Consistency and Completeness of Tableau

Consistency

- Every term proveable by the tableau method is a tautology.
- Build the tree for ~p, show that every branch is closed
- Then the starting term p is a tautology

Build a tableau by the rules

- If it is closed, then it must be a tautology

```
tabTree [] tree = tree
tabTree (x:xs) tree =
  case discrim x of
    Lit p -> tabTree xs tree
    Alpha a b -> tabTree (a:b:xs)
                  (extendTree (double a b) tree)
    Beta a b -> extendTree
                  (Branch
                    (tabTree (a:xs)(single a))
                    (tabTree (b:xs) (single b)))
                  tree
```

Branches in a Tableau Tree

- A tableau tree has a number of branches
- Let v be an assigment to all the variables mentioned any where in the tree.
- A branch is defined to be True under v, if every term on the branch is True under v.
- A tableau is true under v, if some branch of its tree is true under v

Property of algorithm

- Note that in every case, the tree grows by extending the existing tree
- -- invariant: elements of the list are in
- -- the tree but not yet "used"

```
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                  tree
```

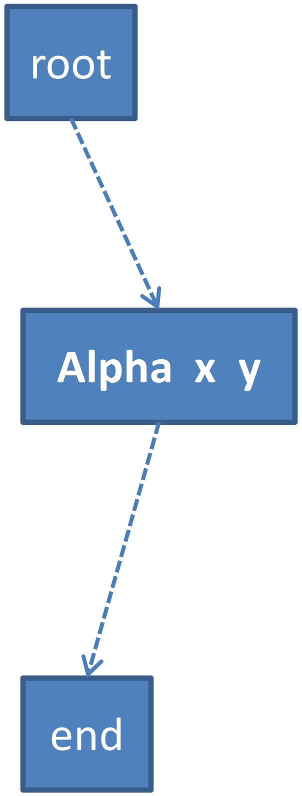
Strategy

- Show that if a tree T is true, and it is extended by the rules of the algorithm, then the new tree is true too!
- Recall a tree is only extended by examining some node already in the tree.
- Thus that node must already be true!

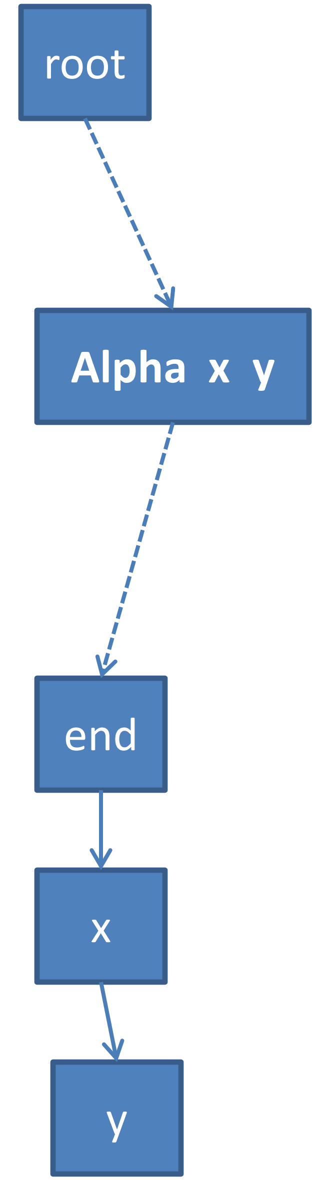
Where is the tree extended?

- Recall the tree is true, so at least one of its paths is true, call this path A
- The tree is extended along some path, call it B.
  - If B is distinct from A, then the new node does not affect path A, and so the whole Tree is still True.
  - If B is the same path as A then we must conside the two cases that are possible. The Alpha and Beta cases

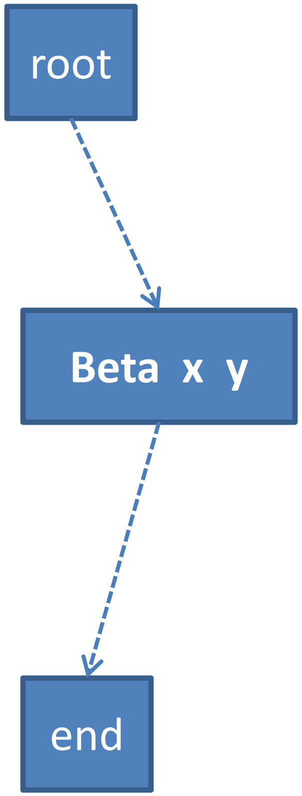
Alpha case



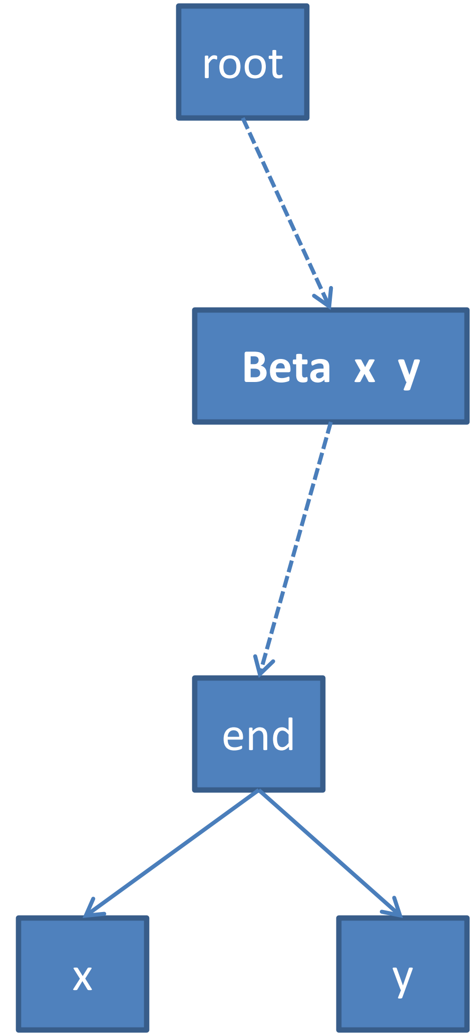
We know (Alpha x y) is True, so by prop1 both x and y are true, so the the new path on the extended tree is also True.



Beta case



We know (Beta x y) is True, so by prop1, either x or y are true, There are 2 new paths on the extended tree. One of which must be true so the tree remains True.



By induction on the number of steps

- If the initial tree node is True, then the tree returned will also be True.
- A closed tableau cannot be true (since every path has at least one conjugate pair), thus the original root node must be unsatisfiable.
- But the original node was `~p`

```
 solveT p =
    (tabTree [NotP p]
             (single (NotP p)))
```

- So `p` must be a tautology.

Completeness

- Here we must show that every tautology has a a closed tableau tree
- And that the algorithm will find it.
- This is about being sure we have enough rules to complete a closed tableau for every kind of formula.

- If X is a tautology, will every complete tableau for ~X close?

Definition of complete path

- Consider a path in a tableau: $P = p_1\ p_2\ ..\ p_n$
- We say P is *complete,* if for every p_i,
  - if p_i is an (Alpha x y) then both x and y are in the path
  - If p_i is a (Beta x y), then either x is in P or y is in P
- *completed,* if every path is either *closed* or *complete*
- The algorithm always constructs complete paths

Strategy

- Let T be a tableau
- If T is an open completed Tableau
  - i.e. T is completed, but at least one path is still open
- Then the root (or origin) of T is satisfiable. I.e. we can extend the open path (in fact we can extend all the open paths) to keep the root satisfiable.

Theorem

- Let P be an open complete path in T
- Let S be the set of terms in the path P
- The the set S satisfies the 3 following conditions for every (Alpha, Beta term) in S.
  - No signed variable and its conjugate are in S
  - If (Alpha x y) in S, then x in S and y in S
  - If (Beta x y) is in S, then either X in S or y in S

Hintikka Sets

- Any set obeying the 3 rules
  - No signed variable and its conjugate are in S
  - If (Alpha x y) in S, then x in S and y in S
  - If (Beta x y) is in S, then either X in S or y in S
- Is called a Hintikka set.

Hintikka's lemma

- Let S be a Hintikka set, then there exists and interpretation (assignment to its variables) in which every set in S is True.

- Start by constructing the following assignment for every variable v that appears in the set.
  1. If v in S, then assign v True
  2. If ~v in S then assign v False
  3. Otherwise give it any assignment you want (we will choose True for concreteness)

Comments

- 1 and 2 are not inconsistent, because S is a Hintikka set, and by definitions both v and ~v cannot be in S

- We will now show that every p in S is true under this assignment
- We do this by induction over the structure of p

Case v or ~v

- If the term is a varaible or a negated variable then it is clearly True, since we designed the assignment v to be True in this case.

Other cases

- If p is ImpliesP, AndP, or OrP, or a Negation of one of these, then it is either an (Alpha x y) or a (Beta x y)
- So by structural induction both x and y evaluate to True under the assignment v

(Alpha x y) Case

- Because S is a Hintikka set, then both x and y are in S, and by induction x and y evaluate to True under v
- So by the structure of discrim (there are three cases)
  - `discrim (AndP x y) = Alpha x y`
  - `discrim (NotP (OrP x y)) = Alpha (NotP x)  (NotP y)`
  - `discrim (NotP (ImpliesP x y)) = Alpha x (NotP y)`

- (Alpha x y) must also evaluate to True by the definition of Hintikka set.

(Beta x y) Case

- Because S is a Hintikka set, then either x or y are in S, and by induction the one in S must evaluate to True under v
- So by the structure of discrim (there are three cases)
  - **`discrim (OrP x y) = Beta x y`**
  - **`discrim (ImpliesP x y) = Beta (NotP x) y`**
  - **`discrim (NotP (AndP x y)) = Beta (NotP x) (NotP y)`**
- (Beta x y) must also evaluate to True by the definition of Hintikka set.

Completeness Theorem

- If X is a tautology then every tableau rooted with ~X must close.

- Suppose T is a complete tableau rooted at ~X.
- If T is open, then by Hinitkka's lemma we can find an assignment where ~X is satisfiable, that means X cannot be a tautology since there is an assignment that makes ~X True.
- Thus if X is a tautology, then the tableau for X must close.