Terminology

- This text distinguishes between systems and the sequences (processes) that result when a WN input is applied
- Systems: AZ, AP, PZ
- Processes
  - Moving Average (MA)
  - Autoregressive Moving-Average (ARMA)
  - Autoregressive (AR)
- The processes are assumed to have a WN input signal: \( x(n) \sim \text{WN}(0, \sigma_w^2) \)
- In some cases, the input is assumed to be a sum of sinusoids
- In this case, the output consists of line spectra
- Goal: estimate frequencies and magnitudes of spectral components

Introduction

- Many researchers use signal models to analyze stationary univariate time series
- Goal: estimate the process from which the signal was generated
- Called signal modeling
- Related to, but different from, system identification
- Popular assumptions:
  - \( x(n) \) is ergodic and WSS
  - The system is LTI and stable
  - The input signal is WN
  - The input signal is Gaussian
  - The system is a finite-order PZ system

Linear Signal Models Overview

- Introduction
- Linear nonparametric vs. parametric models
- Equivalent representations
- Spectral flatness measure
- PZ vs. ARMA models
- Wold decomposition

Nonparametric and Parametric Defined

- Nonparametric Models: LTI system models that are specified by the impulse response
  - System is completely specified by \( h(n) \)
  - Even if the system is causal, \( h(n) \) may be infinitely long in general
  - Requires an infinite number of parameters to specify
- Parametric Models: system models that can be specified with a finite number of parameters
  - Almost always finite-order AP, AZ, or PZ
  - Easier to deal with in practical applications
  - Constrains \( h(n) \) and \( H(z) \)
Nonrecursive Representation

\[ x(n) = \sum_{k=-\infty}^{\infty} h(k)w(n-k) \]
\[ r_x(\ell) = \sigma_w^2 r_h(\ell) \]
\[ R_x(z) = \sigma_w^2 H(z)H^*(z^{-1}) \]
\[ R_x(e^{j\omega}) = \sigma_w^2 |H(e^{j\omega})|^2 \]

• Note that this is a non-recursive representation
• Any LTI system can be written in this form
• The shape of \( r_x(\ell) \) and \( R_x(e^{j\omega}) \) are completely determined by the system

Nonparametric versus Parametric

• We will focus on nonparametric estimators
• These generally have greater variability but less bias than parametric estimators. Why?
• However, they do not give a compact representation of the process
• Will discuss parametric models in detail next term

Why Assume Minimum-Phase?

\[ w(n) \rightarrow h(n) \rightarrow x(n) \]

• Recall that from \( R_x(z) \) alone we cannot distinguish between minimum and non-minimum-phase systems
• This is true in general
• For any stable, finite-order ARMA process, \( \{H(z), w(n)\} \) where \( H(z) \) has no zeros or poles on the unit circle, there exists a white noise process \( \bar{w}(n) \) such that \( X(z) = H_{\text{min}}(z)\bar{W}(z) \) and \( H_{\text{min}}(z) \) is minimum-phase
• If we have no other information but can only observe the signal, there is no reason not to assume \( h(n) \) is stable and minimum-phase!
• Very important assumption
• If \( w(n) \) is IID, it is not true in general that \( \bar{w}(n) \) is IID
Recursive Representation

\[ w(n) \rightarrow h(n) \rightarrow x(n) \rightarrow h_1(n) \rightarrow w(n) \]

\[ w(n) = \sum_{k=0}^{\infty} h_1(k)x(n-k) = x(n) + \sum_{k=1}^{\infty} h_1(k)x(n-k) \]

\[ x(n) = w(n) - \sum_{k=1}^{\infty} h_1(k)x(n-k) \]

- Let us choose (without loss of generality) that \( h(0) = 1 \)
- If we assume the inverse system is causal and stable, then \( h_1(0) = 1 \) (why?)
- In this case, the output is a function of the current (unknown) input and all past values of \( x(n) \)
- This is identical to an infinite-order \( (P = \infty) \) AP model!
- Equivalent representation of the nonrecursive form

Innovations Representation

If we assume \( H(z) \) is minimum-phase (reasonable), both \( h(n) \) and \( h_1(n) \) exist and both are causal and stable.

\[ x(n) = \sum_{k=0}^{\infty} h(k)w(n-k) = \sum_{k=-\infty}^{n} h(n-k)w(k) \]

\[ x(n+1) = w(n+1) + \sum_{k=-\infty}^{n} h(n+1-k)w(k) \]

\[ x(n+1) = \underbrace{w(n+1)}_{\text{New Information}} + \sum_{k=-\infty}^{n} h(n+1-k) \left( \sum_{j=-\infty}^{n} h_1(k-j)x(j) \right) \]

Past values of \( x(n) \)

Comments on Innovations Representation

\[ x(n+1) = \underbrace{w(n+1)}_{\text{New Information}} + \sum_{k=-\infty}^{n} h(n+1-k) \left( \sum_{j=-\infty}^{n} h_1(k-j)x(j) \right) \]

- If the system generating \( x(n) \) is minimum-phase, \( w(n+1) \) carries all the new information needed to generate \( x(n+1) \)
- Thus, \( w(n+1) \) is sometimes called the innovation
- All other information can be predicted from past observations of the output
- Only holds if \( h(n) \) is minimum-phase
- In some contexts, \( h(n) \) is called the synthesis or coloring filter
- The inverse is called the analysis or whitening filter

PZ, AZ, and AP Representations

- We just saw that a nonparametric model can be represented as either white noise driving an AZ(\( \infty \)) system or a PZ(\( \infty \)) system
- Any causal PZ, AZ, or PZ system with finite order can be represented as either a causal AZ(\( \infty \)) or PZ(\( \infty \)) system
- If the system is stable, then \( h(n) \rightarrow 0 \) as \( n \rightarrow \infty \)
- Thus, if the order of the system is sufficiently large, we can represent any of these systems accurately with an AZ(\( Q \)) or AP(\( P \)) system
- We’ll see next term that AP(\( P \)) are much easier to estimate than PZ(\( Q, P \)) or AZ(\( Q \)) systems
- Thus, it is very good news that AP(\( P \)) systems can represent any PZ(\( Q, P \)) or AZ(\( Q \)) system if \( P \) is large enough
- See Example 4.2.1 in the text and discussion in preceding paragraph
Spectral Flatness

\[ \text{SFM}_x \triangleq \frac{\exp \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln |R_x(e^{j\omega})| \, d\omega \right)}{\sqrt{\frac{1}{2\pi} \int_{-\pi}^{\pi} R_x(e^{j\omega}) \, d\omega}} \]

- Single measure of the spectral flatness
- Bounded:
  \[ 0 \leq \text{SFM}_x \leq 1 \]
- If \( \text{SFM}_x = 1 \), then \( x(n) \) is a white process
- Numerator is the geometric mean, denominator is the arithmetic mean

Cepstrum

The cepstrum is the inverse Fourier transform of \( \ln R_x(e^{j\omega}) \)

\[ c(k) \triangleq \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln |R_x(e^{j\omega})| \, e^{jk\omega} \, d\omega \]

- The minimum-phase component of the spectrum is the causal part
  \[ c_+(k) \triangleq \frac{1}{2} c(0) + c(k) u(k-1) \]
  \[ h_{\text{min}}(n) = \mathcal{F}^{-1} \{ \exp \mathcal{F} \{ c_+(k) \} \} \]
- The maximum-phase component is the anticausal part
  \[ c_-(k) \triangleq \frac{1}{2} c(0) + c(k) u(-k-1) \]
  \[ h_{\text{max}}(n) = \mathcal{F}^{-1} \{ \exp \mathcal{F} \{ c_-(k) \} \} \]
- This is rarely used in practice.
- If \( R_x(z) \) is a rational function, spectral factorization is straightforward

Parametric Signal Models

\[ \sum_{k=0}^{P} a_k x(n-k) = \sum_{k=0}^{Q} b_k w(n-k) \]

\[ H(z) = \frac{X(z)}{W(z)} = \frac{\sum_{k=0}^{Q} b_k z^{-k}}{\sum_{k=0}^{P} a_k z^{-k}} = \frac{B(z)}{A(z)} \]

- Parametric models have rational (finite-order) system functions
- Each can be specified by a linear constant-coefficient difference equation
- To make the specifications unique, we always set \( a_0 = 1 \) and usually \( b_0 = 1 \)
- The model is then defined by \( \{a_1, a_2, \ldots, a_P, b_1, \ldots, b_Q, \sigma^2_w\} \)
Parametric Signal Model Theory

The models are generally divided into three categories
- **Moving-Average Model**: MA($Q$), $P = 0$
- **Autoregressive Model**: AR($P$), $Q = 0$
- **Autoregressive Moving-Average Model**: ARMA($P$, $Q$)

All models assume the systems are BIBO stable
In general, when estimating, we constrain ourselves to minimum-phase systems

WOLD Decomposition

A general stationary random process can be written as $x(n) = x_r(n) + x_p(n)$ where $x_r(n)$ is a regular process with a continuous PSD and $x_p(n)$ is a predictable process with a discrete spectrum. Further

$$E[x_r(n_1)x_p^*(n_2)] = 0 \quad \forall n_1, n_2$$

In general, stationary random processes consist of a continuous PSD $R_x(e^{j\omega})$ and a discrete power spectrum with DTFS coefficients $R_x(k)$

These processes are called **mixed**
Continuous PSD is due to regular processes (unpredictable)
Discrete is due to harmonic or almost periodic processes (predictable)
The autocorrelation is $r_x(\ell) = r_{x_r}(\ell) + r_{x_p}(\ell)$
Proof is very difficult (see references in text)