Last Lecture

We began to show CFL = PDA

Theorem 1. Every context-free language is accepted by some PDA.

Theorem 2. For every PDA M, the language L(M) is context-free.

We showed how a PDA could be constructed from a CFL. Given a CFG G=(V,T,P,S), we define a PDA M=({q},T, T $\cup$ V, δ,q,S), with δ given by

– If A $\in$ V, then $\delta(q,\Lambda,A) = \{ (q,\alpha) \mid A \to \alpha \text{ is in } P\}$
– If a $\in$ T, then $\delta(q,a,a) = \{ (q,\Lambda) \}$

1. The stack symbols of the new PDA contain all the terminal and non-terminals of the CFG
2. There is only 1 state in the new PDA
3. Add transitions on Λ, one for each production
4. Add transitions on a $\in$ T, one for each terminal.

Transitions simulate left-most derivation

$$S \Rightarrow SS \Rightarrow (S)S \Rightarrow ((S))S \Rightarrow (())S \Rightarrow (())(S) \Rightarrow (())()$$

1.	$\delta(q, \Lambda, S)$	=	(q, SS)	$S \to SS$
2.	$\delta(q, \Lambda, S)$	=	(q, (S))	$S \to (S)$
3.	$\delta(q, \Lambda, S)$	=	(q, Λ)	$S \to \Lambda$
4.	δ(q, (, ( )	=	(q,Λ)	
5.	δ(q,),))	=	(q,Λ)	

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(q, "(())()" ,S       )  |- [1]
(q, "(())()" ,SS      )  |- [2]
(q, "(())()" ,(S)S    )  |- [4]
(q, "())()"  ,S)S     )  |- [4]
(q, "())()"  ,(S))S   )  |- [4]
(q, "))()"   ,S))S    )  |- [3]
(q, "))()"   ,))S     )  |- [5]
(q, ")()"    ,)S      )  |- [5]
(q, "()"     ,S       )  |- [2]
(q, "()"     ,(S)     )  |- [4]
(q, ")"      ,S)      )  |- [3]
(q, ")"      ,)       )  |- [5]
(q, ε        ,ε       )
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Note there is an entry in δ for each terminal and non-terminal symbol. The stack operations mimic a top down parse, replacing Non-terminals with the rhs of a production.

Proof Outline

To prove that every string of L(G) is accepted by the PDA M, prove the following more general fact:

If $S \Rightarrow_{\text{left-most}}^{*} \alpha$ then $(q,uv,S) \mid\!-^{*} (q,v,\beta)$

where $\alpha = u\beta$ is the "leftmost factorization" of α (u is the longest prefix of α that belongs to T^*, i.e. all terminals).

For example: if α = abcWdXa then u = abc, and β = WdXa, since the next symbol after abc is W$\in$V (a non-terminal or Λ)

$S \Rightarrow_{lm}^{*}$ abcW... then (q, abcV,S) |-* (q,V, W...)

The proof is by induction on the length of the derivation of α.

We also need to prove that every string accepted by M belongs to L(G). Again, to make induction work, we need to prove a slightly more general fact:

If (q,w,A) $\vdash^*$ (q, Λ, Λ), then A $\Rightarrow^*$ w

For all Stacks A, letting A = Start we have our proof.

This time we induct on the length of execution of M that leads from the ID (q,w,A) to (q, Λ , Λ).

A Grammar from a PDA

Assume the $M = (Q,\Sigma,\Gamma,\delta,q_0,Z_0)$ is given, and that it accepts by empty stack. Consider execution of M on an accepted input string.

If at some point of the execution of M the stack is $Z\zeta$ (Z is on top, ζ is the rest of stack)

In terms of instantaneous descriptions

(state$_i$, input, $Z\zeta$) $\vdash \ldots$

Then we know that eventually the stack will be ζ.

Why? Because we assume the input is accepted, and M accepts by empty stack, so eventually Z must be removed from the stack

$$(state_i,\ \alpha X,\ \ Z\zeta)\ \ \vdash^* \ \ (state_j,\ X,\ \ \zeta)$$

The sequence of moves between these two instants is the “net popping” of Z from the stack.

During this sequence of moves, the stack may grow and shrink several times, some input will be consumed (the α), and M will pass through a sequence of states, from $state_i$ to $state_j$.

Net Popping

Net popping is fundamental for the construction of a CFG G equivalent to M.

We will have a variable (Non-terminal) [qZp] in the CFG G for every triple in (q,Z,p) $\in Q \times \Gamma \times Q$ from the PDA. Recall

1. Q is the set of states
2. Γ Is the set of stack symbols

We want the rhs of a production whose lhs is [qZp] to generate precisely those strings w $\in \Sigma^*$ such that M can move from q to p while reading the input w and doing the net popping of Z. A production like [qZp] -> ?

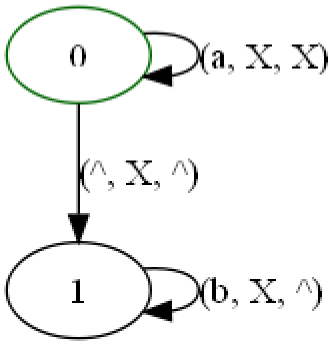
This can be also expressed as (q,w,Z) |-* (p, Λ , Λ)

Productions of G correspond to transitions of M.

If $(p,\zeta) \in \delta(q,a,Z)$, then there is one or more corresponding productions, depending on complexity of ζ.

1. If $\zeta = \Lambda$, we have $[qZp] \rightarrow a$
2. If $\zeta = Y$, we have $[qZr] \rightarrow a[pYr]$ for every state r
3. If $\zeta = YY'$ we have $[qZs] \rightarrow a[pYr][rY's]$, for every pair of states r and s.
4. You can guess the rule for longer ζ.

Example



$Q = \{0,1\}$

$S = \{a,b\}$

$\Gamma = \{X\}$

$\delta(0,a,X) = \{ (0,X) \}$

$\delta(0,\Lambda,X) = \{ (1,\Lambda) \}$

$\delta(1,b,X) = \{ (1,\Lambda) \}$

$Q_{0 = 0}$

$Z_0 = X$

$F = \{\}$, accepts by empty stack

Non-terminals

$(q,Z,p) \in Q\times\Gamma\times Q$

(0,'X',0)

(0,'X',1)

(1,'X',0)

(1,'X',1)

Productions, At least one from each element in delta

$(p,z) \in \delta(q,a,Z)$

(0,a,X,0,X)

$(1,b,X,1,\Lambda)$

$(0,\Lambda,X,1,\Lambda)]$

0X0 -> a 0X0

0X1 -> a 0X1

1X1 -> b

0X1 -> Λ