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WOLFRAM'S CLASS IV AUTOMATA AND A GOOD LIFE

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A comprehensive discussion of Wolfram's four classes of cellular automata is given, with the intention of relating them to Conway's criteria for a good game of *Life*. Although it is known that such classifications cannot be entirely rigorous, much information about the behavior of an automaton can be gleaned from the statistical properties of its transition table. Still more information can be deduced from the mean field approximation to its state densities, in particular, from the distribution of horizontal and diagonal tangents of the latter. In turn these characteristics can be related to the presence or absence of certain loops in the de Bruijn diagram of the automaton.

1. Introduction

It all began with John Horton Conway's search for an "interesting" cellular automaton; one which would perhaps have the complexity of John von Neumann's universal constructor, but with far fewer states, maybe only two. Having observed that the field of activity of an automaton tended either to increase without limit or to dwindle away to nothing, he settled for a delicately balanced combination which gained widespread publicity through its introduction in Martin Gardner's monthly column in *Scientific American* as the game of "*Life*" [1,2].

Interest in the game persisted several years, leading to a most surprising series of artifacts – oscillators, shuttles, glider guns, puffer trains, very orderly collisions and eminently predictable transformations – culminating in the demonstration that *Life* was capable of universal computation [3]. Even though *Life*'s computer shares an extraordinarily sprawling layout with such predecessors as von Neumann's [4] and Codd's [5], nevertheless they all provided concrete testimony that arbitrarily complicated mathematics could be performed within a system whose basic organization was thoroughly rudimentary.

There was no lack of awareness that those cellular automata which seemed to have interesting or useful properties had been plucked out of an environment populous beyond any normal concept of multitudes. Nor is it surprising that comparative studies of all the automata possible within some given class were not undertaken until computing facilities commensurate with the task were available, and even then not until the study of chaotic systems had gained a certain popularity [6]. But it was just this combination which enabled Stephen Wolfram to study and attempt to classify all possible automata into the four well known categories which now bear his name [7]:

class I: evolution to a constant state;
class II: evolution to isolated periodic segments;
class III: evolution which is always chaotic;
class IV: evolution to isolated chaotic segments.

A recently published reprint volume [8] contains a rather complete account of his work on automata, several closely related papers and an appendix showing sample evolutions. His scheme for numbering rules of evolution is now generally followed, as is the notation (k, r) for a k-state linear cellular automaton whose cells are surrounded by r neighbors on each side. When occasion demands a neighborhood with an even number of cells, r can be taken to be half-integral.

Even though relatively rare, the fourth class was the one which attracted attention because it was the one whose evolution could be regarded as accomplishing some purpose. It was thereby a generalization of the concept of a good *Life*, which is supposed to be one with a quiescent state, for which bounded regions of nonquiescent cells would neither die out nor grow without limit.

Other rules, variants of Conway's, have been tried without any having been reported as being worthy of further attention; it is not unnatural to inquire whether the one game is in fact unique. For example, Packard and Wolfram [9] studied two-dimensional cellular automata after the spirit of Wolfram's survey of one-dimensional automata, declaring there were no class IV automata with the exception of "trivial variants on *Life*".

Preston and Duff [10] mention experiments with hexagonal lattices, some using a rule proposed by Marcel Golay; others with a rule of their own devising. William Poundstone [11] reports a "3–4 *Life*", which had been occasionally mentioned in Robert T. Wainwright's newsletter [12]. The newsletter also carried intermittent reports of a variant of *Life* with three states; the live cells were colored red and blue, with supplementary rules for determining the color of an offspring.

The most recent variation on this theme has been a series of articles by Carter Bays [13–15] seeking a worthy three-dimensional variant of *Life*. His latest article [16] discusses the interrelation of Conway's criteria and Wolfram's classes.

2. Statement of the problem

The novelty of Wolfram's classification sufficed for a time, but further scrutiny raised some interesting questions. Increasing the number of states, the size of the neighborhoods, or even just the length of the row of cells can lead to configurations which do not reach quiescence until larger and larger numbers of generations have elapsed.

Not only do randomly selected automata exhibit this tendency; careful representation of the states and selection of the evolutionary transitions allows the fabrication of automata which can perform simple computations. Configurations of an automaton which behaves as a counter can be designed to become quiescent after an exponential number of generations relative to the number of its cells, thereby approaching the upper limit for the lengths of transients in a finite automaton. Thus it could require a very long time for an experimental decision as to whether an automaton belonged to class I or not.

Even worse, as Karel Culik and Sheng Yu [17] have shown, given that it is possible to simulate a universal Turing machine within a linear cellular automaton having a sufficient number of states, and given that it is undecidable whether such a machine will ever halt with a blank tape, it only requires a certain amount of care in arranging the details of the proof to see that membership in class I is undecidable for arbitrary automata.

Continuing with class II, there can be uncertainty whether an assignment should not be made to class IV instead, if the period is very long. That is, the activity in a fairly large patch of the automaton's field may look chaotic, but if such patches are bounded, the activity must ultimately become periodic. Therefore a true class IV automaton would have to contemplate the possibility of unlimited growth, accompanied by either coalescence of neighboring chaotic patches or some kind of coordinated dilation.

Even class III is not immune to further scrutiny. Not all states need occur with comparable frequencies, nor are complex but regular textures excluded. Indeed, if the hallmark of class IV is taken to be the occurrence of quiescent intervals of arbitrary length, there is no reason to exclude similar stretches composed of other states or even of intricate designs.

Finally, there are many interesting fringe cases, even with the simplest automata. If a binary rule consists primarily of complementation, every second generation may remain relatively constant, so that the apparent class of interleaved generations may be different from the one that was originally evident. A variant of this theme has the uniform neighborhoods alternating generations (or following a longer cycle when there are sufficient states) rather than one of them being quiescent. Interesting activity may develop at the interfaces between constant patches in a configuration, which one might have classified as class IV were it not for the alternation of backgrounds.

The problem, then, is that Wolfram's classification is really not susceptible to a precise definition; even if it were, it would be better derived directly from descriptive parameters than from behavioral observations. Conversely, it would be nice to deduce the parameters from observation. The experimental data which are easiest to observe and accumulate are statistical properties, both regarding frequencies of cell states, and of the patterns into which they can organize themselves.

Consequently there has been a growing feeling that Wolfram's classification is really a scale measuring the distribution of periodic cycles and their possible interconnections, class IV forming a transition region between class II and class III. Thus class I automata admit only cycles of length and period 1, those of class II very short periods and lengths, class IV requires the quiescent state plus long periods, while class III rejects a quiescent state or analogous cycles.

The observable characteristics of the probability distributions, which would supposedly follow the presence or absence of this substructure, would be the occurrence stable fixed points, limiting configurations lying on the border between stability and instability, or some similar arrangement.

In order to discuss any of these concepts adequately, it is necessary to review their bases, first probability and then de Bruijn diagrams.

3. Probabilities and measures

At first, interest in automata consisted in observing their evolution and collecting examples of interesting configurations, but eventually the precepts of information theory were applied to *Life*, first in a 1975 paper by Dresden and Wong [18], later in 1978 by Schulman and Seiden [19].

The simplest procedure would be to judge the probability of a state by the number of neighborhoods generating it. Assuming them all to be equally likely often yields good results; an estimate which can be sharpened by weighting each neighborhood according to the frequency of its constituents, especially if the evolution is used to solve for self consistency. Since the neighborhoods from which adjoining cells evolve overlap, there are doubts as to whether their probabilities are independent; mean field theory results from suppressing such doubts, while other theories arise from the way in which they are taken into account.

An alternative to working out the cumulants required by Schulman and Seiden is to use the probabilities of sequences of cells rather than those of individual cells; just recently Wilbur, Lipman and Shamma [20] calculated some densities in linear cellular automata using this approach. Gutowitz, Victor and Knight [21] made a more detailed analysis along the same lines, basing their approach on Kolmogoroff's theory of probability measures, justifying their estimates in terms of Bayes theorem, and even describing an extension applicable in two dimensions and beyond [22].

In order to fix our ideas, consider the one-dimensional (2,1) cellular automaton evolving by Wolfram's rule 22, which also happens to be the rule of evolution of a cross section of Conway's *Life.* Moreover, it is totalistic rule 2 by the same scheme of reckoning, which means that a cell lives in the next generation if there is a single live cell anywhere in the neighborhood during the present generation. Since three of eight transitions produce a live cell, the *a priori* probability of live cells would be 0.375.

According to mean field theory, if p is the probability that a cell lives, $q = 1 - p$ is the probability that it is dead or inactive, and altogether $3pq^2$ that the neighborhood meets the requirement for a live cell in the following generation.

The requirement for self-consistency is then

$$p = 3pq^2$$

or equivalently, $p(1 - 3q^2) = 0$, whose roots are $p = 0.0$ and $q = 1/\sqrt{3}$ or $p = 0.42$. The derivative of the right-hand side at the origin is 3.0; an unstable fixed point there is clearly consistent with the fact that at low densities isolated cells persist and give birth to two neighbors, multiplying the total population by three. The derivative at the other fixed point is approximately -0.46, which implies a moderate alternating approach to stability.

Fig. 1 shows a graph of this probability (the upper curve) together with the diagonal line of unchanged probability; the two fixed points and their properties are quite clearly visible. Displaying the self-consistency equations graphically is always illuminating; even more so is plotting the empiri-

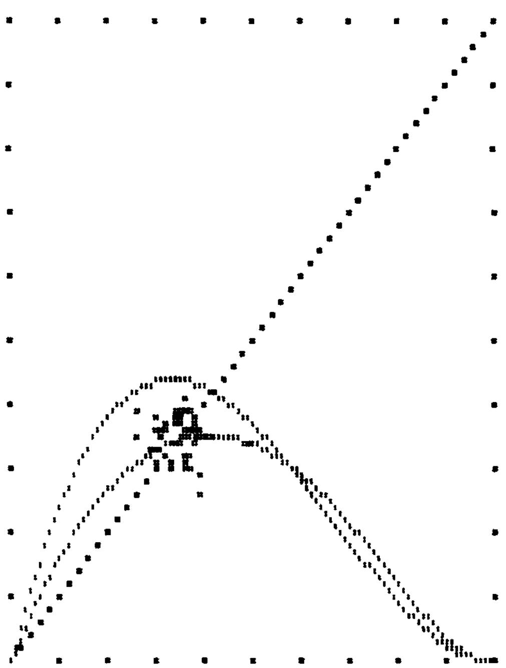

Fig. 1. Self-consistent mean field probability for rule 22. The top curve is for one generation of evolution, the second for two generations. Empirical evolutionary pairs for a 320-cell ring form the cloud of dots.

cal pairs representing successive generations, such would be encountered in an iterative approach to the fixed point.

The most striking observation is that the empirical points do not strictly follow the mean field curves, tending to cluster about a sort of average value, frequently near a fixed point; fig. 1 exhibits a cloud of 50 pairs, showing this effect clearly. As might be expected, in similar environments with unstable fixed points two separate clusters straddle the diagonal.

It can also be noted that the variance per unit length of automaton is quite large, strips of several thousand cells being required before the variance in the mean of the large sample allows predictions of three-figure accuracy to be obtained. Perhaps the intrinsic variance for a given rule could somehow be computed, but serviceable estimates can be obtained from the general knowledge that the variance of a sample of size n scales according to $1/\sqrt{n}$. Thus rows with hundreds of cells, typical of video displays, have enough variance to be conspicuous, yet not overwhelming; automata of fifty cells or less, common when results are presented on a printed line, leave considerable doubt that any probabilistic influences are at work at all.

Careful simulations of rule 22 agree with the stability of its fixed points, but not with the precise location of 0.42. Gutowitz et al. examine this situation at considerable length, concluding that block probabilities for blocks of length six or more are required before the self-consistent numerical probability agrees with empirical observations, which favor a value of 0.35.

Fig. 2 shows a contour plot for the self-consistency of pair, or 2-block, probabilities according to their formulae. Often the equations have multiple solutions; many can be identified with specific periodic configurations, but one will be related to a limiting measure. Longer blocks require so many parameters that the overview afforded by a graphical presentation is no longer feasible; nevertheless numerical solutions by iteration can always be sought.

Another manifestation of correlations lies in the difference between iterated one-generation probabilities (having the same fixed points) and genuine two-generation probabilities taken from the compounded rule of evolution (with their own fixed points); the lower curve in fig. 1 shows the mean field probability corresponding to two generations of rule 22.

4. Further interpretation

Since there are several levels of refinement available for working with probabilities, one wonders which to use. The *a priori* estimate is surely the easiest to obtain; in any event it is the first step in iterating the mean field approximation, starting out with initially uniform probabilities.

Mean field theory works with polynomials; indeed the Bernstein polynomials of probability and spectral theory constitute an appropriate basis. Strictly speaking, a "Bernstein polynomial" is one which has been derived from an arbitrary polynomial by a process of dissection intended to facilitate the calculation of moments and their convergence. When applied to ordinary monomials typical probabilistic expressions result, such as $p^i q^{n-i}$ with $p + q = 1$; these special cases yield what

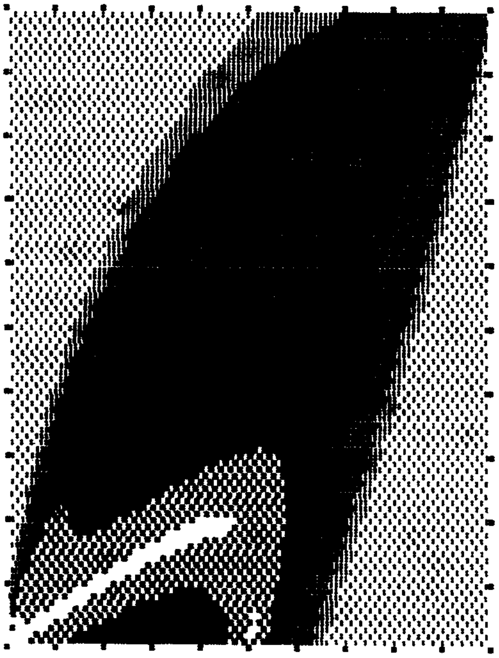

Fig. 2. Contours showing the degree of self-consistency for the evolution of pairs of cells according to rule 22, using local structure theory. Horizontal: probability of a live cell; vertical: probability of a live pair. Only the lower right triangular region is physically significant.

is here called a Bernstein polynomial. The general theory is described in a book by G.G. Lorentz [23]; applications to the moment problem can be found in the memoir of Shohat and Tamarkin [24].

As regards accuracy and convenience, block probability theories, committed as they are to rational fractions, are much more complicated and time consuming; negative factors to be balanced against their superior accuracy. Mean field theory is not only a workable compromise, it is recommended by numerous antecedents from statistical mechanics by which difficult probabilistic situations have been satisfactorily approximated by the same technique.

This does not mean that what one understands by mean field theory is the same for automata as for disordered phenomena (it is not), only that the kinds of approximation involved are similar; not only are they regarded as successful, they are well calibrated through comparison with more exact theories. It would lead too far afield to discuss probabilistic automata, where the analogy is closer; but they can be borne in mind as another source of support for mean field theory.

Whatever the reasons for choosing to work with mean field theory, it turns out that the results are generally quite reasonable; even when evidently lacking in precision, they still lie within the range of 10% accuracy. No less impressive are the qualitative features, such as the observation that some automata have a variety of fixed points while others do not.

For example, it is hardly surprising that automata which would be judged to lie in class III tend to possess stable fixed points away from the origin (the origin, if fixed, would be unstable), in contrast to the others. Consequently the details of any analysis tend to focus on the degree to which the empirical densities actually correspond to values given by the fixed points, and the interpretation of observed behavior when they do not.

It is likewise not unexpected that a quiescent state occupies a stable fixed point for class I automata; it is often observed that class IV automata are associated with tangencies to the diagonal of the fixed point graphs, and that class II sometimes goes with near misses. However these are mere visual impressions, although formed by prolonged observation, whose validity one would like to either explain or disprove.

The easiest aspect to describe is the ocurrence of the quiescent state. As far as binary linear cellular automata go, quiescence evidently requires the absence of the term q^n from the evolution equation for live states (zero is traditionally a quiescent state, n is the size of the neighborhood). The term pq^{n-1} arises two ways – if isolated live cells survive, and if cells are born from contagion: having a single live cell nearby. The first alternative alone produces a curve with a diagonal tangent at the origin; denying both possibilities means that regions of live cells will not expand and that the curve has a horizontal tangent at the origin.

The distance of contagion is an important parameter. It will always extend to the radius of the neighborhood, the number of cells thereby affected varying with the dimension of the automaton. Gutowitz and Victor [25] have noticed this effect while comparing two seven-cell neighborhoods, one taken from a hexagonal lattice with nearest neighbor interactions, the other a linear lattice with third neighbor interactions. In the hexagonal case, it is sufficient to exclude the terms q^7 and pq^6 from the self-consistent probability, but in the linear automaton p^2q^5 and p^3q^4 also have to be excluded to observe class II or class IV behavior. They found similar contrasts between the nine-cell neighborhoods representing two-dimensional *Life* and (2,4) linear automata, respectively.

Bays [13] has proven theorems regarding the behavior of three-dimensional automata with various constraints on the neighborhoods (in his semitotalistic notation). Thus there seems to be a general agreement that the evolution of certain neighborhoods containing large numbers of quiescent states must preserve quiescent states, which is half of Conway's original argument that isolated or relatively isolated states must die out in order to provide an interesting automaton.

Although the other half of his argument – how cells die from overpopulation – still has to be taken into account, the implications of quiescent domains and similar structures can be explored even further.

5. De Bruijn diagrams

Whether working with probabilities, measures, or evolution, overlap between adjoining neighborhoods is the greatest technical obstacle to computations; unless this problem is resolved adequately, further progress is almost impossible. Fortunately, a diagrammatic technique lying at the heart of shift register theory [26] saves the situation; the diagrams are called de Bruijn diagrams, but are merely simple graphs showing the possible ways for neighborhoods to overlap [27]. Erica Jen has shown how many properties of automata can be extracted from such diagrams, especially the static and periodic configurations on a cylinder of fixed circumference [28]. Wolfram used these subdiagrams earlier in his theoretical analysis [29] of cellular automata; although he called it a "regular expression diagram," Wentian Li [30] used exactly the same concept to locate the periodic strings in an automaton; there is no doubt that the ideas have also been used elsewhere with greater or lesser degrees of formality.

In principle, such a diagram could be extended to automata of higher dimensions, but a problem arises from selecting partial neighborhoods that will join to form full neighborhoods in all directions. The straightforward approach of building up strips of successively higher dimension runs afoul of Post's correspondence principle when arbitrary intermediate strips have to be matched to form the layers of the next higher dimension. If only periodic solutions are required, the problem is still soluble. Hopcroft and Ullman [31, ch. 8] give a good explanation of the difficulties involved, which are related to the fact that the halting problem for a Turing machine can be worked into a similar context. This, of course, is the prototypical example of a problem without an algorithmic solution; Minsky [32, pp.273ff] also contains a short description.

At least in one dimension, there is nothing difficult about a de Bruijn diagram; as applied to cellular automata it is simply a graph in which partial neighborhoods are the nodes, with links connecting those which may overlap to form a full neighborhood. Given this correspondence between links and full neighborhoods, each link is also associated with the evolved cell belonging to the neighborhood. Consequently characteristics of the evolution can be used to select subgraphs of the de Bruijn diagram; for example, there is a subgraph composed of the neighborhoods whose central cell never changes. Global properties of the automaton can be read off from the chains inherent in the subdiagram; in this example, the chains yield all the static configurations.

De Bruijn diagrams transform automata problems into known path tracing problems. For instance, no loop can be longer than the total number of nodes in the graph without repeating some segment; but the way is open for still other loops in which the repeated segment is traversed any arbitrary number of times. As an example, since a binary automaton depending upon nearest neighbors has eight distinct neighborhoods, representable as eight links connecting four nodes, it follows that no static configuration can be more than five cells long without repeating some two-cell partial neighborhood.

Many more characteristics than the static, or "still life", configurations of an automaton can be deduced from its de Bruijn diagram. Translational

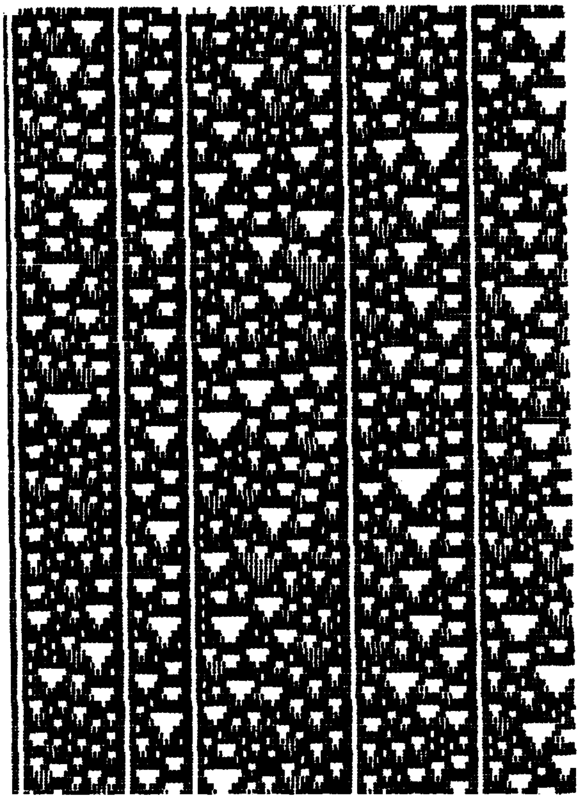

Fig. 3. Sample evolution of a typical (3,1) macrocell rule. For this rule, the state sequence 102 creates a barrier, visible as a vertical streak. Macrocells form in the intervals between barriers; small ones quickly become periodic.

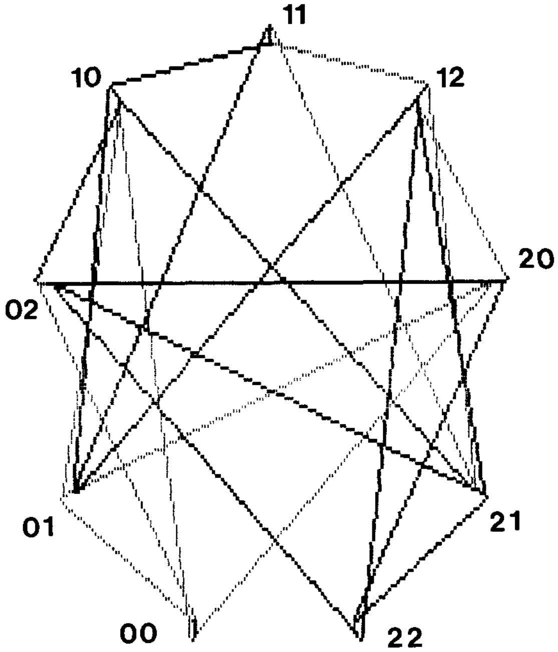

Fig. 4. A de Bruijn diagram which results in macrocells: (3,1) rule 022221211221211012222111120. The sequence 102 defines a cell membrane. Darker lines represent static links; all incoming links at node 10 and all outgoing links at node 02 define still lifes.

uniformity having been built into the very definition of cellular automata, shifting configurations which cover a distance d each generation are as easily found as the static ones. By working instead with the compound rule governing several generations of evolution, and enlarging the "neighborhood" under consideration, the characteristic can be extended to a shift of d cells every g generations; among these are the very interesting class of *superluminal* configurations – those which move further than an average of r cells per generation.

Although rarely repeating beyond a single generation, any Boolean (for $k = 2$) combination of the states of a neighborhood and its evolutes can be detected via the de Bruijn diagram; for example configurations which evolve into constants, or into their complement.

Sometimes the de Bruijn diagram reveals information about localized aspects of a configuration. If an acceptable path terminates at a node in which *all* the outgoing links are acceptable, it need continue no further. Likewise if all the incoming links are acceptable, the path may begin just as though it had been part of a loop. Thus semi-infinite structures may be located, or even finite ones if both ends have such universal terminations. This leads to the phenomenon of membranes and macrocells which Wolfram noticed during his investigations. David Brown [33] has also shown a nice example of such barriers and sheltered regions. That is, an automaton may have patches which are isolated from one another by static regions; the patches evolve quite independently.

That the (3,1) automaton whose evolution is depicted in fig. 3 meets the requirements may be discerned through examination of the de Bruijn diagram presented in fig. 4.

Conversely, a rule of evolution of an automaton is definable by postulating that the de Bruijn diagram have prescribed properties; for example that the sequence (01)* must be static. To the extent the requirements do not contradict one another, and all the alternatives are covered, automata may be created to fulfill ones wishes.

Enumerating the paths through a graph is a classical task, to which many papers have been devoted, but which has a particularly elegant solution in terms of regular expressions. Conway's book on regular algebra [34] expounds the technique; a later article of Backhouse and Carré [35] gives a very thorough presentation.

The only requirement for partial neighborhoods is that all be similar, yet overlap sufficiently to create complete neighborhoods. The usual choice of a linear segment just one cell short of a (k, r) automaton's quota of $2r + 1$ cells per neighborhood – creating the maximum overlap possible – yields k^{2r} different partial neighborhoods. Each defines a node in the de Bruijn diagram, all of them interconnected by k^{2r+1} links, associated with the neighborhoods themselves. Omitting the right cell from a link reveals its source, excluding its left cell shows the destination node, making the relationship to a $2r$-stage shift register apparent.

Many attempts at artistic representations of de Bruijn diagrams have been made; extensive line crossings cannot be avoided except for the very simplest of them. The most systematic representation may be through selected chords of a regular k^{2r}-gon; arranged around the circumference of a circle, the vertices can be numbered sequentially as though the states of the cells were digits in a k-adic number. The connectivity matrix of the diagram would have the matrix elements

$$M_{ij} = 1, \quad j = \begin{cases} ki \\ ki+1 \\ \cdots \\ ki+k-1 \end{cases} \pmod{k^{2r}}$$
$$= 0, \quad \text{otherwise.}$$

Traces of powers of the de Bruijn matrix readily show that there are k^n loops of length n, counting each loop once for each distinct node. However, a loop for which no node is repeated can have at most k^{2r} links. Long sequences necessarily repeat some of their subsequences, rendering really long loops redundant. Such counting arguments yield rapid derivations of various interesting results, such as the bounds obtained by Guan and He [36] for the number of cycles in border decisive automata.

6. Loops, tangencies, Wolfram classes

There are many kinds of subdiagrams in the de Bruijn diagram; indeed the diagram's most important role is that of a common upper bound for all the diagrams, whether constructed from formal language theory or for other reasons, that are then used in automata theory. Fairly evident for subdiagrams which are nearly full subsets, the relationship can be readily overlooked or missed entirely in the case of sparse diagrams, especially if they have been extensively simplified. Yet this is the situation in which knowledge of their existence may well be the most valuable.

For example, it is generally understood that finite cellular automata must evolve into periodic cycles, whose temporal length is bounded by the total number of configurations, albeit exponentially large in terms of the numbers of cells and states. The finiteness of the de Bruijn diagram similarly bounds the spatial periodicity of an infinite or periodic automaton, in the sense that the period must be constructed from a closed loop within the diagram appropriate to the period under consideration. Again the bound may be exponentially large in terms of the number of states and length of the partial neighborhoods; in practice both bounds are extremely liberal.

So it is that many kinds diagrams are pertinent to automata theory, not a few arising from the description of configurations or their evolution via a formal language. Thus diagrams are often constructed to realize some regular expression; really the two concepts are coextensive. The particular role of *de Bruijn* diagrams lies in their ability to describe overlapping sequences; for example, all the segments of a regular expression of a given length. In turn, the algebraic properties of the de Bruijn matrix – the connectivity matrix of a de Bruijn diagram – lead to precise numerical estimates of such things as the number of possible sequences of given type, or to the bounds which have been mentioned.

One application consists of marking all the links evolving into one specific state, say the quiescent state. Varying the evolved state partitions the links into equivalence classes. Interestingly, such a subdiagram consists exclusively of closed loops when the evolutionary rule is totalistic. A totalistic rule depends only on the sum of the values of the cells in a neighborhood, causing all its permutations to evolve the same way. Because passing from one link to the next implies dropping a cell from the left and adding another to the right, a cyclic shift will always yield a continuation preserving the given sum.

The precursors of a static configuration would be described by a path through the appropriate subdiagram, but the individual links can be used for other purposes. For example the probabilities represented by the links evolving to live cells could simply be summed, to obtain the Bernstein polynomial to be used in self-consistent comparisons. Many other arrangements of links into subdiagrams are also possible, such as those which correspond to a static central cell (still lifes), or to uniformly shifting configurations (some of which are called gliders).

If there is to be either a class II or a class IV region, the still life diagram should contain a self loop to the quiescent neighborhood, together with some other loop. If the self loop is not present, quiescent regions of arbitrary length cannot exist, and if there are no loops at all, there are no extended still lifes. If there are additional loops connected to the quiescent self loop, there is a correspondingly greater variety of still life possible; otherwise every island of still life would be identical and no one would call it chaotic. Since it is a property of diagrams that a loop which may be traversed at all may be traversed an arbitrary number of times, the variable separation between live regions in these classes of automata becomes understandable, as well as the minimum separation which is sometimes required.

The self loop to the quiescent partial neighborhood corresponds to the Bernstein monomial q^n; incoming and outgoing links to the rest of the diagram belong to the monomials pq^{n-1}. The self loop cannot be associated with the live state, by definition. Connecting links establish whether contagion is to be avoided, and in some cases how isolated cells behave. If the decision is for quiescence, all these monomials must be excluded from the self-consistent probability of the live state. Thus requiring a horizontal tangent at the origin of the probability curve relates directly to some of the finer details of the de Bruijn diagram.

The quiescent state places zero as a fixed point of the probability curve, but expansivity or virulent contagion render it unstable. Stability, maybe even superstability, results whenever quiescent regions are not so readily invaded, which is surely one of the hallmarks of Wolfram's class IV; but the rest of the probability curve has to be accounted for, as well as the degree of internal chaos necessary to distinguish class II from class IV. Whether there are additional fixed points, and the nature of their stability should be investigated.

If the mean field probabilities were reliably followed during the evolution of an automaton, well developed theories of functional iteration could be applied. One of the best known of these utilizes the Mandelbrot set and related concepts to parameterize the behavior of iteration in the complex plane as well as the real line. As it is, the empirical cell frequencies observed in each generation do not track the functional iteration of the probability curve very well. Nevertheless gross properties of the fixed points appear to be valid; where the slope of the curve is negative, higher and lower densities tend to alternate, for example. Points on the borderline of stability, which is to say, those for which there is a diagonal tangency or near tangency at the fixed point, seem to be especially significant. In such instances, there is more of a fixed interval than a fixed point, permitting a greater variety of permanent structures than otherwise.

If there are not many ways of producing or conserving a live cell, the probability curve will never reach very high values, especially if it is already constrained to begin with a horizontal tangent. Thus even its maximum value may not even be sufficient to sustain its own density for the next generation, a reasonable conclusion if the curve does not even cross the diagonal. Alternatively, the production of live cells in the new generation may just barely meet the break even point, represented by a tangency. Finally it may be that modest densities proliferate freely, as would be expected from a curve whose values lay well on the other side of the diagonal.

This reasoning could well describe the transition from class II, where maintaining a live population is difficult and the regions which survive are nonexpansive, through class IV, where it is just barely possible over a limited range but with quiescent regions freely interspersed, until the opposite extreme of a class III, in which quiescent regions do not last long, and never dominate the field of evolution.

7. A survey of small binary automata

In summary, it is proposed to explain Wolfram's classes by a mixture of probability theory and de Bruijn diagrams, resulting in a classification based on readily visible properties of the mean field theory curve:

class I: monotonic, entirely on one side of diagonal;
class II: horizontal tangency, never reaches diagonal;
class IV: horizontal plus diagonal tangency, no crossing;
class III: no tangencies, curve crosses diagonal.

Classes being undecidable in principle, and mean field theory being approximate, the utility of this classification must be judged on empirical evidence. Although reasons have been given for its justification, the part concerning horizontal tangencies at the origin has the best foundation, being an accurate description of both automata and desirable mean field curves, whatever the actual relationship between the two. To illustrate the procedure, consider some simple examples:

7.1. (2,1/2) automata

The simplest automata are probably the sixteen of radius 1/2 corresponding to the sixteen primitive Boolean functions of two variables. Their Bernstein polynomials are formed from a basis of p^2, pq, and q^2, which does not provide much variety for tangencies; nevertheless much of the mischief possible in rendering classifications is already visible in this elementary example. In terms of the mean field curves, there are essentially four groups to consider:

eight noncrossing rules, 0, 2, 4, 8, 11, 13, 14, 15;
four diagonal and antidiagonal rules, 3, 5, 10, 12;
two superstable rules, 6, 9;
two with unstable crossings rules, 1, 7.

Visual inspection places the noncrossing rules in class I or class II according to whether some nonquiescent cells remain after a few generations or not. Diagonals result from copying one of the ancestors, antidiagonals to its complementation; the former produce shifts, the latter dramatic differences from generation to generation which could be assigned to class III.

The four remaining rules qualify for class III through fixed points interior to the interval (0,1); in two cases the fixed point is stable, in the other two it is not. Assignments to class III on the basis of the widespread complementation due to an unstable fixed point are always dubious; Wolfram [7] would probably sidestep the issue by classifying the composite rule for which enough generations have elapsed to possess a quiescent state.

By this criterion, only rule 6 (EXCLUSIVE OR) and rule 9 (EXCLUSIVE NOR) belong strictly to class III, none to class IV, the rest to class I or class II.

7.2. (2,1) automata

The next simplest comparisons are worthwhile because the (2,1) automata have been extensively studied. Arranging the eight possible neighborhoods in sequence corresponding to Wolfram's codes, the transitions required by quiescence and guaranteed contagion can be inserted.

111	110	101	100	011	010	001	000
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Thirty two rules arise from the five free choices remaining:

18	22	26	30	50	54	58	62
82	86	90	94	114	118	122	126
146	150	154	158	178	182	*186	*190
210	214	*218	*222	*242	*246	*250	*254

Computer simulation or examination of Wolfram's tables [6,8], shows that these are Wolfram's class III automata, although not quite. The rules whose numbers are prefixed with stars resemble class II more closely than class III, quite in agreement with the failure of their curves to cross the diagonal. Rule 58 might not be considered part of class III because of evolving towards a very strongly shifting pattern, albeit space filling; its curve has a superstable fixed point.

Among the remaining rules, there is a group complementary to the ones just considered, for which live cells would constitute the quiescent background. Beyond this, there are many nonsymmetrical with respect to contagion which can mostly be classified according to the combination of their one-sided behaviors, leaving the nonexpansive rules to fall into Wolfram's class I or class II, with very little remarkable behavior.

The de Bruijn diagram shows how to form macrocells, for example with rule 73. All one-cell environments of the segment 0110 yield still lifes, yet not all form loops; instead they form barriers enclosing macrocells. Similarly all single cell extensions to the segment 11 in rule 154 shift left, bounding a moving macrocell. Boundaries which were not originally present may jell during the course of evolution, remaining thereafter. For automata with more than two states, partial barriers as well as complete barriers may exist, be formed, or dissolve, according to how many terminal links share the properties of the interior of a segment.

As an example of automaton design, one could postulate that the quiescent loop 0^* be linked to the loop $(01)^*$, fixing five of the eight links in the de Bruijn diagram. Of the three links and eight choices remaining, rule 4, of class II, results from rejecting any additional still lifes. Its Bernstein polynomial, pq^2, always lies below the diagonal, with a diagonal tangency at the origin. The rule is nonexpansive, but low concentrations of live cells are conserved, so that it is noncontractive as well. The discrepancy with the proposed classification arises from the ocurrence of pq^2 through the survival of isolated cells rather than from contagion. This distinction must often be made to reconcile the correct classification.

Some of the other extensions of rule 4 incorporate additional loops, up to the extreme of the identity rule, rule 204, which is best left unclassified. Is it class II for extensive still life, or class IV for its blatant tangency? Its Bernstein polynomial is $p = p(p+q)^2 = pq^2 + 2p^2q + p^3$.

Rule 104 is totalistic rule 4 for (2,1) automata, which places it in an interesting category. For (k,r) automata with integer r, totalistic rule $pk^{p(r+1)}$ has an isolated still life consisting of the string $0^r p^{r+1} 0^r$, as well as for all rules whose Wolfram number is a nonzero multiple (mod k) of this basic number. For example, (2,1) totalistic rules 4 and 12, which are rules 104 and 232. A similar effect is even seen in Conway's *Life*, for which any large region in which every 3×3 square contains exactly four live cells is a still life, and will persist until its border is eaten away.

Let us call this particular block an α-block. A rule whose still lifes include α-blocks should be expected to behave anomalously with respect to block probabilities calculated for blocks shorter than $2r+1$-blocks, a conclusion which is borne out by experience; in this instance, significant probabilities for live cells do not appear until 4-blocks are considered.

The Bernstein polynomial for an α-block rule would be

$$\frac{(2r+1)!}{(r+1)!r!}\, p^{r+1} q^r ,$$

which is slightly asymmetrical and agrees with class II by approaching but not crossing the diagonal. Its presence in the de Bruijn diagram is manifested by a closed loop with incoming and outgoing links to the quiescent loop.

7.3. (2,3/2) automata

Four cells per neighborhood signify an automaton which can be one-dimensional with radius 3/2 or two-dimensional with radius 1/2, for which the effects of a difference in dimensionality can be observed. The neighborhoods of automata with half-integer radii lack a central cell, so the survival of isolated cells is not a direct issue, although "still lifes" can span two generations.

Interestingly, totalistic rule 4 yields the Bernstein polynomial p^2q^2, shown in fig. 5, which almost displays a diagonal tangency. In two dimensions, this rule possesses several still lifes of period 2 which coincide with actual *Life* still lifes on alternate generations. Gliders have not yet been observed; the five cell gliders which work with *Life* do not glide in this environment.

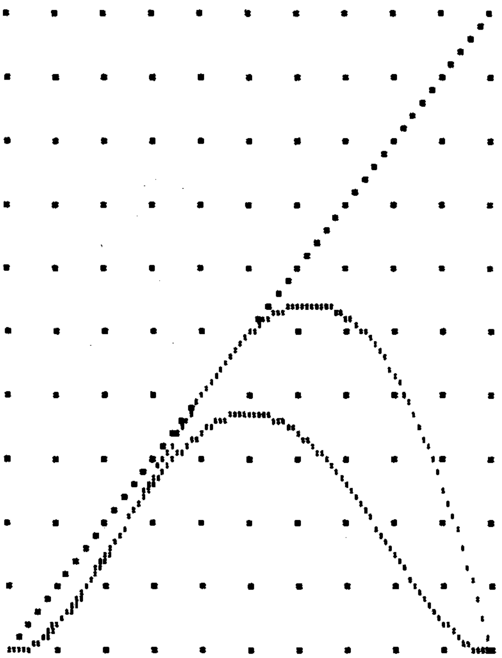

Fig. 5. Mean field probability curves. Upper: generic rule 30376, lower: totalistic rule 4 (rule 5736).

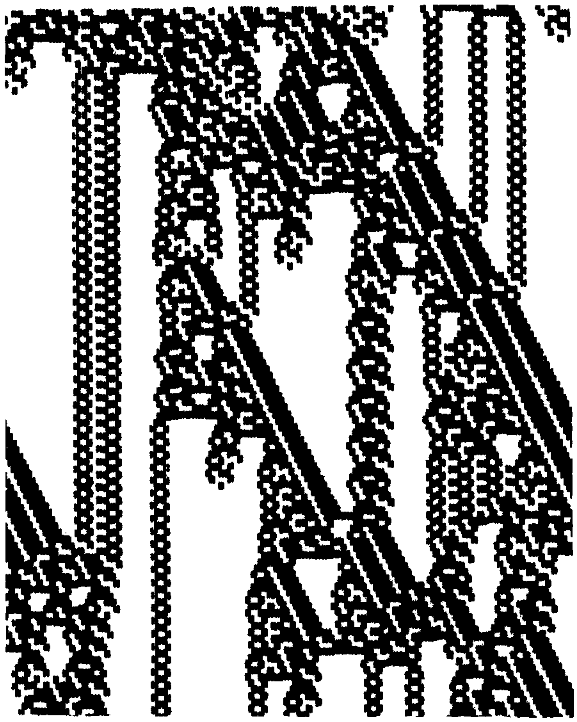

Fig. 6. The one-dimensional (2,3/2) totalistic rule 4 belongs to class II, its mean field curve falling short of the diagonal. The temporal evolution of the nearby class IV rule 30376 is shown.

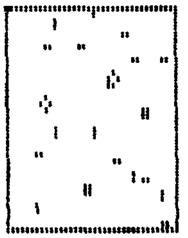

Fig. 7. Sample field of evolution for the two-dimensional (2,1/2) (just barely) class IV totalistic rule 4, while rule 30376 is class III.

A systematic way to find diagonal tangencies is to expand the Lagrange interpolation polynomials of numerical analysis in a basis of Bernstein polynomials, or better yet, to use Lagrange–Hermite interpolation. Lagrange polynomials are defined as polynomials of nth degree which vanish at n points and take unit value at one additional point. With Lagrange–Hermite interpolation, derivatives may be specified in addition. Unfortunately, all such polynomials are extremely non-orthogonal, exposing the determination of coefficients to a very sensitive dependence on parameters. Furthermore, positive integral coefficients are required, creating a problem in integer programming.

All these difficulties beset the selection of a polynomial whose interpretation is not all that precise; much better results are obtained empirically by graphing the polynomial belonging to a given rule, and then tinkering with individual neighborhoods to alter the tangencies which are visible. Since the monomial $p^i q^{n-i}$ has a single, not too broad, maximum at i/n, it is not hard to decide which values to adjust, while simultaneously evaluating the position of the neighborhood in the de Bruijn diagram. Indeed, the classical reason for using Bernstein polynomials, in their general sense, was the ability to work in a given interval with functions which were entirely positive and well localized, becoming even more so in the limit of high degree.

Fig. 6, several generations of one-dimensional evolution, and fig. 7, a typical two-dimensional field, contrast the differences in dimensionalities. Low dimensions require high degrees of tangency at the origin to produce class IV. The classes are also sensitive to the closeness of the diagonal tangency.

7.4. (2,2) automata

Wolfram has classified the 32 quiescent totalistic (2,2) linear automata [7] as follows:

class I: 0, 4, 16, 32, 36, 48, 54, 60, 62;
class II: 8, 24, 40, 56, 58;
class III: 2, 6, 10, 12, 14, 18, 22, 26, 28, 30, 34, 38, 42, 44, 46, 50;
class IV: 20, 52.

By and large, class I follows Wolfram's description, although many of the rules listed have simple structures of low period. For example, the regular expression $(0011)^*$ describes a still life for totalistic rule 4 and some of the others, while $(01)^*$ and $(000111)^*$ have period 2. While not evolution to a quiescent state, neither is it evolution into disconnected periodic regions, the province of class II. Wolfram dismisses such anomalies by invoking exceptions "of measure zero", but one suspects that they are inherent in the classification system and part of the reason for its approximate nature.

Class II consists principally of the rules with α-blocks, except for rule 58, and including rule 24, which has an exemplary class IV tangency. Its excellence notwithstanding, graphing the ninth-degree polynomial for *two* generations of evolution confirms the assignment to class II.

The class III automata behave as expected; their rules are all expansive. Whenever long gaps arise they begin to fill up immediately.

The two class IV rules are similar in their behavior, but rule 52 is especially interesting for being self-complementary; its evolution can resemble certain Escher prints at times. On account of nonorthogonality its mean field polynomial, $10p^2q^3 + 5p^4q + p^5$, gives a fairly good representation of the function p; its graph is symmetric by 180° rotation. All automata have similar rules, especially those of larger radii.

7.5. (2,7/2) automata

Eight cells per neighborhood allows another dimensional comparison: one-dimensional automata of radius 7/2, or three-dimensional of radius 1/2. The three-dimensional diagonally tangent totalistic rule yields better candidates for class II automata than class IV, but it is possible to locate such artifacts as a cube–octahedron pair which oscillates with period 2, following totalistic rule 24. Larger composite structures of similar form also exist. Variants on the theme of semitotalistic rules yield other candidates for class IV.

In any event, the experience of Gutowitz and Victor [25], that similar patterns do not transfer from one dimension to another, and that a higher degree of tangency is required to obtain class IV in the lower dimension, is confirmed. Likewise Bays' experience of finding small artifacts is repeated, without encountering the larger structures characteristic of *Life*.

8. Other automata

8.1. Totalistic rules

Totalistic rules give some fairly strange results, until it is understood that they do not form a representative sampling of all the possible rules. For binary automata, a totalistic sum can only remain constant, increase, or decrease by 1 from one neighborhood to the next, so that adjoining cells assuredly do not enjoy independent probabilities. Were the probabilities truly independent, the sums would approximate a Gaussian distribution, but with such a restriction they would follow a much flatter distribution characteristic of the dominant eigenvector of a Markov chain. By favoring quiescent domains, this distortion of statistics enhances class IV automata among totalistic rules relative to the general population.

Another characteristic of totalistic rules is the fact that they lead to closed loops in the de Bruijn diagram. Summing the cells of a neighborhood defines an equivalence relation for neighborhoods, thus for links also. Cyclic shift assures every link of a continuation with the same sum in either direction, so that an equivalence class contains only closed loops. Totalistic rules map equivalence classes, and the result follows. In turn there is a closer affinity between totalistic rules and mean field probabilities, because there is no need to choose between permutations in assigning a Bernstein polynomial to an entire equivalence class.

8.2. Near totalistic rules

When totalistic rules do not allow for sufficient variation in parameterizing an automaton, the so-called semitotalistic rules are useful, sharing many of the statistical characteristics of totalistic rules. One reason for their use is that they arise naturally when small neighborhoods are submitted to symmetry requirements. Then the central cell may distinguish two kinds of evolution according to its value, as in *Life*. Automata with half-integer radii lack such a central cell, but there are substitutes, such as summing separately the even-parity cells and the odd-parity cells, or choosing a two-cell central region.

8.3. Automata which are not binary

When there are more than two states per cell, the Bernstein polynomials become multinomials, and the probabilities for all the states have to be fit at once, complicating the discussion of tangencies. Two correctives seem to be efficacious; one of them is to calculate a sum of squares of differences between the individual probabilities from one generation to the next – a distance between vectors of probabilities. Even so, it is not a function which is easily graphed except perhaps for trinary automata and with considerably greater difficulty of visual presentation for quaternary automata.

The second technique is to use the parameter λ introduced by Christopher Langton [37], which is the average probability of all the nonquiescent states. If the probability space is supposed to be a simplex, λ is the distance from the quiescent vertex to the opposite face, measured along an altitude. As such it may miss some of the twistings and turnings of the contours of constant deviation; it has the advantage of being a single representative parameter, but interpreting crossings of the diagonal must take into account its following a direct line rather than the gradients. Experience shows that it is a reasonable guide, but that it must be used rather cautiously. Fig. 8 shows an instance of a curved arc in a class IV automaton.

8.4. Probabilistic automata

Variants on the theme of cellular automata are interesting topics of study, either for modeling some particular physical process or for more intrinsic reasons. One is the coupled map lattice, another the probabilistic automaton. In the former an assemblage of iterative functions of a real variable (not necessarily probabilities) is coupled together in some way, typically as lattice neighbors. In the latter the state of the cells of the lattice is not known, only the probability that each is in one of its possible states.

In both cases, the law of evolution is exactly known, not approximated by mean field theory or one of the block structure theories. This was how Schulman and Seiden [19] brought estimates of the density of live cells in *Life* into closer agreement with their calculations. Recently Bidaux, Boccara, and Chaté [38] used the same approach to study the effect of dimension on phase transitions in a class of probabilistic automata. If rigorously probabilistic automata behave reasonably, any discrepancies observed in cellular automata can be pinpointed as having arisen from correlations among

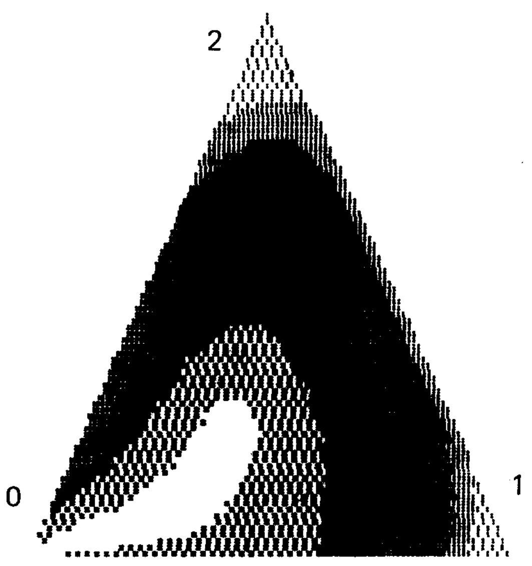

Fig. 8. Contours for mean field probability differences after one generation for (3,1) totalistic rule 792. There is an arc of tangency, which will be poorly sampled by Langton's parameter λ.

discrete states; useful limiting behavior might also be expected.

8.5. Automata designed to order

It is instructive to invert the process of seeking out interesting automata by examining the statistical properties of automata with known characteristics. This reverse process is heavily biased toward quiescent automata with large inactive regions because those are the ones which our experience tells us how to handle; highly parallel processes interacting closely are not at all commonplace. It is easy enough to devise simple automata which behave as counters, parenthesis balancers, or the like. Comparison of their statistics with the description of class IV allows a judgement of whether that is an appropriate niche for them; many examples do meet the requirements.

9. Discussion

There does not seem to be a single, infallible criterion as to what makes a good "*Life*", in part because the concept itself is somewhat subjective. Nevertheless there is a certain amount of experience in several contexts which indicates that the existence of independent isolated structures is fundamental, and that the structures which do exist should have a certain stability.

Isolation is easily tied to the existence of certain loops in the de Bruijn diagram for the automaton. Knowing the de Bruijn diagram for the still lifes does not determine the diagrams for higher periods, but their behavior is obviously part of the criterion that one would like to establish. In practice, the still lifes seem to give good guidance; moreover if the still life diagram does not behave properly it is assured that those for longer periods will not do so either.

Tying the stability of extant structures to tangencies in the mean field probability curve likewise seems to be fairly speculative. Nevertheless this curve seems to give a fairly good approximation to probabilities calculated via more complex approaches. Additionally, it is possible to alter the rule of evolution slightly and to observe a certain continuity of behavior as the probability curve changes from tangency to greater stability for one of the fixed points. For example, the (2,3) totalistic rule 88 seems to stretch the limits of what one ought to call a class IV automaton, which is partly why it supports a glider gun and not just gliders. Also note that tangency of the probability curve only implies constant density – not constancy of evolution – but the contrary assures a varied evolution.

The question of the stability of classification against minor changes is an important one. Kunihiko Kaneko [39] has discussed variations in the basin of attraction of a configuration whose individual cells have been altered; Li, Packard and Langton [40] consider the effect of alterations at the level of the de Bruijn diagram. An alteration in the evolution of one neighborhood results in reassigning its link from one subdiagram to another. This will break some loops and close others, but will not affect badly fractured loops. Some rules will be more sensitive to such shifting than others, depending on the fracture patterns arising when the subdiagrams are created; relocating one or a few links could affect many loops or none at all.

10. Conclusion

Conway's *Life* is the prototypical example of a class IV automaton, and it remains an interesting question as to why it shows such remarkable behavior relative to variants which have been studied. In a sense it is almost unique: if a rotationally and reflectively invariant rule is required, there are 102 symmetry classes of *Life* neighborhoods, and it is possible to tabulate the number of these which are used in the evolution of any of *Life*'s artifacts. If gliders, glider guns, and the two shuttles used in forming glider guns are counted, the evolution of all but a few neighborhoods is determined. What is surprising is that all the transitions required have such a succint description, via an easily described semitotalistic rule.

Of course there is no principle so far known excluding the existence of alternative collections of artifacts, with still another style of operation, so the uniqueness of *Life* is only relative to the present state of our experience. Maybe the search for alternatives has been insufficiently vigorous, or

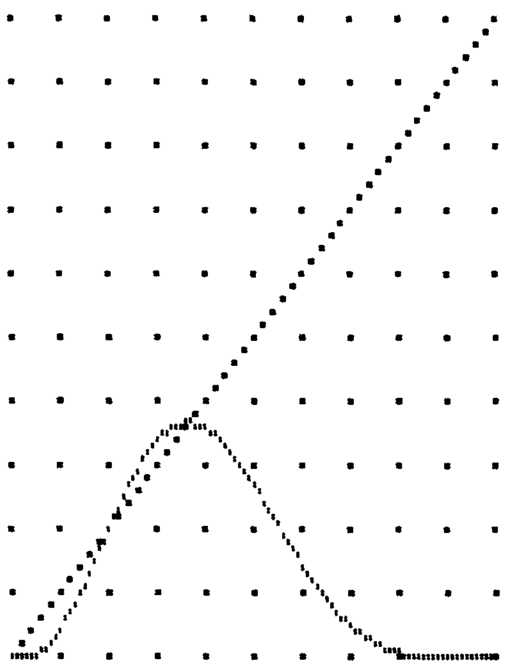

Fig. 9. Mean field curve for Conway's *Life*; crossing the diagonal slightly more than a tangency, quadratic at the origin.

possibly binary automata with close neighbors operate in just one way. Knowledge and systematic usage of de Bruijn diagrams has spread slowly; heretofore the discovery of artifacts has largely been one of ingenuity and inspiration coupled with detailed analysis. Even so, the exponential growth of the diagrams still makes it impractical to search for very large structures.

With present equipment, only first-generation patterns in strips of width less than ten cells are feasible for the nine-cell two-dimensional neighborhoods that *Life* or its variants utilize, yet reproducing results already known would need four generations. That is one reason to work with half-integer neighborhoods of four cells, whereby two generations can be searched. One pays for the privilege of a more comprehensive search by working in an arena with fewer possibilities.

The foregoing analysis may well have to be judged by the degree to which it applies to this one case, *Life*, whose curve is shown in fig. 9. It would have been nice to say that a clear cut class of automata had been identified, in which some new specimens had been found exhibiting outstanding behavior. They may yet be found; meanwhile progress is more likely to come from inventing ingenious mechanisms which can be implemented as cellular automata. And above all, from carrying out some of the searches which are now possible.

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