Tableau Theorem Prover for Intuitionistic Propositional Logic

Larry Diehl

Portland State University

CS 510 - Mathematical Logic and Programming Languages

- If $\neg A$ is contradictory in all paths, then $A \vee \neg A$ lets us conclude A is a *tautology*.
- For *satisfiability*, running tableau on A yield a (classical model) evaluation context σ.
- Tableau seems awfully tied to classical logic, is intuitionistic tableau doomed!?

Classical vs Intuitionistic Logic

Classical Logic

- The *meaning* of a proposition is its truth value.
- **Satisfiability:** Does evaluating it yield true?
- $A \vee \neg A$
- $\neg\neg A \supset A$
- $A \supset \neg\neg A$

Intuitionistic Logic

- The *meaning* of a proposition is its constructive content.
- **Satisfiability:** Can you write it as a program?
- $A \supset \neg\neg A$

Proof Theory for Intuitionistic Logic

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B} \supset_I \qquad \frac{\Gamma \vdash A \supset B \quad \Gamma \vdash A}{\Gamma \vdash B} \supset_E$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge_I \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \wedge_{E_1} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \wedge_{E_2}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \vee_{I_1} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \vee_{I_2} \qquad \frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C} \vee_E$$

$$\frac{\Gamma, A \vdash \bot}{\Gamma \vdash \neg A} \neg_I \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash \neg A}{\Gamma \vdash \bot} \neg_E$$

$$\frac{}{\Gamma \vdash \top} \top_I \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash A} \bot_E$$

Proof Theory for Classical Logic

... intuitionistic rules plus ...

$$\frac{\Gamma \vdash A}{\Gamma \vdash \neg\neg A}\ \neg\neg_I \qquad \frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A}\ \neg\neg_E$$

...or...

$$\frac{}{\Gamma \vdash A \vee \neg A}$$

Model Theory for Classical Logic

Boolean Algebra $\langle \mathbb{B}, \text{false}, \text{true}, \&\&, ||, ! \rangle$
Classical truth is a boolean value.

Satisfiability

$$\sigma \vDash A \;\Leftrightarrow\; \sigma \triangleright A \equiv \text{true}$$
$$\sigma \nvDash A \;\Leftrightarrow\; \sigma \triangleright A \equiv \text{false}$$

Evaluation

$$\begin{aligned}
\sigma \triangleright p &\Leftrightarrow \sigma\ p \\
\sigma \triangleright A \wedge B &\Leftrightarrow \sigma \triangleright A \;\&\&\; \sigma \triangleright B \\
\sigma \triangleright A \vee B &\Leftrightarrow \sigma \triangleright A \;||\; \sigma \triangleright B \\
\sigma \triangleright A \supset B &\Leftrightarrow !(\sigma \triangleright A) \;||\; \sigma \triangleright B \\
\sigma \triangleright \neg A &\Leftrightarrow !(\sigma \triangleright A)
\end{aligned}$$

Kripke Model $\langle \mathbb{C}, \leq, \varnothing, \Vdash \rangle$
Intuitionistic truth is constructive evidence, or a program.

Forcing (intuitionistic satisfiability)

$$
\begin{aligned}
\Gamma \Vdash p \;&\Leftrightarrow\; \Gamma \Vdash^{p} p \\
\Gamma \Vdash A \wedge B \;&\Leftrightarrow\; \Gamma \Vdash A \times \Gamma \Vdash B \\
\Gamma \Vdash A \vee B \;&\Leftrightarrow\; \Gamma \Vdash A \;\uplus\; \Gamma \Vdash B \\
\Gamma \Vdash A \supset B \;&\Leftrightarrow\; \Gamma \leq \Delta \Rightarrow \Delta \Vdash A \Rightarrow \Delta \Vdash B \\
\Gamma \Vdash \neg A \;&\Leftrightarrow\; \Gamma \leq \Delta \Rightarrow \Delta \Vdash A \Rightarrow \bot \\
\Gamma \nVdash A \;&\Leftrightarrow\; \Gamma \Vdash \neg A
\end{aligned}
$$

Classical vs Intuitionistic Model Theory

Many more intuitionistic models than classical models because intuitionistic implication and negation allow arbitrary intrinsically distinct functions.

Much bigger search space for an intuitionistic theorem prover!

Evaluation

$$\sigma \triangleright A \supset B \;\Leftrightarrow\; !(\sigma \triangleright A) \;||\; \sigma \triangleright B$$

$$\sigma \triangleright \neg A \;\Leftrightarrow\; !(\sigma \triangleright A)$$

Forcing

$$\Gamma \Vdash A \supset B \;\Leftrightarrow\; \Gamma \leq \Delta \Rightarrow \Delta \Vdash A \Rightarrow \Delta \Vdash B$$

$$\Gamma \Vdash \neg A \;\Leftrightarrow\; \Gamma \leq \Delta \Rightarrow \Delta \Vdash A \Rightarrow \bot$$

$$\frac{S, \mathrm{T}(A \wedge B)}{S, \mathrm{T}A, \mathrm{T}B}\ \mathrm{T}\wedge \qquad \frac{S, \mathrm{F}(A \wedge B)}{S, \mathrm{F}A \mid S, \mathrm{F}B}\ \mathrm{F}\wedge$$

$$\frac{S, \mathrm{T}(A \vee B)}{S, \mathrm{T}A \mid S, \mathrm{T}B}\ \mathrm{T}\vee \qquad \frac{S, \mathrm{F}(A \vee B)}{S, \mathrm{F}A, \mathrm{F}B}\ \mathrm{F}\vee$$

$$\frac{S, \mathrm{T}(A \supset B)}{S, \mathrm{F}A \mid S, \mathrm{T}B}\ \mathrm{T}\supset \qquad \frac{S, \mathrm{F}(A \supset B)}{S_{\mathrm{T}}, \mathrm{T}A, \mathrm{F}B}\ \mathrm{F}\supset$$

$$\frac{S, \mathrm{T}(\neg A)}{S, \mathrm{F}A}\ \mathrm{T}\neg \qquad \frac{S, \mathrm{F}(\neg A)}{S_{\mathrm{T}}, \mathrm{T}A}\ \mathrm{F}\neg$$

Intuitionistic Tableau Calculus

$$S_{\mathrm{T}} \Leftrightarrow \{\mathrm{T}A \mid \mathrm{T}A \in S\}$$

$$\frac{S, \mathrm{T}(A \wedge B)}{S, \mathrm{T}A, \mathrm{T}B}\ \mathrm{T}\wedge \qquad \frac{S, \mathrm{F}(A \wedge B)}{S, \mathrm{F}A \mid S, \mathrm{F}B}\ \mathrm{F}\wedge$$

$$\frac{S, \mathrm{T}(A \vee B)}{S, \mathrm{T}A \mid S, \mathrm{T}B}\ \mathrm{T}\vee \qquad \frac{S, \mathrm{F}(A \vee B)}{S, \mathrm{F}A, \mathrm{F}B}\ \mathrm{F}\vee$$

$$\frac{S, \mathrm{T}(A \supset B)}{S, \mathrm{F}A \mid S, \mathrm{T}B}\ \mathrm{T}\supset \qquad \frac{S, \mathrm{F}(A \supset B)}{S_{\mathrm{T}}, \mathrm{T}A, \mathrm{F}B}\ \mathrm{F}\supset$$

$$\frac{S, \mathrm{T}(\neg A)}{S, \mathrm{F}A}\ \mathrm{T}\neg \qquad \frac{S, \mathrm{F}(\neg A)}{S_{\mathrm{T}}, \mathrm{T}A}\ \mathrm{F}\neg$$

Classical Tableau Interpretation

Gradually build an evaluation context σ for A (such that $\sigma \vDash A$), until tableau is finished or the model is contradictory.

Judgments

- $\mathrm{T}A$ means A is true in the model.
- $\mathrm{F}A$ means A is false in the model.

Inference Rules

If the premise is true, then the conclusion is true.

Intuitionistic Tableau Interpretation

Gradually build a "proof" of A (an "element" of $\Gamma \Vdash A$), until tableau is finished or the model is contradictory.

Judgments

- $\mathrm{T}A$ means we have a proof of A.
- $\mathrm{F}A$ means A we do not (yet) have a proof of A.

Inference Rules

- If the premise is true, then the conclusion may be true.
- The conclusion is logically consistent with the premise.

Intuitionistic Tableau Calculus

$$S_{\mathrm{T}} \;\Leftrightarrow\; \{\mathrm{T}A \mid \mathrm{T}A \in S\}$$

$$\frac{S, \mathrm{T}(A \wedge B)}{S, \mathrm{T}A, \mathrm{T}B}\ \mathrm{T}\wedge \qquad \frac{S, \mathrm{F}(A \wedge B)}{S, \mathrm{F}A \mid S, \mathrm{F}B}\ \mathrm{F}\wedge$$

$$\frac{S, \mathrm{T}(A \vee B)}{S, \mathrm{T}A \mid S, \mathrm{T}B}\ \mathrm{T}\vee \qquad \frac{S, \mathrm{F}(A \vee B)}{S, \mathrm{F}A, \mathrm{F}B}\ \mathrm{F}\vee$$

$$\frac{S, \mathrm{T}(A \supset B)}{S, \mathrm{F}A \mid S, \mathrm{T}B}\ \mathrm{T}\supset \qquad \frac{S, \mathrm{F}(A \supset B)}{S_{\mathrm{T}}, \mathrm{T}A, \mathrm{F}B}\ \mathrm{F}\supset$$

$$\frac{S, \mathrm{T}(\neg A)}{S, \mathrm{F}A}\ \mathrm{T}\neg \qquad \frac{S, \mathrm{F}(\neg A)}{S_{\mathrm{T}}, \mathrm{T}A}\ \mathrm{F}\neg$$

Closed Example $A \supset A$

$$[\{\mathrm{F}(A \supset A)\}],$$
$$[\{\mathrm{T}A, \mathrm{F}A\}].$$

$$\frac{S, \mathrm{T}(A \wedge B)}{S, \mathrm{T}A, \mathrm{T}B}\ \mathrm{T}\wedge \qquad \frac{S, \mathrm{F}(A \wedge B)}{S, \mathrm{F}A \mid S, \mathrm{F}B}\ \mathrm{F}\wedge$$

$$\frac{S, \mathrm{T}(A \vee B)}{S, \mathrm{T}A \mid S, \mathrm{T}B}\ \mathrm{T}\vee \qquad \frac{S, \mathrm{F}(A \vee B)}{S, \mathrm{F}A, \mathrm{F}B}\ \mathrm{F}\vee$$

$$\frac{S, \mathrm{T}(A \supset B)}{S, \mathrm{F}A \mid S, \mathrm{T}B}\ \mathrm{T}\supset \qquad \frac{S, \mathrm{F}(A \supset B)}{S_{\mathrm{T}}, \mathrm{T}A, \mathrm{F}B}\ \mathrm{F}\supset$$

$$\frac{S, \mathrm{T}(\neg A)}{S, \mathrm{F}A}\ \mathrm{T}\neg \qquad \frac{S, \mathrm{F}(\neg A)}{S_{\mathrm{T}}, \mathrm{T}A}\ \mathrm{F}\neg$$

Closed Example $A \supset (A \wedge A)$

$$[\{\mathrm{F}(A \supset (A \wedge A))\}],$$
$$[\{\mathrm{T}A, \mathrm{F}(A \wedge A))\}],$$
$$[\{\mathrm{T}A, \mathrm{F}A\}, \{\mathrm{T}A, \mathrm{F}A\}].$$

$$\frac{S, \mathrm{T}(A \wedge B)}{S, \mathrm{T}A, \mathrm{T}B}\ \mathrm{T}\wedge \qquad \frac{S, \mathrm{F}(A \wedge B)}{S, \mathrm{F}A \mid S, \mathrm{F}B}\ \mathrm{F}\wedge$$

$$\frac{S, \mathrm{T}(A \vee B)}{S, \mathrm{T}A \mid S, \mathrm{T}B}\ \mathrm{T}\vee \qquad \frac{S, \mathrm{F}(A \vee B)}{S, \mathrm{F}A, \mathrm{F}B}\ \mathrm{F}\vee$$

$$\frac{S, \mathrm{T}(A \supset B)}{S, \mathrm{F}A \mid S, \mathrm{T}B}\ \mathrm{T}\supset \qquad \frac{S, \mathrm{F}(A \supset B)}{S_{\mathrm{T}}, \mathrm{T}A, \mathrm{F}B}\ \mathrm{F}\supset$$

$$\frac{S, \mathrm{T}(\neg A)}{S, \mathrm{F}A}\ \mathrm{T}\neg \qquad \frac{S, \mathrm{F}(\neg A)}{S_{\mathrm{T}}, \mathrm{T}A}\ \mathrm{F}\neg$$

Open Example $A \vee \neg A$

$$[\{\mathrm{F}(A \vee \neg A)\}],$$
$$[\{\mathrm{F}A, \mathrm{F}(\neg A)\}],$$
$$[\{\mathrm{T}A\}].$$

$$\frac{S, \mathrm{T}(A \wedge B)}{S, \mathrm{T}A, \mathrm{T}B}\ \mathrm{T}\wedge \qquad \frac{S, \mathrm{F}(A \wedge B)}{S, \mathrm{F}A \mid S, \mathrm{F}B}\ \mathrm{F}\wedge$$

$$\frac{S, \mathrm{T}(A \vee B)}{S, \mathrm{T}A \mid S, \mathrm{T}B}\ \mathrm{T}\vee \qquad \frac{S, \mathrm{F}(A \vee B)}{S, \mathrm{F}A, \mathrm{F}B}\ \mathrm{F}\vee$$

$$\frac{S, \mathrm{T}(A \supset B)}{S, \mathrm{F}A \mid S, \mathrm{T}B}\ \mathrm{T}\supset \qquad \frac{S, \mathrm{F}(A \supset B)}{S_{\mathrm{T}}, \mathrm{T}A, \mathrm{F}B}\ \mathrm{F}\supset$$

$$\frac{S, \mathrm{T}(\neg A)}{S, \mathrm{F}A}\ \mathrm{T}\neg \qquad \frac{S, \mathrm{F}(\neg A)}{S_{\mathrm{T}}, \mathrm{T}A}\ \mathrm{F}\neg$$

Closed Example $A \supset (A \supset B) \supset B$

$$[\{\mathrm{F}(A \supset (A \supset B) \supset B)\}],$$
$$[\{\mathrm{T}A, \mathrm{F}((A \supset B) \supset B)\}],$$
$$[\{\mathrm{T}A, \mathrm{T}(A \supset B), \mathrm{F}B\}],$$
$$[\{\mathrm{T}A, \mathrm{F}A, \mathrm{F}B\}, \{\mathrm{T}A, \mathrm{T}B, \mathrm{F}B\}].$$

$$\frac{S, \mathrm{T}(A \wedge B)}{S, \mathrm{T}A, \mathrm{T}B}\ \mathrm{T}\wedge \qquad \frac{S, \mathrm{F}(A \wedge B)}{S, \mathrm{F}A \mid S, \mathrm{F}B}\ \mathrm{F}\wedge$$

$$\frac{S, \mathrm{T}(A \vee B)}{S, \mathrm{T}A \mid S, \mathrm{T}B}\ \mathrm{T}\vee \qquad \frac{S, \mathrm{F}(A \vee B)}{S, \mathrm{F}A, \mathrm{F}B}\ \mathrm{F}\vee$$

$$\frac{S, \mathrm{T}(A \supset B)}{S, \mathrm{F}A \mid S, \mathrm{T}B}\ \mathrm{T}\supset \qquad \frac{S, \mathrm{F}(A \supset B)}{S_{\mathrm{T}}, \mathrm{T}A, \mathrm{F}B}\ \mathrm{F}\supset$$

$$\frac{S, \mathrm{T}(\neg A)}{S, \mathrm{F}A}\ \mathrm{T}\neg \qquad \frac{S, \mathrm{F}(\neg A)}{S_{\mathrm{T}}, \mathrm{T}A}\ \mathrm{F}\neg$$

Classical vs Intuitionistic Tableau Search

When looking for a closed tableau:

Classical

You can prioritize **any** rule to apply to S to shrink the search space.

Intuitionistic

You must try applying **all** rules to S, but can still prioritize some and backtrack if they fail.

"Open" Example $\neg A \supset \neg A$

$$[\{\mathrm{F}(\neg A \supset \neg A)\}],$$
$$[\{\mathrm{T}(\neg A), \mathrm{F}(\neg A)\}],$$
$$[\{\mathrm{F}A, \mathrm{F}(\neg A)\}],$$
$$[\{\mathrm{T}A\}].$$

$$\frac{S, \mathrm{T}(A \wedge B)}{S, \mathrm{T}A, \mathrm{T}B}\ \mathrm{T}\wedge \qquad \frac{S, \mathrm{F}(A \wedge B)}{S, \mathrm{F}A \mid S, \mathrm{F}B}\ \mathrm{F}\wedge$$

$$\frac{S, \mathrm{T}(A \vee B)}{S, \mathrm{T}A \mid S, \mathrm{T}B}\ \mathrm{T}\vee \qquad \frac{S, \mathrm{F}(A \vee B)}{S, \mathrm{F}A, \mathrm{F}B}\ \mathrm{F}\vee$$

$$\frac{S, \mathrm{T}(A \supset B)}{S, \mathrm{F}A \mid S, \mathrm{T}B}\ \mathrm{T}\supset \qquad \frac{S, \mathrm{F}(A \supset B)}{S_{\mathrm{T}}, \mathrm{T}A, \mathrm{F}B}\ \mathrm{F}\supset$$

$$\frac{S, \mathrm{T}(\neg A)}{S, \mathrm{F}A}\ \mathrm{T}\neg \qquad \frac{S, \mathrm{F}(\neg A)}{S_{\mathrm{T}}, \mathrm{T}A}\ \mathrm{F}\neg$$

Closed Example $\neg A \supset \neg A$

$$
\begin{aligned}
&[\{\mathrm{F}(\neg A \supset \neg A)\}],\\
&[\{\mathrm{T}(\neg A), \mathrm{F}(\neg A)\}],\\
&[\{\mathrm{T}(\neg A), \mathrm{T}A\}],\\
&[\{\mathrm{F}A, \mathrm{T}A\}].
\end{aligned}
$$

$$\frac{S, \mathrm{T}(A \wedge B)}{S, \mathrm{T}A, \mathrm{T}B}\ \mathrm{T}\wedge \qquad \frac{S, \mathrm{F}(A \wedge B)}{S, \mathrm{F}A \mid S, \mathrm{F}B}\ \mathrm{F}\wedge$$

$$\frac{S, \mathrm{T}(A \vee B)}{S, \mathrm{T}A \mid S, \mathrm{T}B}\ \mathrm{T}\vee \qquad \frac{S, \mathrm{F}(A \vee B)}{S, \mathrm{F}A, \mathrm{F}B}\ \mathrm{F}\vee$$

$$\frac{S, \mathrm{T}(A \supset B)}{S, \mathrm{F}A \mid S, \mathrm{T}B}\ \mathrm{T}\supset \qquad \frac{S, \mathrm{F}(A \supset B)}{S_{\mathrm{T}}, \mathrm{T}A, \mathrm{F}B}\ \mathrm{F}\supset$$

$$\frac{S, \mathrm{T}(\neg A)}{S, \mathrm{F}A}\ \mathrm{T}\neg \qquad \frac{S, \mathrm{F}(\neg A)}{S_{\mathrm{T}}, \mathrm{T}A}\ \mathrm{F}\neg$$

Using Classical vs Intuitionistic Tableau

Classical

To show that A is true:

1. Assume that A is false.
2. Build a tableau for $\neg A$.
3. If some sub-proposition is true and false, A must be true.

Intuitionistic

To show that A is provable:

1. Assume that A has not been proven.
2. Build a tableau for $\neg A$.
3. If some sub-proposition is proven and not yet proven, it must be impossible that A has not been proven.

Classical vs Intuitionistic Tableau Soundness

Classical

Have a model σ from the tableau *conclusion*, so **check** that $\sigma \vDash A$.

Intuitionistic

Have a tableau *derivation* of A, so **construct** an element of $\Gamma \Vdash A$.

Intuitionistic Tableau Soundness

Theorem

Have a tableau *derivation* of A, so **construct** an element of $\Gamma \Vdash A$.

Fitting's Proof

By showing the contrapositive.
Sadly, $(\neg B \supset \neg A) \not\Rightarrow (A \supset B)$ intuitionistically.

Classical Tableau Book

First Order Logic - Smullyan'68

Intuitionistic Tableau Book

Intuitionistic Logic: Model Theory and Forcing - Fitting'69

Intuitionistic Tableau Optimization Papers

- *An O(n log n)-Space Decision Procedure for Intuitionistic Propositional Logic* - Hudelmaier'93
- *A Tableau Decision Procedure for Propositional Intuitionistic Logic* - Avellone et. al.'06