

The Induction Principle

To prove that a statement $S(n)$ is true for every natural number n , it suffices to:

1. *Base Case*: Prove that the statement $S(0)$ is true;
2. *Induction Step*: Assuming $S(n)$ is true, prove that $S(n+1)$ is true.

When proving the induction step, the assumption $S(n)$ is called the *induction hypothesis*.

Often we need to prove that a statement $S(n)$ is true not exactly for every n , but for every n starting from a given number k . The base case is then $S(k)$; the induction step is the same.

Example 1

Problem. Prove that the sum of first n odd numbers is equal to n^2 .

Proof. The statement $S(n)$ is:

$$1 + 3 + \dots + (2n-1) = n^2 ,$$

I.e. $(\sum_{i=1,n} (2i-1) = n^2)$ and we want to prove it is true for every $n \geq 1$.

Base Case. $S(1)$ is the statement $1=1$.

Induction Step. Assume the induction hypothesis

$$1 + 3 + \dots + (2n-1) = n^2$$

The goal is to prove

$$1 + 3 + \dots + (2n-1) + (2n+1) = (n+1)^2$$

Using the IH, the goal can be rewritten as

$$n^2 + (2n+1) = (n+1)^2,$$

which is directly verified.

qed

Complete (Strong) Induction

To prove that a statement $S(n)$ is true for every natural number n , it suffices to:

1. *Base Case*: Prove that the statement $S(0)$ is true.
2. *Induction Step*: Assuming $n > 0$ and that $S(k)$ is true for all numbers k smaller than n , prove that $S(n)$ is true.

Example 2

Problem. Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be defined recursively by

$$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ 2f\left(\frac{n}{2}\right) & \text{if } n \text{ is even} \\ f(n-1) + 1 & \text{if } n \text{ is odd} \end{cases}$$

Prove that $f(n) = n$ for every n .

Proof.

Base Case. $f(0) = 0$; true by definition of f .

Induction Step. Suppose $n > 0$ and $f(k) = k$ for all $k < n$. To derive $f(n) = n$, we consider separately the cases when n is even and odd.

- If n is even, we have $f(n/2) = (n/2)$ by IH (note $(n/2) < n$). Therefore, $f(n) = 2f(n/2) = 2 * (n/2) = n$.
- If n is odd, the IH gives us $f(n-1) = n-1$, so we get
 $f(n) = f(n-1) + 1 = (n-1) + 1 = n$.

qed

Example 3

Problem. Suppose two strings u and v satisfy the relation $uv=vu$. Prove that u and v are powers of the same string.

Proof. Induction on $|u|+|v|$. Strictly speaking, the statement $S(n)$ is this: If $uv=vu$ and $|u|+|v|=n$ then u and v are powers of the same string.

Base Case. $|u|+|v|=0$. This implies $u=v=\varepsilon$, and the statement is true.

Induction Step. We're arguing by complete induction. Suppose $|u|+|v|=n$ and $n>0$ and suppose that the statement is true for every u',v' such that $|u'|+|v'|<n$.

Proof continued

If $|u|=|v|$, the statement is true. Assume $|u|<|v|$.
(The third case $|u|>|v|$ is symmetric and does not need to be considered separately.)

Then $v=uw$ for some w and we have $uw=uw$.

This implies $uw=wu$. Since $|w|<|v|$, we have $|u|+|w|<|u|+|v|$ and the IH applies giving us that u and w are powers of the same string z .

Clearly then, $v=uw$ is also a power of z .

qed

Structural Induction

A method for proving properties of objects (trees, expressions, etc.) defined recursively. Such recursive definitions have a number of base cases defining the simplest objects and a number of rules telling how a bigger object is build from smaller ones.

To prove that a statement $S(x)$ is true for every object it suffices to prove:

Base Case: $S(x)$ is true for the basic objects.

Induction Step: For every rule telling us how to build a bigger object x from smaller objects x_1, \dots, x_k , prove that $S(x)$ is true, assuming as the IH that $S(x_1), \dots, S(x_k)$ are true.

Structural induction is induction on the size of the object.

Example: Balanced Parentheses

Parenthesis expressions (pexps) are defined recursively by the following rules:

- [1.] The empty string ε is a pexp.
- [2.] If w is a pexp, then (w) is a pexp.
- [3.] If u and v are pexps, then uv is a pexp.

Note: pexps define a language over the alphabet $\Sigma = \{ (,) \}$.

Problem 1. Every pexp has equal number of left and right parentheses.

Pexp proof

Problem 1. Every pexp has equal number of left and right parentheses.

For a string x over the alphabet $\Sigma = \{ (,) \}$, let $E(x)$ denote the property “ x has equal number of left and right parentheses”.

Proof.

1. $x = \varepsilon$. True for ε .
2. $x = (v)$. Assume v has the same number of left and right parentheses, The induction hypothesis. Then the same is true of (v)
3. $x = uv$. Assume u and v both have equal number of left and right parentheses. Then the same holds for uv . ($E(u)$ and $E(v) \Rightarrow E(uv)$)

qed

Problem 2

Problem 2. If x is a pexp, then every prefix of x has at least as many left as right parentheses.

Proof. Let $S(x)$ stand for “every prefix of x has at least as many left as right parentheses”.

1. $x = \varepsilon$. $S(\varepsilon)$ is true.
2. $x = (w)$. If $S(w)$ (the i.h.) is true, then $S((w))$ is true.
3. $x = uv$. If $S(u)$ and $S(v)$ are true, then $S(uv)$ is true.

qed

Problem 3

Problem. If a string w satisfies both $S(w)$ and $E(w)$ then w is a pexp.

Proof. Complete induction on $|w|$.

Base case. $|w|=0$ is OK because then we have $w=\varepsilon$, and ε is a pexp.

Induction step. Assume that w satisfies $S(w)$ and $E(w)$, that $|w|>0$, and (the IH) that all strings u shorter than w and satisfying $S(u)$ and $E(u)$ are pexps.

There are two possibilities for w :

- (1) all its prefixes except ε and w itself have strictly greater number of '('s than ')'s; (2) there exist a prefix u of w such that $u \neq \varepsilon$, $u \neq w$, and u has equal number of '('s and ')'s.

Case analysis

Case (1). w must be of the form $w=(u)$ for some u . Clearly, $E(u)$ is true. But $S(u)$ must be true as well (why?). The IH implies that u is a pexp. Then, referring to the second rule for building pexps, we can conclude that w is a pexp.

Case (2). We can write $w=uv$. It follows that both u and v satisfy the properties E and S (why?). Since both u and v are shorter than w , the IH applies to them, so u and v are pexps. The third rule for building pexps implies finally that w is a pexp.

qed