NFA Closure Properties

Sipser pages  pages 58-63
NFAs also have closure properties

- We have given constructions for showing that DFAs are closed under
  1. Complement
  2. Intersection
  3. Difference
  4. Union
- We will now establish that NFAs are also closed under
  1. Reversal
  2. Union
  3. Concatenation
  4. Kleene star
Proof Strategy

• As we did for DFAs, To prove these properties

1. We’ll assume some language (or languages) are recognized by an NFA (or an \( \varepsilon \)-NFA)
2. The then that NFA must be a 5-tuple \( A = (Q, \Sigma, \delta, q_0, F) \)
3. Then we’ll use the pieces of the 5-tuple to create a new 5-tuple that is the NFA we want.
4. It is very similar to writing a program!
Reversal of ε-NFAs

- Closure under reversal is easy using ε-NFAs. If you take such an automaton for $L$, you need to make the following changes to transform it into an automaton for $L^{\text{Rev}}$:

1. Reverse all arcs

2. The old start state becomes the only new final state.

3. Add a new start state, and an ε-arc from it to all old final states.
Example

1. Reverse all arcs

2. The old start state becomes the only new final state.

3. Add a new start state, and an ε-arc from it to all old final states.

This is upside down on purpose
Union

- We showed that DFAs are closed under union by using the product construction. It is much easier to show NFAs closed under union because we have \( \epsilon \) transitions.

- How?
Concatenation

• \( L \cdot R = \{ x \cdot y \mid x \text{ in } L \text{ and } y \text{ in } R \} \)

• To form a new \( \varepsilon \)-NFA that recognizes the concatenation of two other \( \varepsilon \)-NFAs with the same alphabet do the following
  – Union the states (you might have to rename them)
  – Add an \( \varepsilon \)-transition from each final state of the first to the start state of the second.
Formally

• Let
  \[ L = (Q_L, A, T_L, s_L, F_L) \]
  \[ R = (Q_R, A, T_R, s_R, F_R) \]

• \( L \circ R = (Q_L \cup Q_R, A, T, s_L, F_R) \)

Where
\[ T \ s | s \in F_L = S_R \cup T_L \ s \in \varepsilon \]
\[ T \ s \ c | s \in Q_L = T_L \ s \ c \]
\[ T \ s \ c | s \in Q_R = T_R \ s \ c \]
Kleene - Star

• If a language L is recognized by an NFA then so is the language L*  

• Add a new state.  
• Make it the start state in the new NFA.  
• Add an ε-arc from this state to the old start state.  
• Add ε-arcs from every final state to this new state.
Example

- Add a new state.
- Make it the start state in the new NFA, and an accepting state.
- Add an ε-arc from this state to the old start state.
- Add ε-arcs from every final state to this new state