Practical Application

- In practice, time-frequency analysis is most frequently performed by estimating the statistical properties from sliding windows.
- Resulting estimates, e.g., $R(e^{j\omega}, n)$, are a function of two independent variables.
- Typically represented with three-dimensional meshes, contour plots, or pseudo-colored images.
- When used to estimate the time-varying PSD, are called spectrograms or short-time Fourier transforms (STFT).

Introduction and Motivation

\[
\begin{align*}
  r_x(\ell, n) &= E[x(n)x(n-\ell)^*] \quad \text{for } m \leq \ell \leq m \\
  R_x(e^{j\omega}, n) &= \sum_{\ell=-\infty}^{\infty} r_x(\ell, n) \\
  &\approx \sum_{\ell=-m}^{m} r_x(\ell, n)
\end{align*}
\]

- In practical applications, many signals are nonstationary.
- In order to apply the techniques, we usually assume the signal statistics change "slowly".
- "Locally" stationary.
- Rate of change depends on the process.
- Time frequency analysis tries to estimate how these properties change with time.

Example 1: Chirp Spectrogram

Apply the modified periodogram on a chirp signal with a linear frequency sweep from 1–10 Hz over a period of 20 s.
Why the Windowed Periodogram?

- Why not a smoother technique with less variance?
  - The eye can smooth the plots visually
  - Techniques to decrease variance decrease the resolution too much
- Why window the segments? Could use a rectangular window
  - If tapered window is not applied, estimate will vary even if the signal is a constant sinusoid
  - Due to variation of phase of sinusoid that is included in the segment (covered by the window)
  - Taper reduces this effect
  - Also reduces sidelobe leakage

Example 1: Relevant MATLAB Code

```matlab
fs = 20; % Sample rate (Hz)
T = 20; % Signal duration (s)
f = [1 10]; % Chirp frequencies [start stop]
Tv = [0.2 0.5 2 5 15]; % Window lengths
k = 0:T*fs; % Discrete time index
t = (k-0.5)/fs; % Sample times (s)
x = chirp(t,f(1),T,f(2));
for c1=1:length(Tw),
    tw = Tw(c1);
    [S,t,f] = NonparametricSpectrogram(x,fs,tw);figure;
    I = abs(S).^2;
    cr = prctile(reshape(I,prod(size(I)),1),[0.5 99.5]);
    h = axes('Position',[0.14 0.20 0.80 0.73]);
    h = plot(tw/2*[1 1],fs/2,'k',tw/2*[1 1],fs/2,'w');
    h = plot((T-tw/2)*[1 1],fs/2,'k',(T-tw/2)*[1 1],fs/2,'w');
    hold on;
    h = plot(tw/2*[1 1],0,'k',tw/2*[1 1],0,'w');
    h = plot((T-tw/2)*[1 1],0,'k',(T-tw/2)*[1 1],0,'w');
    hold off;
end;
```

Filter Bank Interpretation

\[
X(e^{j\omega},n) = \sum_{\ell=-\infty}^{\infty} x(n+\ell)w(\ell)e^{-j\omega\ell}
\]
\[
= \sum_{m=-\infty}^{\infty} x(m)w(m-n)e^{-j\omega(m-n)}
\]
\[
= \sum_{m=-\infty}^{\infty} x(m)w(-(n - m))e^{j\omega(n-m)}
\]
\[
\hat{R}^{(MP)}(e^{j\omega},n) = \frac{1}{L} |X(e^{j\omega},n)|^2
\]

- The spectrogram can be thought of as the squared magnitude of the output of an LTI system with impulse response \( h_w(n) = w(-n)e^{j\omega n} \)
- The magnitude response evaluated at \( \omega_0 \) is then \( W(e^{j(\omega_0-\omega)}) \)
Primary Tradeoff: Segment Length

- Primary purpose of windowing is to guarantee local stationarity
- Window shape controls tradeoff between sidelobe leakage and main lob width
- Primary parameter is the window length, $L$
- Long window
  - Poor time resolution
  - Good frequency resolution
  - Use with slowly varying signal characteristics
- Short window
  - Good time resolution
  - Poor frequency resolution
  - Use with rapidly changing signal characteristics
- Thus $L$ primarily controls the tradeoff between time and frequency resolution
Uncertainty Principle

It can be shown that
\[ \sigma_{t,x} \times \sigma_{\omega,x} \geq \frac{1}{2} \]
- In this context, it is not actually an expression of uncertainty
  - Signals with short duration have broad bandwidth
  - Signals with long duration can have narrow bandwidth
- Equivalently, if one density is narrow then the other is wide
- They can’t both be arbitrarily narrow
- In quantum mechanics, applies to probability densities of position and velocity so uncertainty is appropriate
- Not aware of an equivalent uncertainty principle for DT signals

Energy Density Functions

- Note that the windowed signal segments have finite energy
- Analogous ideas apply in continuous time
- Consider, for a moment, a continuous time windowed segment \( x(t) \)
- Defined the normalized temporal energy density function and spectral energy density function as
  \[
e_x(t) = \frac{|x(t)|^2}{\int_{-\infty}^{\infty} |x(t)|^2 \, dt}
  \quad E_x(\omega) = \frac{|X(j\omega)|^2}{\int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega}
\]
- Like pdfs, both of these functions are nonnegative and have unit area
- Can be thought of as energy density functions (edfs) in time and frequency
- Like pdfs, we can define energy moments

Bandwidth

\[
\begin{align*}
\mu_{t,x} &= \int_{-\infty}^{\infty} t \, e_x(t) \, dt \\
\mu_{\omega,x} &= \int_{-\infty}^{\infty} \omega \, E_x(\omega) \, d\omega \\
\sigma_{t,x}^2 &= \int_{-\infty}^{\infty} |t - \mu_{t,x}|^2 \, e_x(t) \, dt \\
\sigma_{\omega,x}^2 &= \int_{-\infty}^{\infty} |\omega - \mu_{\omega,x}|^2 \, E_x(\omega) \, d\omega
\end{align*}
\]
- There are many definitions of duration and bandwidth of a signal
- One pair of possible definitions is based on the second-order moments of the energy density functions
- Let us define the signal
  - Duration as the standard deviation of the temporal edf, \( \sigma_{t,x} \)
  - Bandwidth as the standard deviation of the spectral edf, \( \sigma_{\omega,x} \)
MATLAB Code

```matlab
MATLAB Code Continued

for c1=1:nL,
    l = L(c1);
    n = (-l-1)/2:(l-1)/2';
    wn = blackman(l+2);
    wn = wn(2:end-1); % Trim off the zeros
    wn = wn.*sqrt(l/sum(wn.^2)); % Scale appropriately
    x = cos(w0*n).*wn;
    ymx = max(x);
    ymn = min(x);
    yrg = ymx-ymn;
end;
```

```
MATLAB Code

NZ = 2^10;
w0 = pi/5;
L = [32 64 128]+1;

nL = length(L);
k = (0:NZ-1);''
w = k*2*pi/NZ;
kw = 1:NZ/2;
l = 1001;
wn = blackman(l);
w = wn.*sqrt(1/sum(wn.^2));''
N = (-l-1)/2:(l-1)/2';''
x = cos(w0*n).*wn;
ymx = max(x);
ynx = min(x);
yrg = ymx-ynx;
```

J. McNames Portland State University ECE 538/638 Time Frequency Analysis Ver. 1.01 1

```
MATLAB Code Continued

subplot(nL,2,c1*2-1);
h = plot(n,x);
xlim([-max(L) max(L)]);ylim([ymx-0.05*yrg ymx+0.05*yrg]);AxisLines;
box off;
ylabel(sprintf('L=%d',l));if c1==1,
title(sprintf('Windowed Sinusoid \omega_0=%5.3f rad/sample',w0));
elseif c1==nL,
xlabel('Sample Index (n)');end;
subplot(nL,2,c1*2);
h = plot(w(kw),X(kw),'r');xlim([0 pi]);ylim([0 1.05*max(X)]);
AxisLines;
ylabel('X(e^{j\omega}');box off;if c1==1,
title('Frequency Domain');
elseif c1==nL,
xlabel('Frequency (rad/sec)');end;
end;
```

J. McNames Portland State University ECE 538/638 Time Frequency Analysis Ver. 1.01 2

Attaining the Uncertainty Principle

The Gaussian function attains the uncertainty principle with equality

\[ x(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}} \quad X(j\omega) = e^{-\frac{\omega^2}{2}} \]

\[ e_x(t) = \frac{1}{\sqrt{\pi\sigma}} e^{-\frac{t^2}{\sigma^2}} \quad E_x(\omega) = \frac{1}{\sqrt{\pi\sigma}} e^{-\omega^2\sigma^2} \]

\[ \sigma_{t,x} = \frac{\sigma}{\sqrt{2}} \quad \sigma_{\omega,x} = \frac{\sigma}{\sqrt{2\sigma}} \]

\[ \sigma_{t,x}\sigma_{\omega,x} = \frac{1}{2} \]

- If this has the best duration-bandwidth product (i.e., resolution),
  why don’t we use this as a window?
**Practical Implementation of Spectrograms**

- Remove the signal mean (center the signal)
  - Apply a highpass filter, if necessary
- The MATLAB implementation is not very good
- Time and frequency resolution should not exceed that of the display
- Use sufficient zero padding and time resolution
  - Make user-specified parameters
  - Do not ask for degree of overlap - calculate it
- Edge effects
  - Notify user of the extent of edge effects
  - Handle edges elegantly (repeat last value, extrapolate by prediction, etc.)
- Often easier to visualize the square root of the PSD, rather than the PSD

---

**Example 2: Cosine Spectrogram**

Cosine Signal $T = 0.2$ s

**Example 2: Cosine Spectrogram**

Cosine Signal $T = 0.5$ s

---

**Example 2: Cosine, Impulse, and Step Spectrograms**

Repeat the previous example for a 5 Hz sinusoidal signal, the unit impulse at $t = 10$ s, and the unit step starting at $t = 10$ s.
Example 2: Impulse Spectrogram

**Impulse Signal** $T_w = 0.2 \text{ s}$

Example 2: Cosine Spectrogram

**Cosine Signal** $T_w = 2.0 \text{ s}$

Example 2: Cosine Spectrogram

**Cosine Signal** $T_w = 10.0 \text{ s}$

Example 2: Impulse Spectrogram

**Impulse Signal** $T_w = 0.5 \text{ s}$
Periodic Signals

- Periodic signals consist of frequency components only at multiples of the fundamental frequency
- If the window length is smaller than the fundamental period, these areas analyzed as discrete events (insufficient frequency resolution)
- If the window length is several times the fundamental period, these areas appear to be periodic

Example 2: Unit Step Spectrogram

Example 3: Impulse Train Spectrograms
Repeat the previous example for a 1 Hz impulse train.
Example 3: Impulse Train Spectrogram

ImpulseTrain Signal $T_w = 0.5$ s

Time (s)

Frequency (Hz)

ImpulseTrain Signal $T_w = 1.0$ s

Time (s)

Frequency (Hz)

ImpulseTrain Signal $T_w = 2.0$ s

Time (s)

Frequency (Hz)

ImpulseTrain Signal $T_w = 5.0$ s

Time (s)

Frequency (Hz)
Random Signals

- It’s important to be able to distinguish significant features from random fluctuations due to estimation error
- Recall that the periodogram had excessive variance
- Helpful to examine the spectrograms of white noise

Example 4: White Noise Spectrograms

Repeat the previous example for a white noise signal.
Example 5: Detecting Sleep Apnea

- Can obstructive sleep apnea be detected from the electrocardiogram alone?
- If possible, could build simple and cheap, portable diagnostic device
- International competition in 2000
- Data set
  - 25 recordings labelled by expert
  - 10 unlabelled recording
  - Approximately 8 hours each
- Objective: detect OSA for every minute of the 10 recordings

Example 6: Arterial Blood Pressure

Generate the spectrogram of an arterial blood pressure signal from a child in an intensive care unit (ICU) with sepsis. Interpret the figure.
Example 7: ABP and ICP

Generate the cohereogram of arterial blood pressure and intracranial pressure signals from a child with traumatic brain injury.

Example 7: ABP and ICP

Example 6: ABP Spectrogram
Other Estimators

- Time-frequency analysis is a topic that warrants an entire class itself
- There are many estimators of how power is distributed across time and frequency
- Includes wavelets (“scaleograms”)

Summary

- Moving window estimates can be used to perform time-frequency analysis of locally stationary signals
- Conceptually, is the output of a bank of overlapping bandpass filters
- Window length is the most critical parameter
  - Controls tradeoff between time and frequency resolution
  - Duration-bandwidth product is limited by the uncertainty principle
- Can apply to joint estimation techniques as well
- Very useful for observing time-varying statistical properties