Regular Expressions

Sipser pages 63-66

A new Computation System

- DFAs, NFs, ε-NFAs all describe a language in an operational manner. They describe a machine that one can execute in order to recognize a string.
- An alternate method of describing a set of strings is to describe the properties it should have.
- Regular-Expressions are based upon the closure properties we have studied.

Regular Expressions

- Fix an alphabet Σ. We define several ways to form a regular expression and how each of them specifies a language.

- **Definition.** The set of *regular expressions* (with respect to Σ) is defined inductively by the following rules:

  1. The symbols $\varnothing$ and ε are regular expressions
  2. Every symbol $\alpha \in \Sigma$ is a regular expression
  3. If E and F are regular expressions, then (E^*), (EF) and (E+F) are regular expressions.

Juxtaposition of two RE uses an implicit • or concatenation

Sipser uses EUF, but we also may use E+T

Note how the closure properties are used here

Computation system as Data

- We have made a big point that computation systems are just data, regular expressions are no exception.
- We can represent them as data. Here we use Haskell as an example.

```
data RegExp a
  = Epsilon                          -- the empty string ""
  | Empty                            -- the empty set
  | One a                            -- a singleton set {a}
  | Union (RegExp a) (RegExp a)      -- union of two RegExp
  | Cat (RegExp a) (RegExp a)        -- Concatenation
  | Star (RegExp a)                  -- Kleene closure
```

- How would you represent regular expressions in your favorite language?

Regular Expressions as Languages

- **Definition.** For every regular expression E, there is an associated language *L(E)*, defined inductively as follows:

1. $L(\varnothing)=\varnothing$ and $L(\varepsilon)=\{\varepsilon\}$
2. L(a)={'a'}
3. Inductive cases
   1. $L(E^*) = (L(E))^*$
   2. L(EF) = L(E) L(F)
   3. $L(E+F) = L(E) \cup L(F)$

Recall $x^* =$
$\varepsilon \cup x \cup x \bullet x \cup x \bullet x \bullet x \ldots$

recall implicit use of dot
$L(E) \bullet L(F)$

- **Definition.** A language is *regular* if it is of the form L(E) for some regular expression E.

Equivalence

1. We say that regular expressions E and F are *equivalent* iff L(E)=L(F).

2. We treat equivalent expressions as equal (just as we do with arithmetic expressions; (e.g., 5+7 = 7+5).

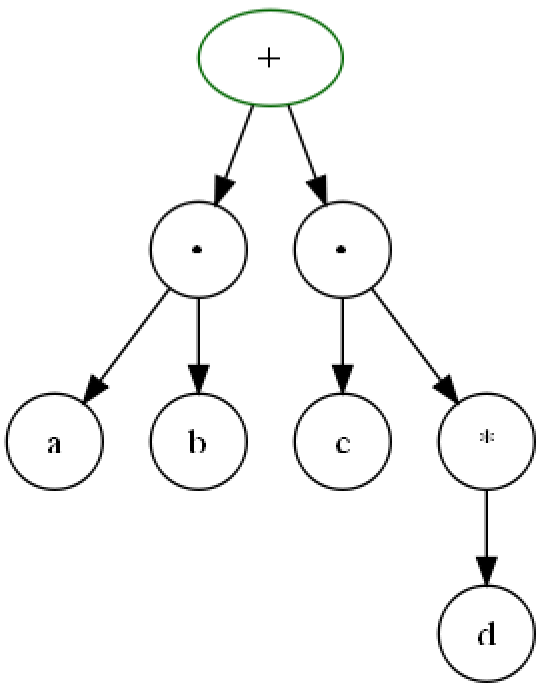
3. Equivalences (E+F)+G = E+(F+G) and (EF)G=E(FG) allow us to omit many parentheses when writing regular expressions.

4. Even more parentheses can be omitted when we declare the precedence ordering of the three operators :
   1. star (binds tightest)
   2. concatenation
   3. union (binds least of all)

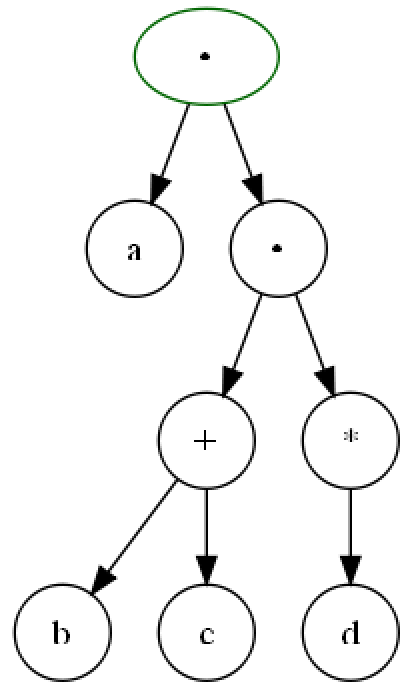
Regular expressions as Trees

- Every RE has a tree like structure. Binding rules specify this structure.

ab+cd*



a(b+c)d*



RE's over {0,1}

- Fill in the blank

- $E_1 =$ **0+11** then $L(E_1)=$______
- $E_2 =$ **(00+01+10+11)*** then $L(E_2)=$_____
- $E_3 =$ **0*+1*** then $L(E_3)=$______
- $E_4 =$ **(00*+11*)*** then $L(E_4)=$______
- $E_5 =$ **(1+ε)(01)*(0+ε)** then $L(E_5)=$______

Computing a language

- We can compute a language by using the definition of the meaning of a regular expression
- L(a+b.c*) = L(a) U L(b.c*)
- L(a+b.c*) = L(a) U (L(b).L(c*))
- L(a+b.c*) = {a} U ({b} . {c}*)
- L(a+b.c*) = {a} U ({b}.{ε,c,cc,ccc,..., c^n})
- L(a+b.c*) = {a} U ({b,bc,bcc,bccc,..., bc^n}
- L(a+b.c*) = {a,b,bc,bcc,bccc,..., bc^n})

Laws about Regular expressions

- The regular expressions form an algebra
- There are many laws (just as there are laws about arithmetic (5+2)=(2+5)

Laws about +

1. $R + T = T + R$
2. $R + \emptyset = \emptyset + R = R$
3. $R + R = R$
4. $R + (S + T) = (R + S) + T$

Laws about .

1. $R . \varnothing = \varnothing . R = \varnothing$
2. $R.\varepsilon = \varepsilon.R = R$
3. $(R.S).T = R.(S.T)$

- With Implicit .

1. $R \varnothing = \varnothing R = \varnothing$
2. $R\varepsilon = \varepsilon R = R$
3. $(RS)T = R(ST)$

Distributive Properties

1. $R(S + T) = RS + RT$
2. $(S + T)R = SR + TR$

Closure Properties *

1. $\emptyset^* = \varepsilon^* = \varepsilon$
2. $R^* = R^*R^* = (R^*)^* = R + R^*$
3. $R^* = \varepsilon + R^* = (\varepsilon + R)^* = (\varepsilon+R)R^* = \varepsilon + RR^*$
4. $R^* = (\varepsilon + \ldots + R^k)^*$ *for all k >= 1*
5. $R^* = \varepsilon + R + \ldots + R^{(k-1)} + R^kR^*$ *for all k >= 1*
6. *$RR^* = R^*R$*
7. *$R(SR)^* = (RS)^*R$*
8. *$(R^*S)^* = \varepsilon + (R + S)^*S$*
9. *$(RS^*)^* = \varepsilon + R(R + S)^*$*

Next section

- We will study how to make recognizers from regular expressions
- We will prove that RE and DFAs describe the same class of languages.