Reasoning with DFAs

Closure properties of DFAs

Languages captured by DFA's are closed under

Union
Concatenation
Kleene Star
Complement
Intersection

That is to say if L_1 and L_2 are recognized by a DFA, then there exists another DFA, L_3, such that

1. L_3 = complement L_1
2. $L_3 = L_1 \cup L_2$
3. $L_3 = L_1 \cap L_2$
4. $L_3 = L_1^*$
5. $L_3 = L_1 \bullet L_2$ (The first 3 are easy, we'll wait on 4 and 5)

Complement

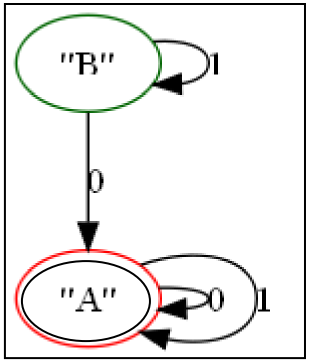
Complementation

Take a DFA for L and change the status - final or non-final - of all its states. The resulting DFA will accept exactly those strings that the first one rejects. It is, therefore, a DFA for the Complent(L).

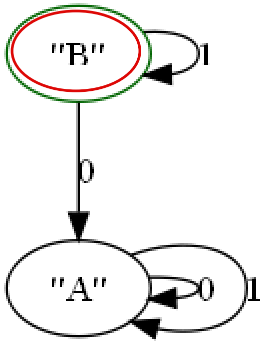
Thus, the complement of DFA recognizable language is DFA recognizable.

Complement Example

Contains a “0”

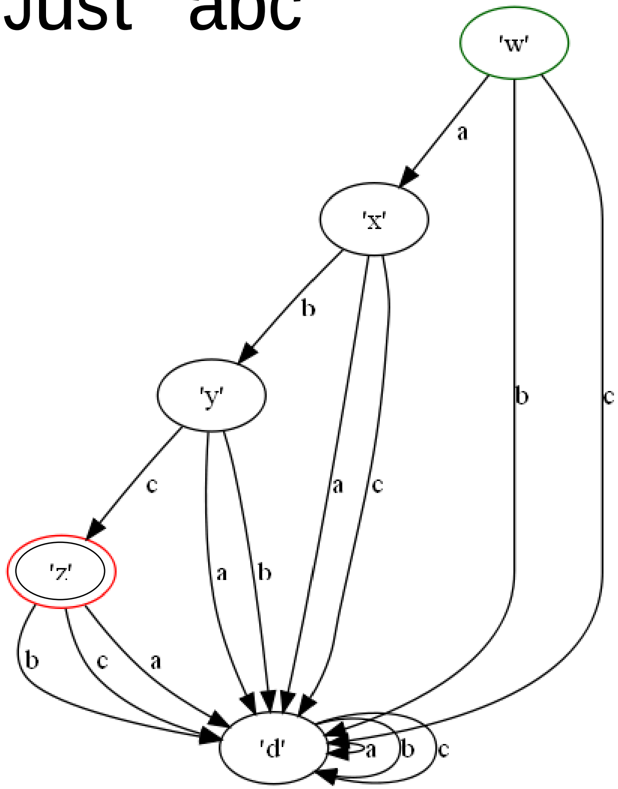


Contains only “1”

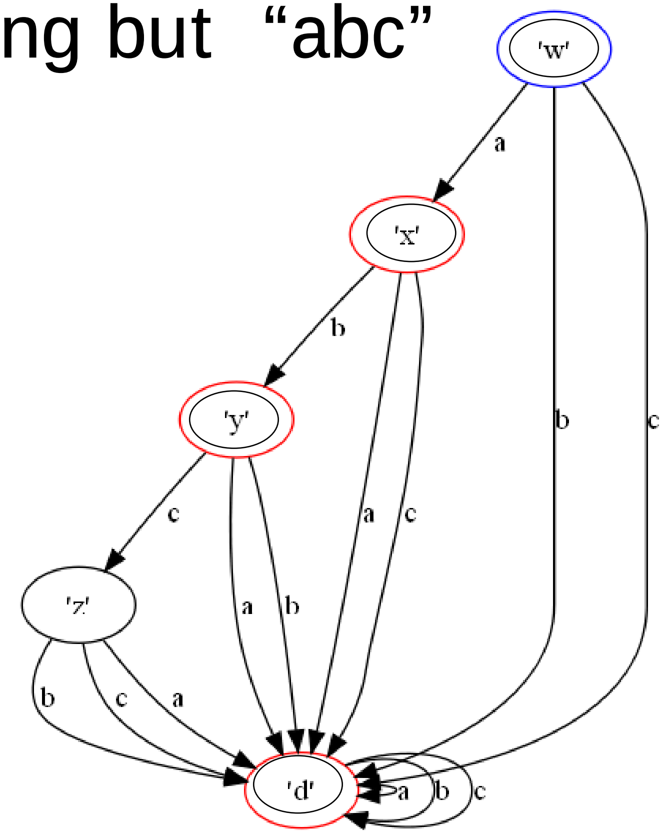


2nd Complement Example

Just "abc"



Anything but "abc"



Intersection

The intersection `L ∩ M` of two DFA recognizable languages must be recognizable by a DFA too. A constructive way to show this is to construct a new DFA from 2 old ones.

Constructive Proof

The proof is based on a construction that given two DFAs A and B, produces a third DFA C such that L(C) = L(A) $\cap$ L(B). The states of C are pairs (p,q) , where p is a state of A and q is a state of B. A transition labeled a leads from (p,q) to (p',q') iff there are transitions

$$p \xrightarrow{a} p' \qquad\qquad q \xrightarrow{a} q'$$

in A and B. The start state is the pair of original start states; the final states are pairs of original final states. The transition function

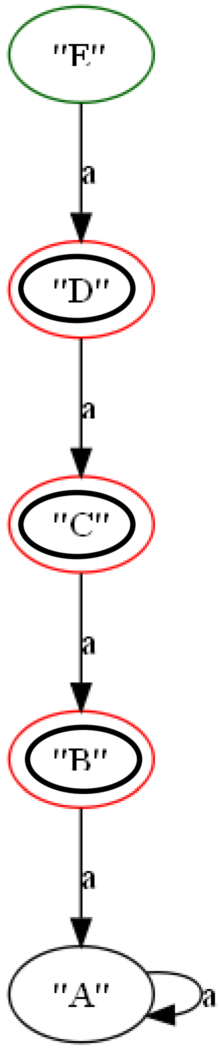
$$\delta_{A \cap B}(q,a) = (\ \delta_A(q,a),\ \delta_B(q,a)\ )$$

This is called the *product construction*.

Example 1

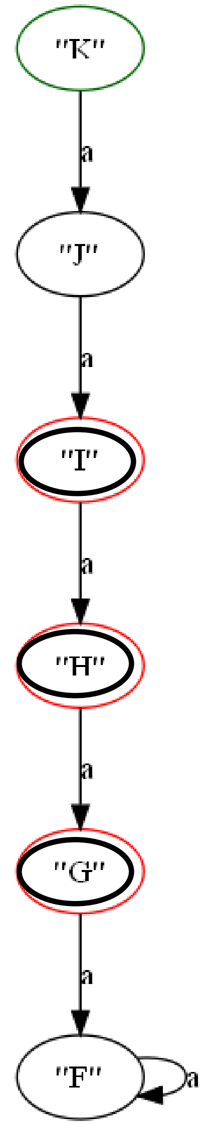
aa+aaa+aaaa

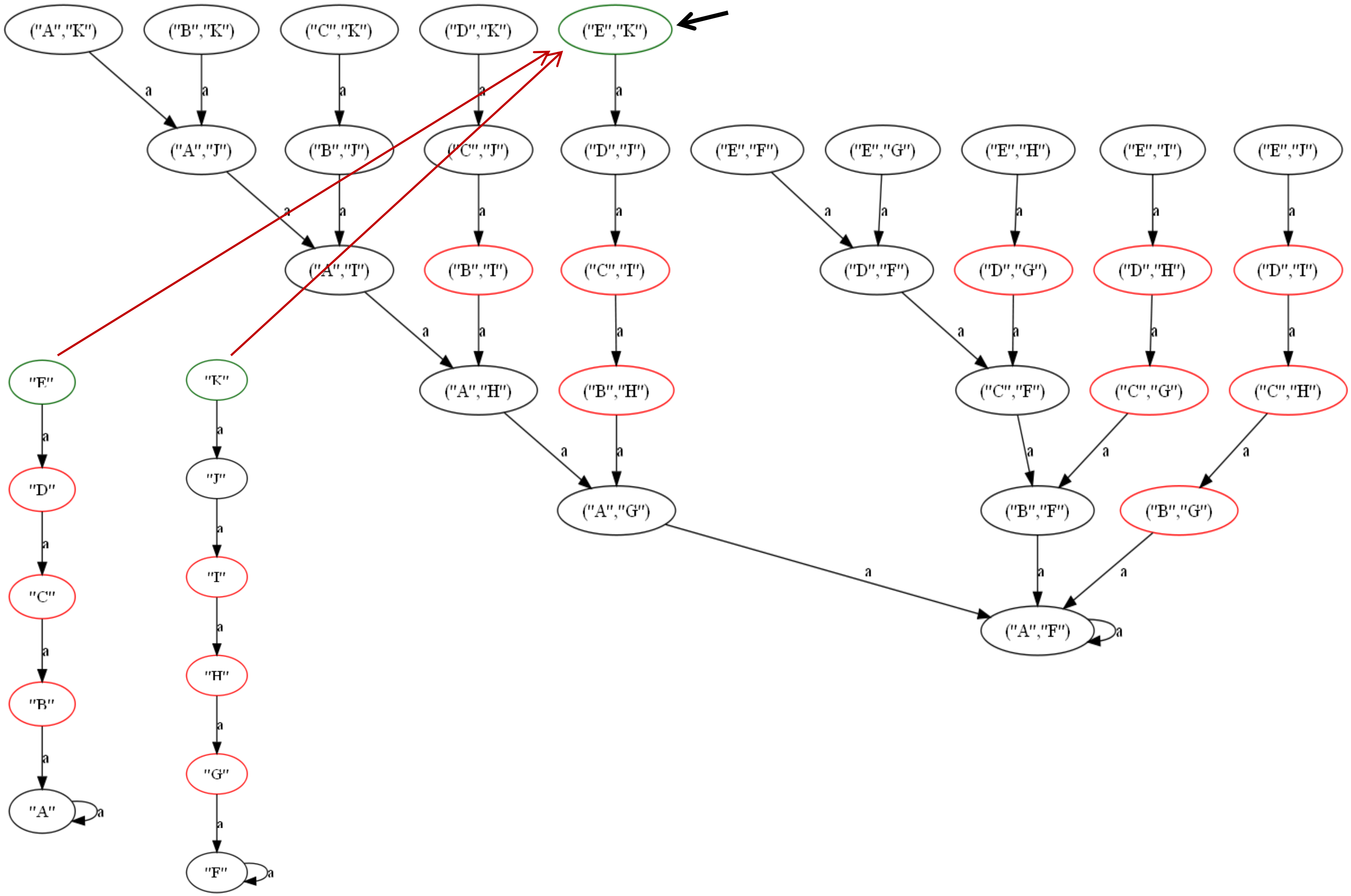
a+aa+aaa



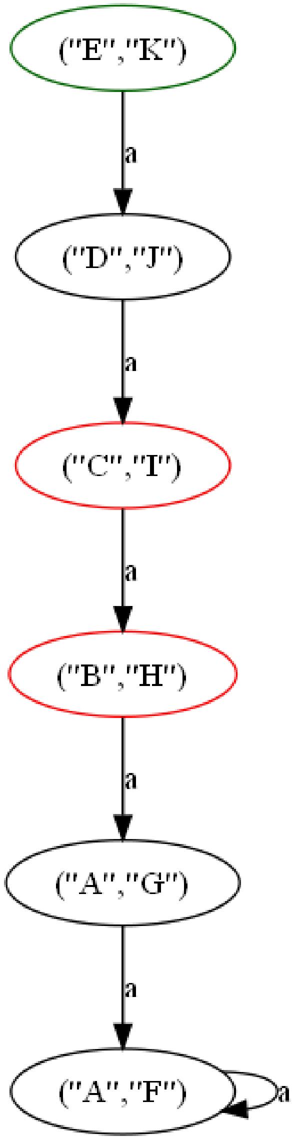
What is the intersection?

Make a new DFA where states of the new DFA are pairs of states form the old ones




("A","K")
("B","K")
("C","K")
("D","K")
("E","K")
("A","J")
("B","J")
("C","J")
("D","J")
("E","F")
("E","G")
("E","H")
("E","I")
("E","J")
("A","I")
("B","I")
("C","I")
("D","F")
("D","G")
("D","H")
("D","I")
"E"
"K"
("A","H")
("B","H")
("C","F")
("C","G")
("C","H")
"D"
"J"
("A","G")
("B","F")
("B","G")
"C"
"I"
("A","F")
"B"
"H"
"G"
"A"
"F"
a

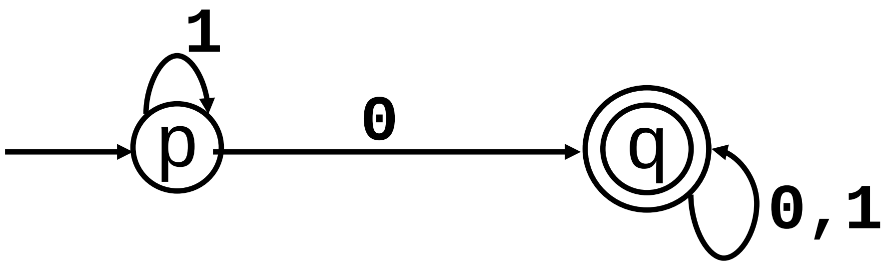
Reachable states only



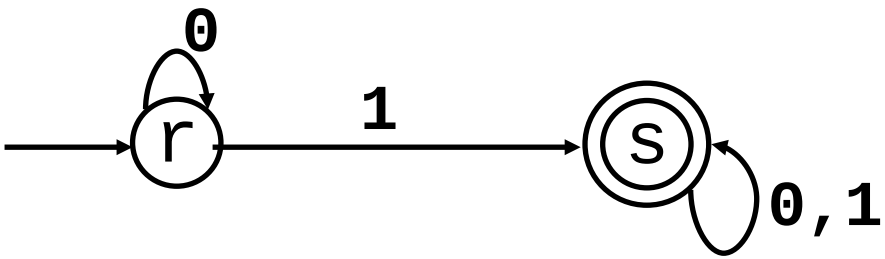
Intersection

{a,aa,aaa} ∩ {aa,aaa,aaaa}

Example 2



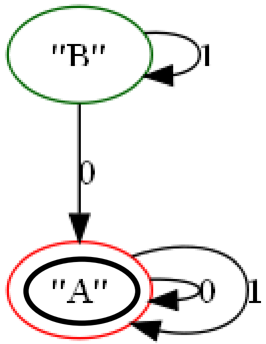
A – string contains a 0



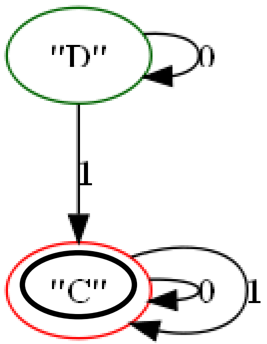
B – string contains a 1

C – string contains a
0 and a 1

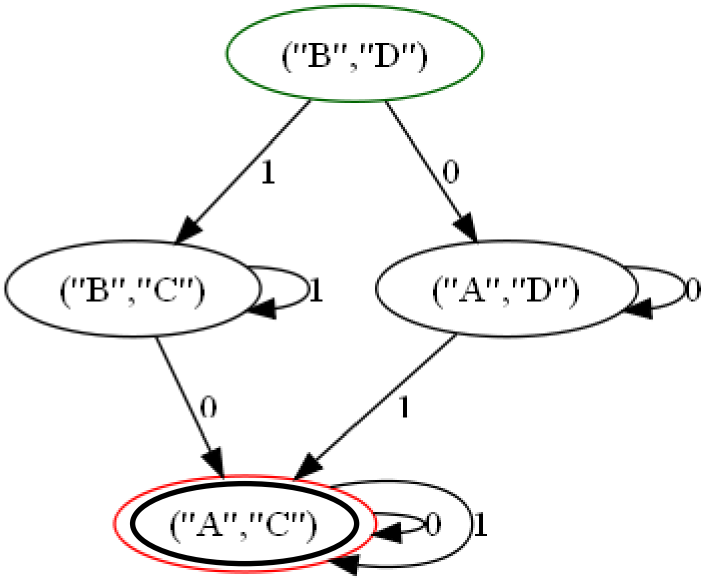
Contains a “0”



Contains a “1”



Contains both a “1” and a “0”



Difference

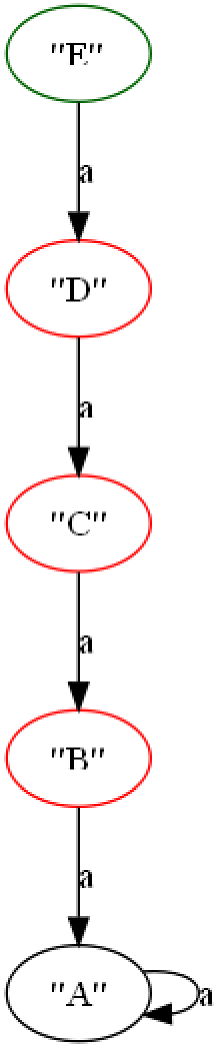
The identity:

$$L - M = L \cap C(M)$$

reduces the closure under set-theoretical difference operator to closure under complementation and intersection.

Example Difference

L={a,aa,aaa}



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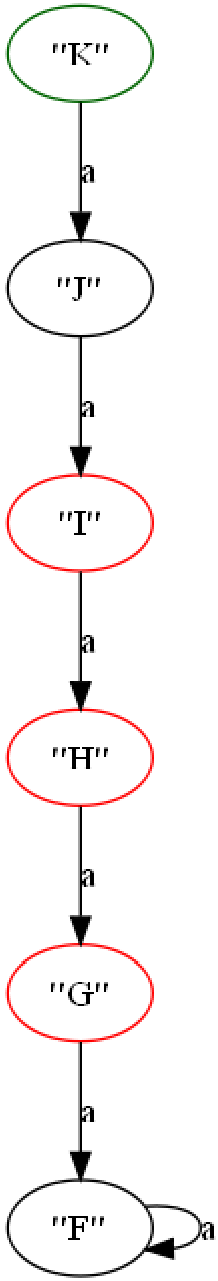
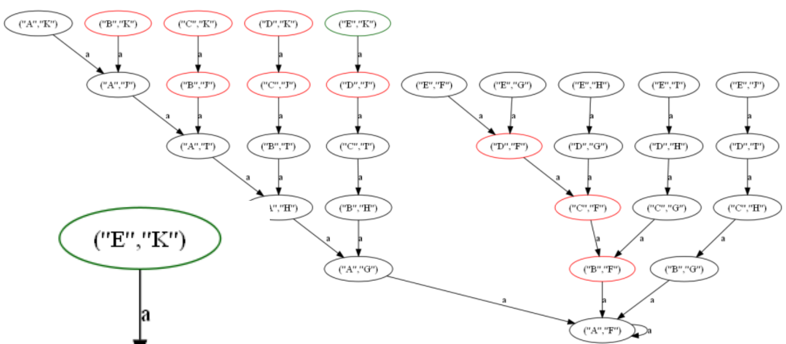
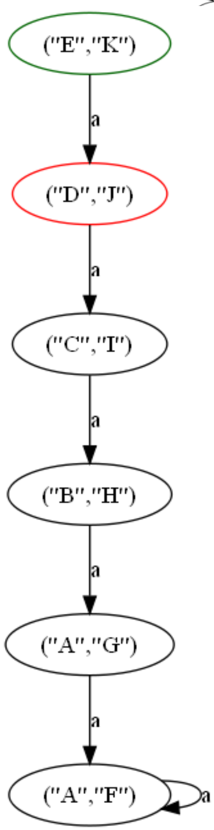
M={aa,aaa,aaaa}



=

$$L - M = L \cap C(M)$$





Union

- The union of the languages of two DFAs (over the same alphabet) is recognizable by another DFA.
- We reuse the product construction of the intersection proof, but widen what is in the final states of the constructed result.

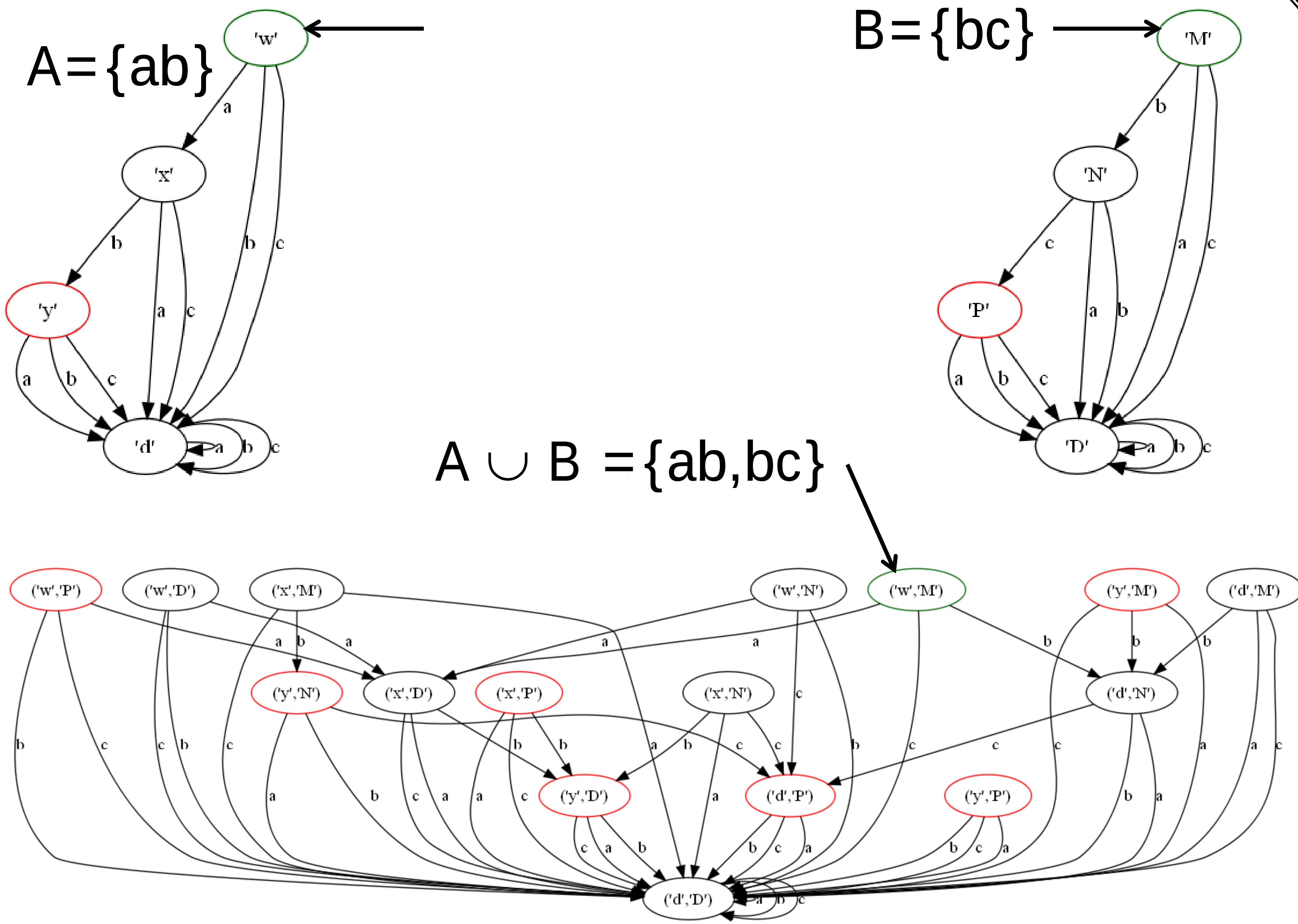
Let $A=(\mathbf{Q_a}, \Sigma, \mathbf{s_a}, \mathbf{F_a}, T_a)$ and

$B = (\mathbf{Q_b}, \Sigma, \mathbf{s_b}, \mathbf{F_b}, T_b)$

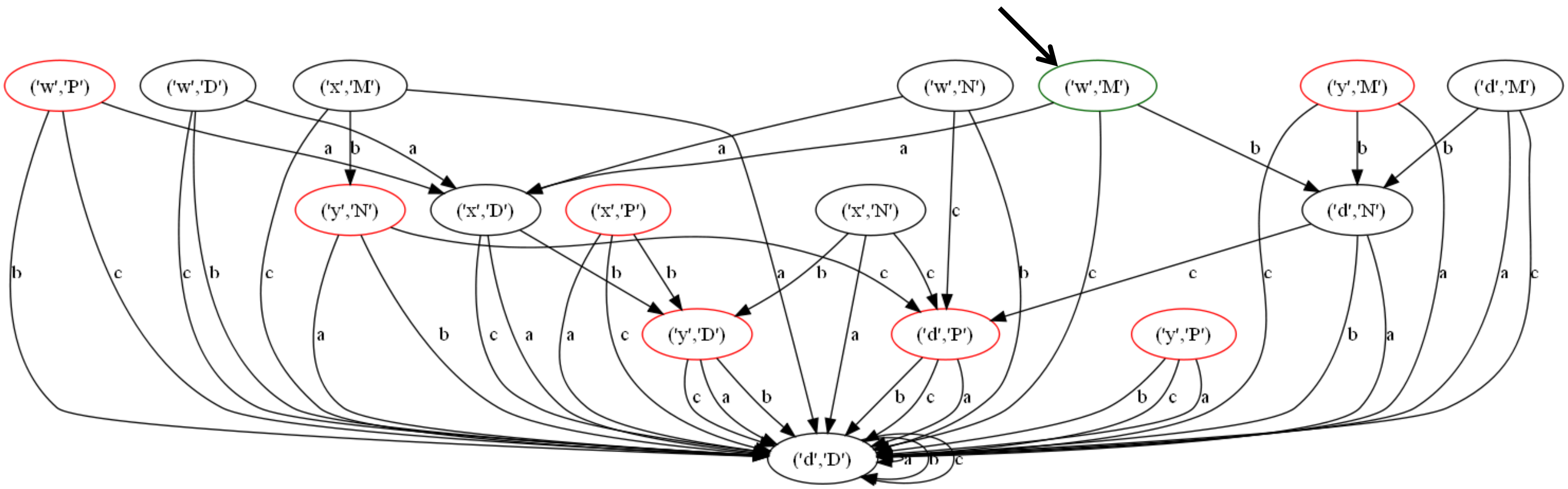
Then: $A \cup B = ((\mathbf{Q_a}, \mathbf{Q_b}), \Sigma, (\mathbf{s_a}, \mathbf{s_b}), \mathbf{Final}, \delta)$

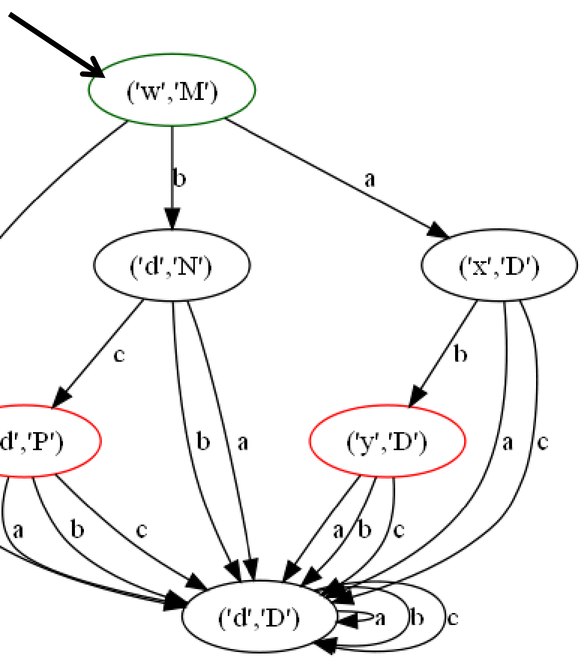
Final = { (p,q) | p from F_a, q from Q_b} $\cup$
{ (p,q) | p from Q_a, q from F_b}

$\delta((a,b),x) = (T_a(a,x), T_b(b,y))$


A={ab}
B={bc}
A ∪ B ={ab,bc}
'w'
'x'
'y'
'd'
'M'
'N'
'P'
'D'
('w','P')
('w','D')
('x','M')
('w','N')
('w','M')
('y','M')
('d','M')
('y','N')
('x','D')
('x','P')
('x','N')
('d','N')
('y','D')
('d','P')
('y','P')
('d','D')

Only reachable from start





Example Closure Construction

Given a language L, let L' be the set of all prefixes of even length strings which belong to L. We prove that if L is regular then L' is also regular.

It is easy to show that prefix(L) is regular when L is (How?). We also know that the language **Even** of even length strings is regular (How?). All we need now is to note that

L' = **Even** $\cap$ prefix(L)

and use closure under intersection.

What's next

We have given constructions for showing that DFAs are closed under

1. Complement
2. Intersection
3. Difference
4. Union

In order to establish the closure properties of

1. Reversal
2. Kleene star
3. Concatenation

We will need to introduce a new computational system.