Notions of Computability

- Many notions of computability have been proposed, e.g.
  - *(Type 0 a.k.a. Unrestricted or context sensitive)* Grammars
  - Partial Recursive Functions
  - Lambda calculus
  - *Markov Algorithms*
  - Post Algorithms
  - Post Canonical Systems,
- • All have been shown equivalent to Turing machines by simulation proofs

Systems we'll study

- Context Sensitive Grammars
- Primitive Recursive Functions
- Partial recursive functions

Context Sensitive Grammars

- We can extend the notion of context-free grammars to a more general mechanism
- An (unrestricted) grammar $G = (V,\Sigma,R,S)$ is just like a CFG except that rules in R can take the more general form $\alpha \rightarrow \beta$ where α,β are **arbitrary strings of terminals and variables.** α must contain at least one variable (or nontermial).
- If $\alpha \rightarrow \beta$ then $u\alpha v \Rightarrow u\beta v$ ("yields") in one step
- Define $\Rightarrow^*$ ("derives") as reflexive transitive closure of $\Rightarrow$.

Classical not-context free language

- $\{ a^n b^n c^n \mid n \geq 0 \}$

S -> aSBC
S -> aBC
CB -> HB
HB -> HC
aB -> ab
bB -> bb
bC -> bc
cC -> cc

Example: $\{a^{2^n}, n \geq 0\}$

- Here's a set of grammar rules

1. $S \rightarrow a$
2. $S \rightarrow ACaB$
3. $Ca \rightarrow aaC$
4. $CB \rightarrow DB$
5. $CB \rightarrow E$
6. $aD \rightarrow Da$
7. $AD \rightarrow AC$
8. $aE \rightarrow Ea$
9. $AE \rightarrow \Lambda$

Try generating 2^3 a's

S

ACaB

AaaCB

AaaDB

AaDaB

ADaaB

ACaaB

AaaCaB

AaaaaCB

AaaaaDB

(Unrestricted) Grammars and Turing machines have equivalent power

- For any grammar G we can find a TM M such that L(M) = L(G).
- For any TM M, we can find a grammar G such that L(G) = L(M).

Computation using Numerical Functions

- We're used to thinking about computation as something we do with **numbers (e.g.** on the natural numbers)

- What kinds of functions from numbers to numbers can we actually compute?

- To study this, we make a very careful selection of building blocks

Primitive Recursive Functions

- The primitive recursive functions from ℕ x ℕ x ... x ℕ → ℕ are those built from these primitives:
  - zero(x) = 0
  - succ(x) = x+1
  - π k,j (x1,x2,...,xk) = xj for 0 < j ≤ k
- using these mechanisms:
  - Function composition, and
  - Primitive recursion

Function Composition

- Define a new function f in terms of functions h and $g_1, g_2, \ldots, g_m$ as follows:

  $f(x_1,\ldots x_n) = h(g_1(x_1,\ldots,x_n), \ldots ,g_m(x_1,\ldots,x_n))$

  Note that f and g_i have arity n but that h has arity m

Example: $f(x) = x + 3$ can be expressed using two compositions as $f (x) = succ(succ(succ(x)))$

Primitive Recursion

- Primitive recursion defines a new function f in terms of functions h and g as follows:

- f(0,x1, ..., xk) = h(x1,...,xk)
- f(Succ(n),x1, ..., xk)= g(n, f(n,x1,...,xk**),** x1,...,xk**)**

Note that the order of arguments here differs slightly from the order in the Hein book (page 832). Here we place the number being analyzed first rather than last.

Many ordinary functions can be defined using primitive recursion, e.g.

add(0,x) = π1,1(x)

add(Succ(y),x) = succ(π3,3(y, add(y,x),x))

More P.R. Functions

- For simplicity, we omit projection functions and write 0 for zero(_) and 1 for succ(0)

```
add(x,0) = x
add(x,succ(y)) = succ(add(x,y))

mult(x,0) = 0
mult(x,succ(y)) = add(x,mult(x,y))

factorial(0) = 1
factorial(succ(n)) =  mult(succ(n),factorial(n))

exp(n,0) = 1
exp(n, succ(n)) = mult(n,exp(n,m))

pred(0) = 0
pred(succ(n)) = n
```

- Essentially all practically **useful arithmetic** functions are primitive recursive, but...

Ackermann's Function is not Primitive Recursive

- A famous example of a function that is clearly well-defined but not primitive recursive

```
A(m, n)=
  if m0 then n+1
  else if n=0 then A(m–1, 1)
  else A(m–1, A(m,n–1))
```

This function grows extremely fast!

Values of *A*(*m*, *n*)

*m**n*	0	1	2	3	4	n
0	1	2	3	4	5	$n + 1$
1	2	3	4	5	6	$n + 2 = 2 + (n + 3) - 3$
2	3	5	7	9	11	$2n + 3 = 2 \cdot (n + 3) - 3$
3	5	13	29	61	125	$2^{(n+3)} - 3$
4	13	65533	$2^{65536} - 3$	$2^{2^{65536}} - 3$	$A(3, A(4, 3))$	$\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{n+3 \text{ twos}} - 3$
5	65533	$\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{65536 \text{ twos}} - 3$	$A(4, A(5, 1))$	$A(4, A(5, 2))$	$A(4, A(5, 3))$	$A(4, A(5, n\text{-}1))$
6	$A(5, 1)$	$A(5, A(6, 0))$	$A(5, A(6, 1))$	$A(5, A(6, 2))$	$A(5, A(6, 3))$	$A(5, A(6, n\text{-}1))$

A is not primitive recursive

- Ackermann's function grows faster than any primitive recursive function, that is:
- for *any primitive recursive function f, there is* an *n such that*
- *A(n, x) > f x*
- So *A can't be primitive recursive*

An Algebra of PR functions

- A grammar for well formed term PR terms

Term $\rightarrow$ Z
| S
| P N — nth projection
| C Term [$Term_1$, … ,$Term_n$] — composition
| PR Term Term — primitive recursion

- N $\rightarrow$ 1 | 2 | 3 | 4 | …

Example 1

- Equations
- `f(x) = succ(succ(succ(x)))`

- Algebra
- `F = C S [C S [S]]`

Term $\rightarrow$ Z		
| S		
| P N		nth projection
| C Term [Term_1, … ,Term_n]		composition
| PR Term Term		primitive recursion

Example 2

- Equations

```
add(0,x) = x
add(succ(y),x) = succ(add(y,x))
```

f(0,x1, ..., xk) = h(x1,...,xk)
f(Succ(n),x1, ..., xk)= g(n, f(n,x1,...,xk**),** x1,...,xk**)**

- Algebra

```
add = PR (P 1) (C S [P 2])
```

Term $\rightarrow$ Z
| S
| P N — nth projection
| C Term [Term_1, ... ,Term_n] — composition
| PR Term Term — primitive recursion

Example 3

f(0,x1, ..., xk) = h(x1,...,xk)
f(Succ(n),x1, ..., xk)= g(n, f(n,x1,...,xk**)**, x1,...,xk**)**

- Equations

```
pred Zero = Zero
pred (Succ n) = n
```

- Algebra

```
pred = PR Z (P 1)
```

Term $\rightarrow$ Z
| S
| P N — nth projection
| C Term [Term_1, ... ,Term_n] — composition
| PR Term Term — primitive recursion

Example 4

```
f(0,x1, ..., xk) = h(x1,...,xk)
f(Succ(n),x1, ..., xk)= g(n, f(n,x1,...,xk), x1,...,xk)
```

- Equations

```
monus Zero x = x
monus (Succ n) x = pred(monus n x)
minus x y = monus y x
```

- Algebra

```
minus = C (PR (P 1)
              (C pred [P 2]))
          [P 2, P 1]
```

```
Term → Z
     | S
     | P N
     | C Term [ Term_1, ... ,Term_n ]
     | PR Term Term
```

Summary

- The algebra denotes functions by combining other functions.
- The simplest functions: Z, S, P n are trivial
- Yet by using
  - Composition - C Term [Term_1, … , Term_n]
  - Primitive recursion - PR Term Term

Many other functions can be built

- Almost every function we use can be built this way

Sanity Check

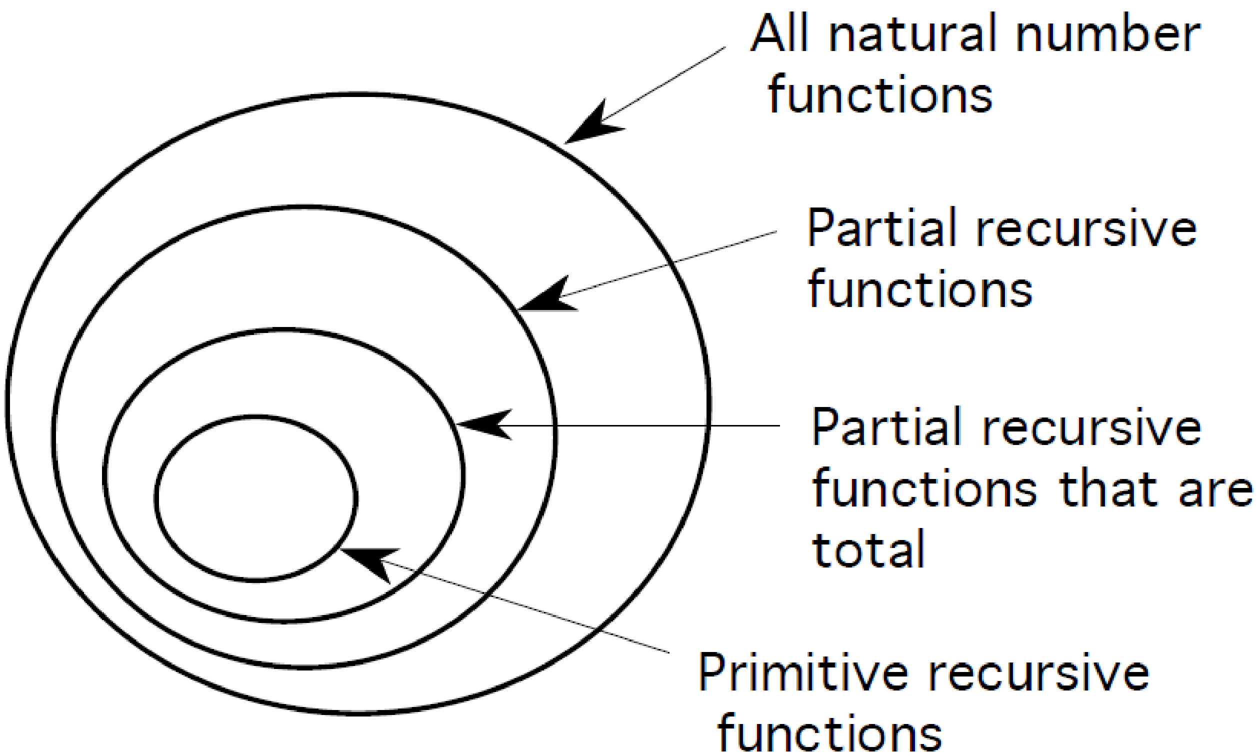
- We can check if a term in the algebra is a function of n arguments

```
check Z _ = True
check S 1 = True
check (P n) m = n <= m
check (C f gs) n =
  check f (length gs) &&
  (all (\g -> check g n) gs)
check (PR g h) n =
  check g (n-1) && check h (n+1)
```

Partial Recursive Functions

- *A belongs to class of **partial recursive*** **functions, a superset of the primitive recursive** functions.
- Can be built from primitive recursive operators & new **minimization operator**
  - Let *g be a (k+1)-argument function.*
  - Define *f(x1,…,xk) as the **smallest m such that g(x1,…,xk,m)*** = 0 (if such an m exists)
  - Otherwise, *f(x1,…,xn) is undefined*
  - We write *f(x1,…,xk) = μm.[g(x1,…,xk,m) = 0]*
  - Example: *μm.[mult(n,m) = 0] = zero(_)*

Hierarchy of Numeric Functions



Turing-computable functions

- To formalize the connection between partial recursive functions and Turing machines, mathematicians have described how to use TM's to compute functions on $\mathbb{N}$.
- We say a function f : $\mathbb{N} \times \mathbb{N} \times \ldots \times \mathbb{N} \rightarrow \mathbb{N}$ is **Turing-computable** if there exists a TM that, when started in configuration $q_0 1^{n1} \sqcup 1^{n2} \sqcup \ldots \sqcup 1^{nk}$, halts with just $1^{f(n1,n2,\ldots nk)}$ on the tape.
- **Fact: f is Turing-computable iff it is partial recursive.**