$

Because one root is known $(s=-3)$ and is greater than the number of units of required relative stability, the polynomial for test purposes becomes

$$P(s) = s^3 + 22s^2 + 156s + 360 + K$$

Shifting the imaginary axis two units to the left,

$$P(\sigma) = (\sigma-2)^3 + 22(\sigma-2)^2 + 156(\sigma-2) + 360 + K$$

$$= \sigma^3 + 16\sigma^2 + 80\sigma + 128 + K$$

The Routh-Hurwitz array is

$$\begin{array}{c|cc} \sigma^3 & 1 & 80 \\ \sigma^2 & 16 & 128+K \\ \sigma^1 & \dfrac{1280-(128+K)}{16} & 0 \\ \sigma^0 & 128+K & 0 \end{array}$$

Stability requires

$$\frac{1280-(128+K)}{16} > 0$$

$$128 + K > 0$$

Combining,

$$\underline{\underline{-128 < K < 1152}} \leftarrow$$

As a check, when $K = -128$

$$p(s) = s^3 + 22s^2 + 156s + 232$$

$$\text{roots} = \begin{array}{l} -10.0000 + j\,4.0000 \\ -10.0000 - j\,4.0000 \\ -2.0000 \end{array}$$

when $K = 1152$,

$$p(s) = s^3 + 22s^2 + 156s + 1512$$

$$\text{roots} = \begin{array}{l} -18.0000 \\ -2.0000 + j\,8.9443 \\ -2.0000 - j\,8.9443 \end{array}$$

KHARITONOV'S THEOREM

CONSIDER THE NOMINAL POLYNOMIAL

$$a_{nom}(s) = a_{0,nom} + a_{1,nom}s + \dots + a_{n,nom}s^n$$

AND AN INFINITE FAMILY OF PERTURBED POLYNOMIALS

$$a_p(s) = a_0 + a_1 s + \dots + a_n s^n$$

WITH EACH a_i VARYING INSIDE THE GIVEN BOUNDS

$$0 < l_i \le a_i \le h_i \quad , \quad \forall i = [0, n]$$

THEN, THE ENTIRE FAMILY OF POLYNOMIALS IS STABLE IF AND ONLY IF FOUR EXTREME POLYNOMIALS $a_{e_1}(s)$ THROUGH $a_{e_4}(s)$ OF DEGREE n ARE STABLE

$$a_{e_1}(s) = h_0 + l_1 s + l_2 s^2 + h_3 s^3 + h_4 s^4 + l_5 s^5 + l_6 s^6 + \dots$$
$$a_{e_2}(s) = h_0 + h_1 s + l_2 s^2 + l_3 s^3 + h_4 s^4 + h_5 s^5 + l_6 s^6 + \dots$$
$$a_{e_3}(s) = l_0 + h_1 s + h_2 s^2 + l_3 s^3 + l_4 s^4 + h_5 s^5 + h_6 s^6 + \dots$$
$$a_{e_4}(s) = l_0 + l_1 s + h_2 s^2 + h_3 s^3 + l_4 s^4 + l_5 s^5 + h_6 s^6 + \dots$$

IF THE DEGREE n OF THE POLYNOMIALS IS SMALL, FEWER CORNER POLYNOMIALS ARE REQUIRED:

FOR $n = 2$ THE COEFF'S ARE REQUIRED TO BE OF LIKE SIGN

$n = 3$ ONLY $a_{e_1}(s)$ IS NEEDED

$n = 4$ ONLY $a_{e_1}(s)$ AND $a_{e_2}(s)$ ARE NEEDED

$n = 5$ ONLY $a_{e_1}(s)$, $a_{e_2}(s)$ AND $a_{e_3}(s)$ ARE NEEDED

EXAMPLE:

$$p(s) = q_0 + q_1 s + q_2 s^2 + q_3 s^3$$

WHERE

$$q_0 \in [2, 3]$$
$$q_1 \in [3, 4]$$
$$q_2 \in [4, 5]$$
$$q_3 \in [5, 6]$$

SOLUTION:

$$a_{e_1}(s) = 3 + 3s + 4s^2 + 6s^3$$
$$= 6s^3 + 4s^2 + 3s + 3$$

$$\begin{array}{c|cc} s^3 & 6 & 3 \\ s^2 & 4 & 3 \\ s^1 & -\frac{3}{2} & 0 \\ s^0 & 3 & \end{array}$$

2 SIGN CHANGES $\Rightarrow$ 2 RHP ROOTS

$\Rightarrow$ NOT ROBUSTLY STABLE