Multiple-Valued Quantum Logic Synthesis

2002 International Symposium on New Paradigm VLSI Computing

December 12, 2002, Sendai, Japan,

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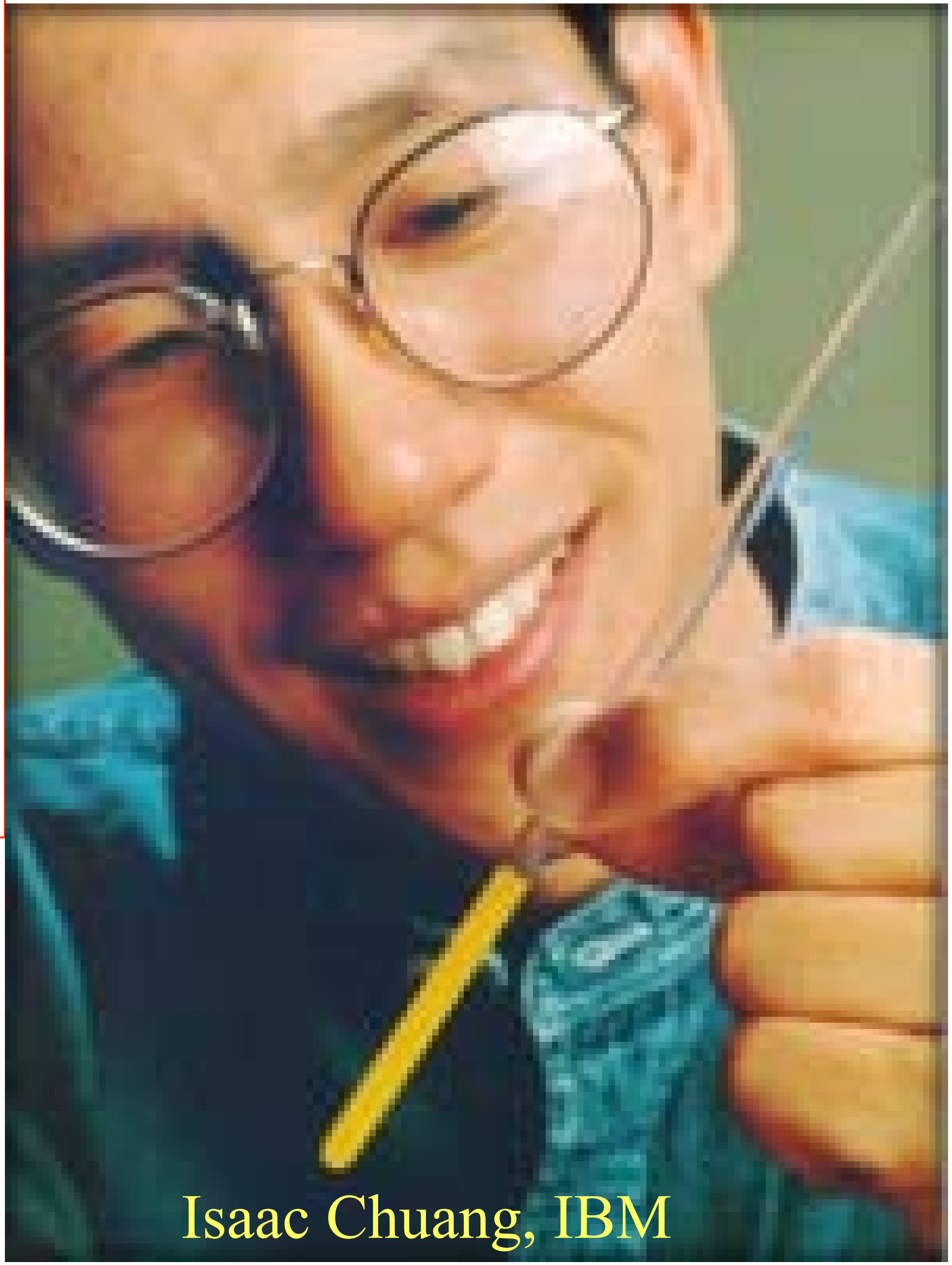
What size of (binary) Quantum Computers can be build in year 2002?

- *7 bits*

Is logic synthesis for quantum computers a practical research subject?

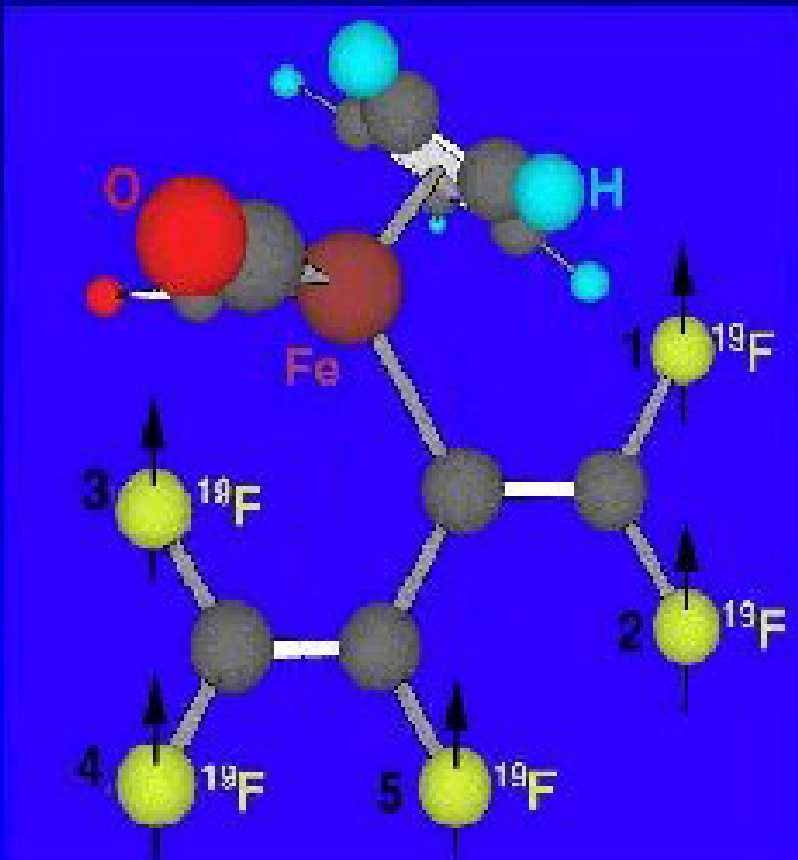
Yes, it is a useful technique for physicists who are mapping logic operations to NMR computers.

***CAD for physicists*.**

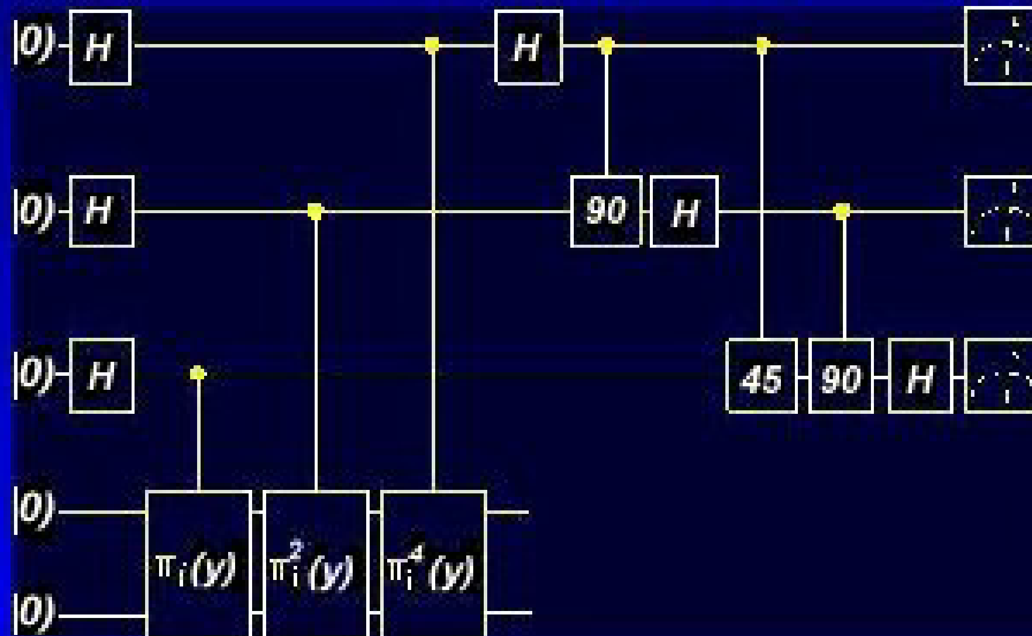


5 qubit 215 Hz Q. Processor

(Vandersypen, Steffen, Breyta Yannoni, Cleve, and Chuang, 2000)



- The Molecule

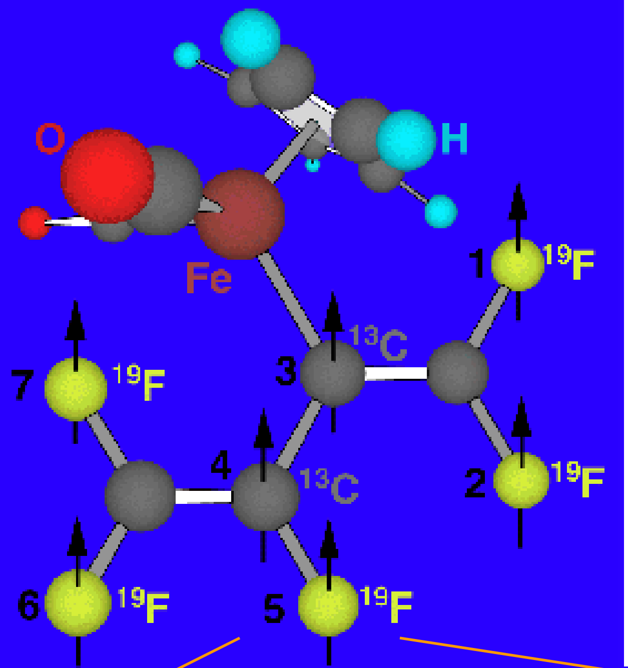


- Quantum Circuit

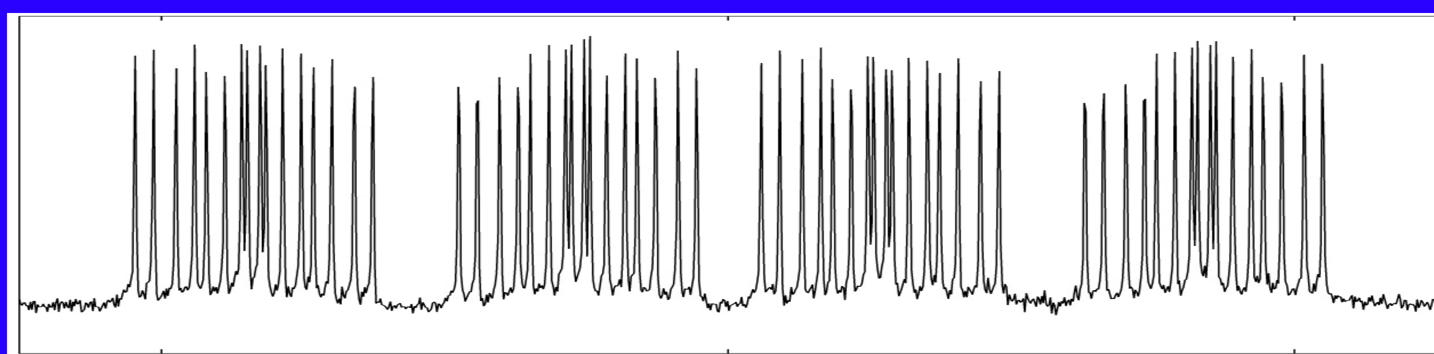
T_2 > 0.3 sec ; ~ 200 gates

Source: IBM, Hot Chips Conference, 2000

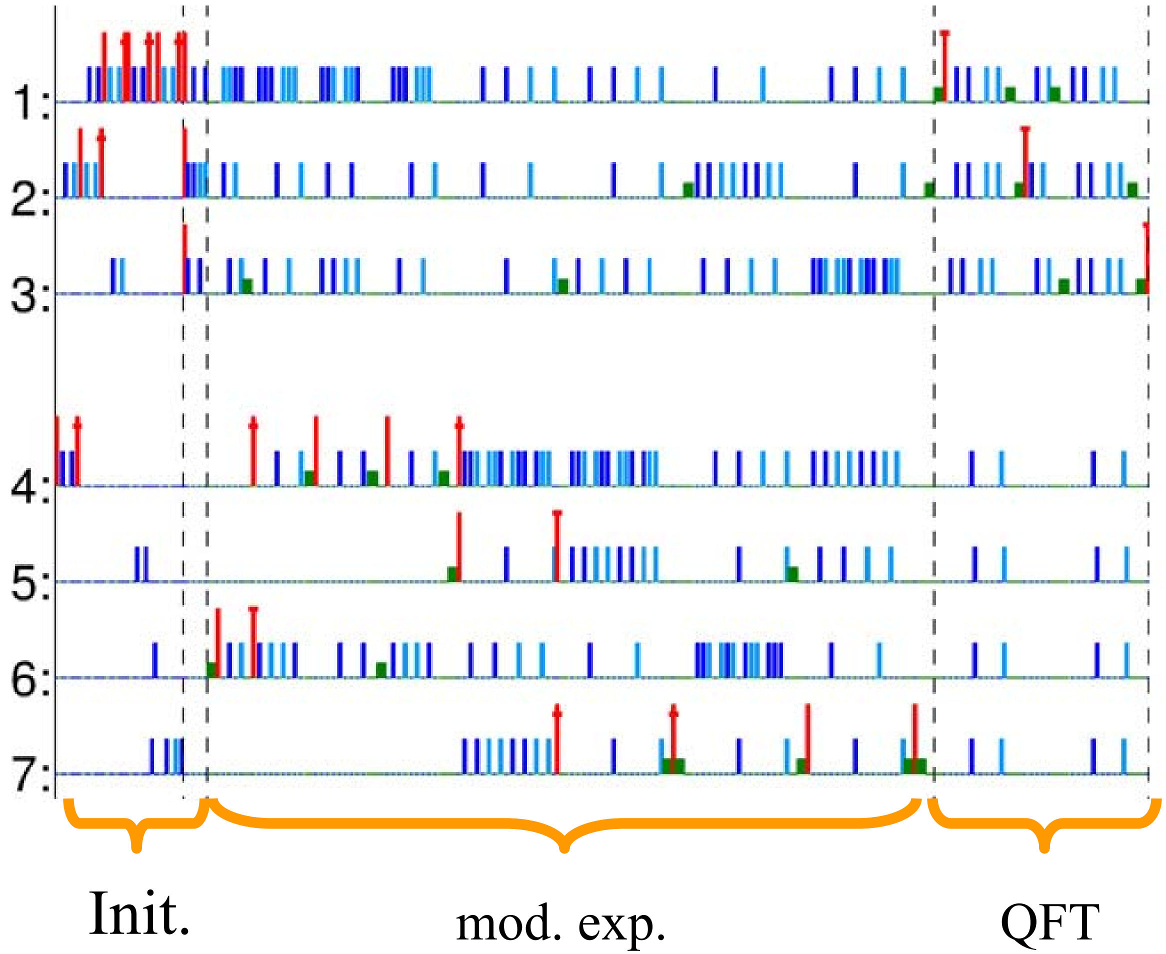
The molecule



i	$\omega_i/2$	$T_{1,i}$	$T_{2,i}$	J_{7i}	J_{6i}	J_{5i}	J_{4i}	J_{3i}	J_{2i}
1	−22052.0	5.0	1.3	−221.0	37.7	6.6	−114.3	14.5	25.16
2	489.5	13.7	1.8	18.6	−3.9	2.5	79.9	3.9	
3	25088.3	3.0	2.5	1.0	−13.5	41.6	12.9		
4	−4918.7	10.0	1.7	54.1	−5.7	2.1			
5	15186.6	2.8	1.8	19.4	59.5				
6	−4519.1	45.4	2.0	68.9					
7	4244.3	31.6	2.0						

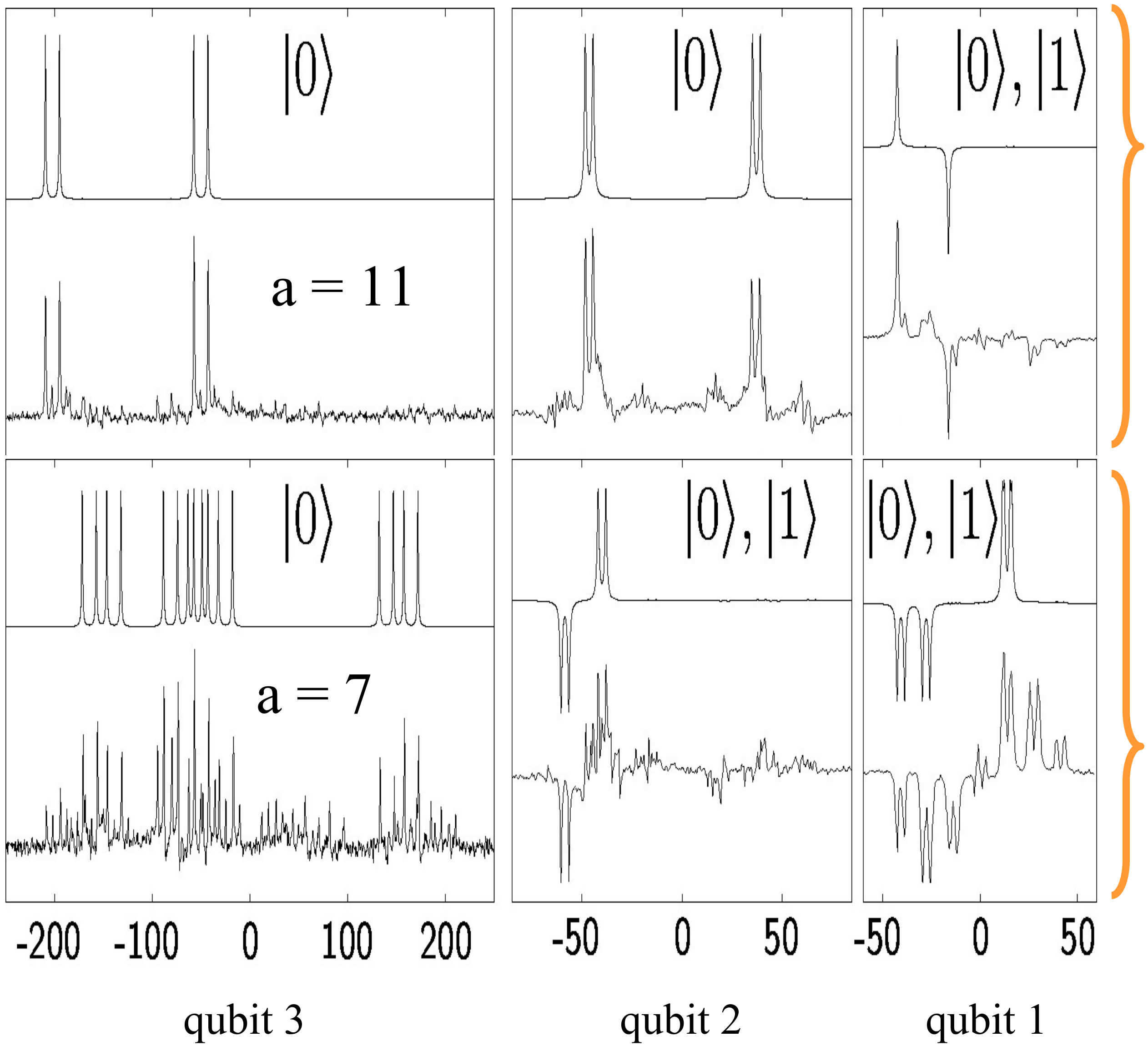


Pulse Sequence



~ 300 RF pulses || ~ 750 ms duration

Results: Spectra



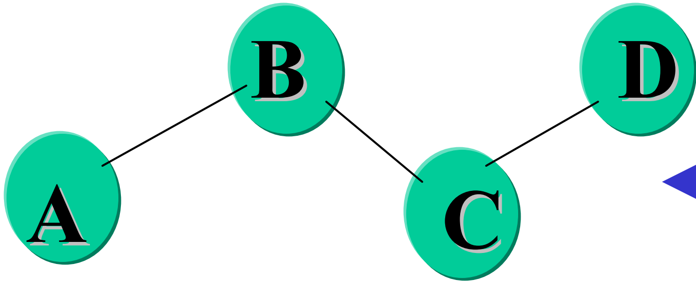
Mixture of $|0\rangle, |4\rangle$
$2^3/4 = r = 2$
$\gcd(11^{2/2} \pm 1, 15) = 3, 5$

$15 = 3 \cdot 5$

Mixture of $|0\rangle, |2\rangle, |4\rangle, |6\rangle$
$2^3/2 = r = 4$
$\gcd(7^{4/2} \pm 1, 15) = 3, 5$

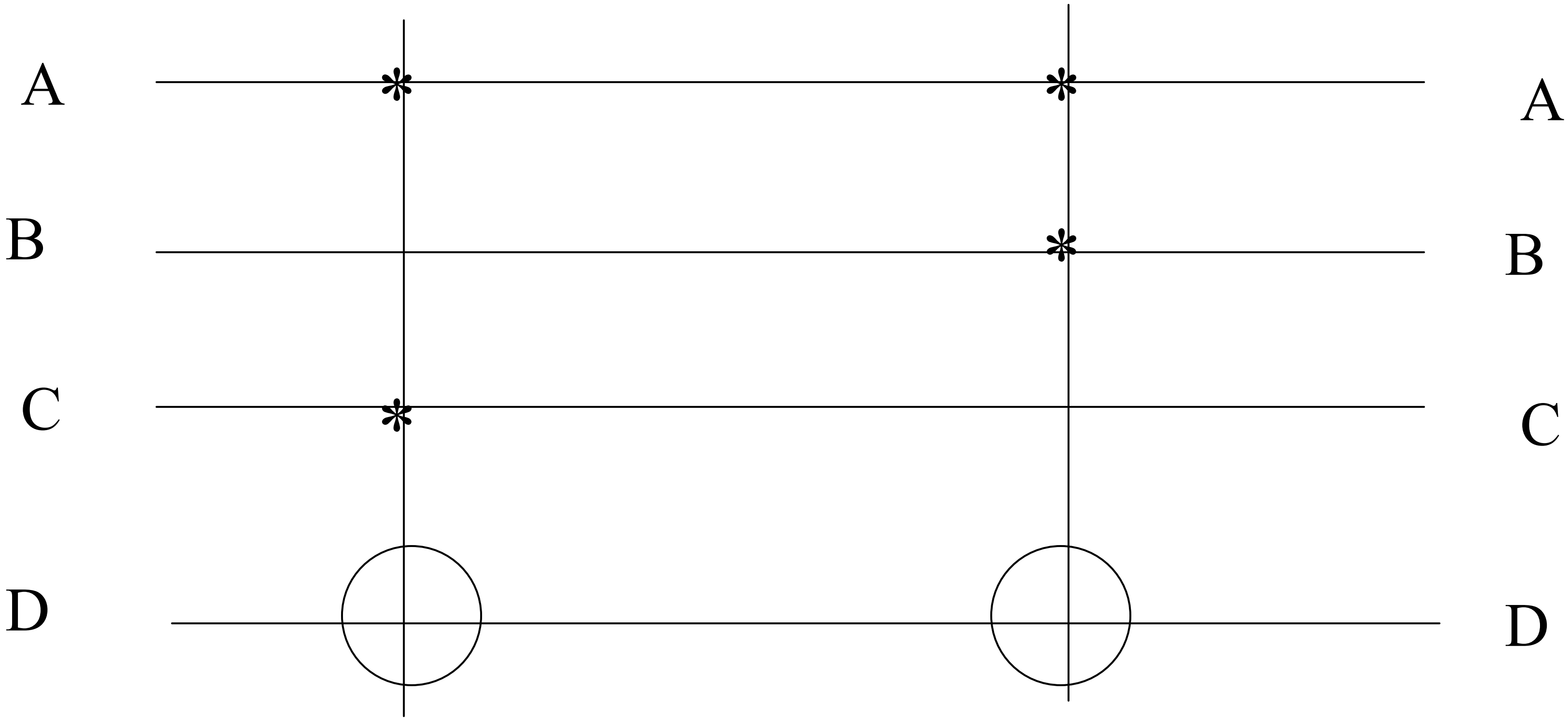
Problem

- We would like to assume that any two quantum wires can interact, but we are limited by the realization constraints
- Structure of atomic bonds in the molecule determines neighborhoods in the circuit.
- This is similar to restricted routing in FPGA layout - *link between logic and layout synthesis* known from CMOS design **now appears in quantum**.
- Below we are interested only in the so-called **"permutation circuits"** - their unitary quantum matrices are permutation matrices



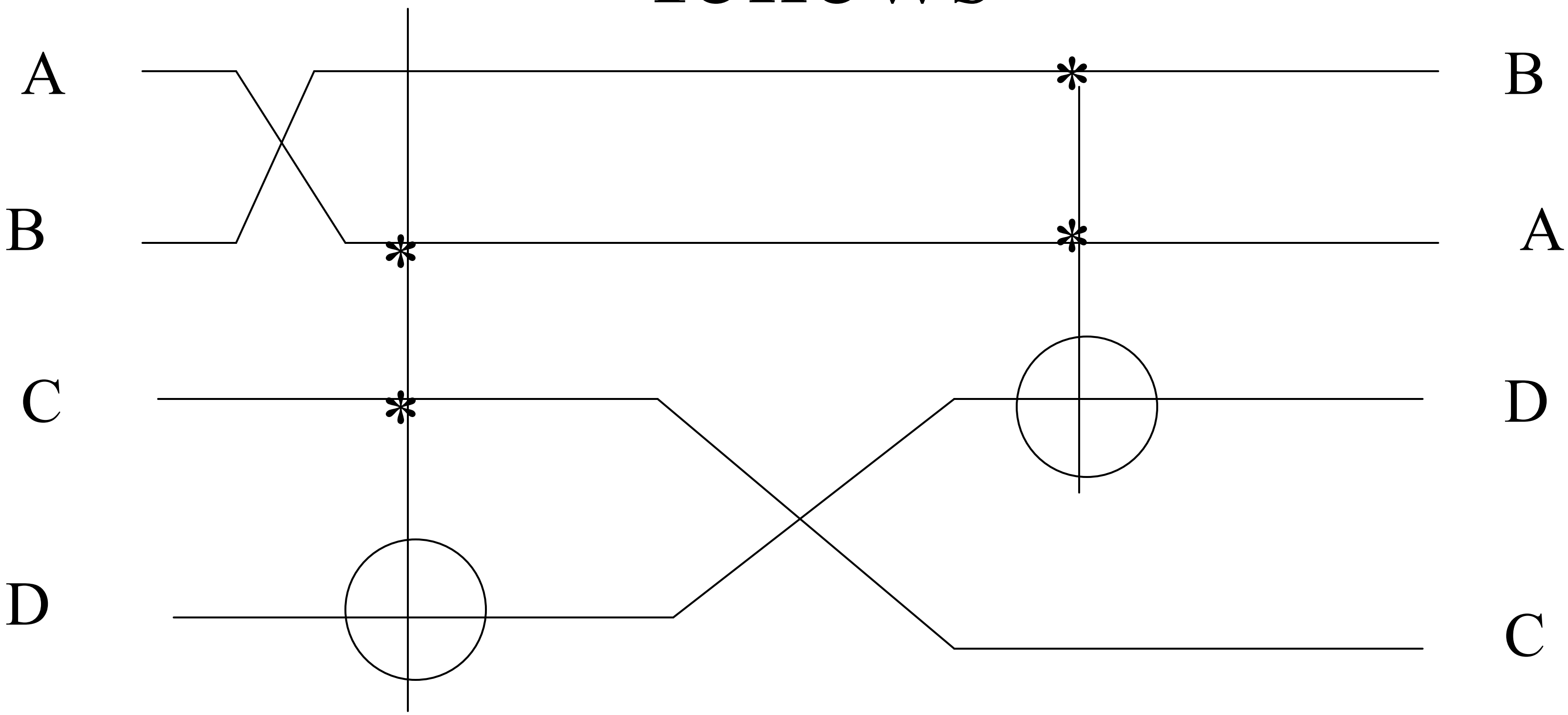
Quantum wires A and C are not neighbors

A schematics with two binary Toffoli gates

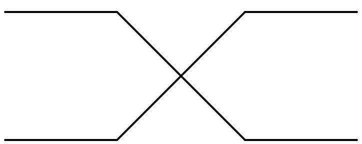


This is a result of our ESOP minimizer program, but this is not realizable in NMR for the above molecule

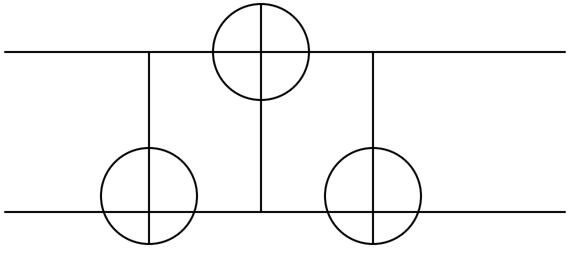
So I modify the schematics as follows



But this costs me two swap gates



Costs 3
Feynmans



Solution

- One solution to connection problem in VLSI has been to increase the number of values in wires.
- Have a **"quantum wire"** have **more than two** eigenstates.
- Increase from superpositions of 2^n to **superpositions of 3^n**
- Basic gate in quantum logic is a **2*2** (2-qubit gate). We have to build from such gates.

Can we build multiple-valued Quantum Computers in year 2002?

- *In principle, yes*

Has one tried?

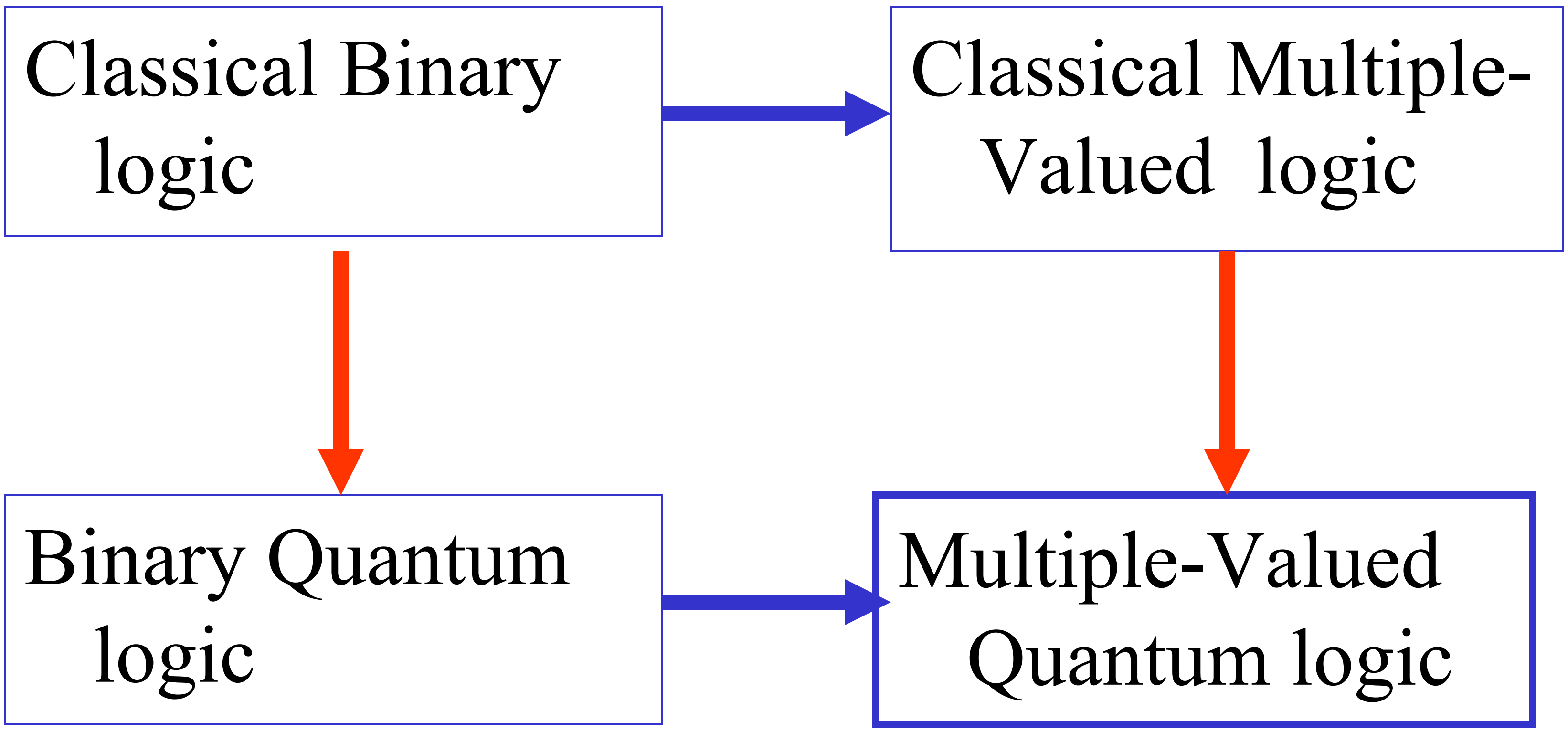
No.

Gates, yes

Qudits not qubits

- In ternary logic, the notation for the superposition is **$\alpha|0> + \beta|1> + \gamma|2>$.**
- These intermediate states cannot be distinguished, rather a **measurement** will yield that the qudit is in one of the basis states, |0>, |1> or |2>.
- The probability that a measurement of a qudit yields state |0> is $|\alpha|^2$, the probability is $|\beta|^2$ for state |1> and $|\gamma|^2$ for state γ . The sum of these probabilites is one. The absolute values are required, since in general, α β and γ are

The concept of Multiple-Valued Quantum Logic



What is known?

- **Mattle** 1996 - *Trit |0>, |1>, |2>*
- Chau 1997 - *qudit, error correcting quantum codes*
- Ashikhmin and Knill 1999, *MV codes*.
- Gottesman, Aharonov and Ben-Or 1999 - *MV fault tolerant gates*.
- Burlakov 1999 - *correlated photon realization of ternary qubit.*
- Muthukrishnan and Stroud 2000 - *multi-valued universal quantum logic for linear ion trapped devices*.
- Picton 2000 - *Multi-valued reversible PLA*.
- *Perkowski, Al-Rabadi, Kerntopf and Portland Quantum Logic Group 2001 - Galois Field quantum logic synthesis*

What is known?

- Al-Rabadi, 2002 - *ternary EPR and Chrestenson Gate*
- De Vos 2002 - *Two ternary 1*1 gates and two ternary 2*2 gates for reversible logic*.
- Zilic and Radecka 2002 - *Super-Fast Fourier Transform*
- Bartlett et al, 2002 - *Quantum Encoding in Spin Systems*
- Brassard, Braunstein and Cleve, 2002 - *Teleportation*
- Rungta, Munro et al *Qudit Entanglement*.

Ternary Galois Field (GF3) operations.

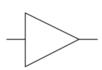
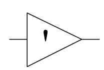
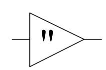
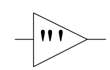
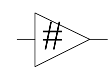
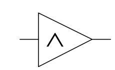
+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

(a) Addition

•	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

(b) Multiplication

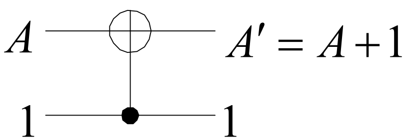
Reversible ternary shift operations.

Operator Name		Buffer	Single-Shift	Dual-Shift	Self-Shift	Self-Single-Shift	Self-Dual-Shift
Operator symbol & equation		A	$A' = A+1$	$A'' = A+2$	$A''' = A+A=2A$	$A^{\#} = 2A+1$	$A^{\wedge} = 2A+2$
Gate symbol							
A	0	0	1	2	0	1	2
	1	1	2	0	2	0	1
	2	2	0	1	1	2	0

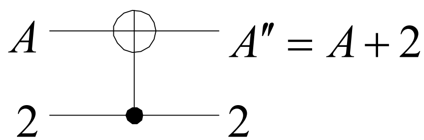
Conversion of one shift form to another shift form using ternary shift gates

Input	Output					
	A	A'	A''	A'''	$A^{\#}$	$A^{\wedge}$
A		′	″	‴	#	^
A'	″		′	#	^	‴
A''	′	″		^	‴	#
A'''	‴	#	^		′	″
$A^{\#}$	#	^	‴	″		′
$A^{\wedge}$	^	‴	#	′	″	

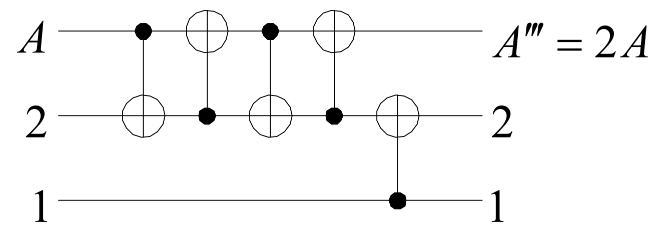
Quantum realization of ternary shift gates.



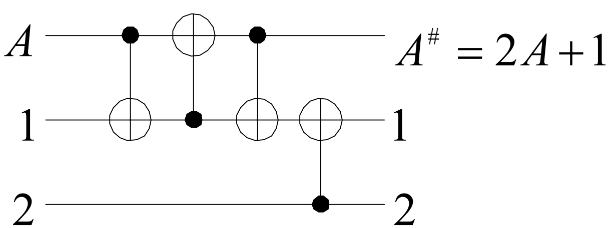
(a) Single-Shift



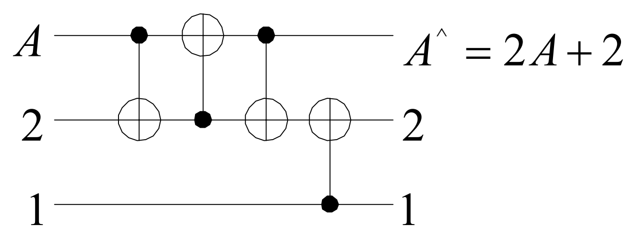
(b) Dual-Shift



(c) Self-Shift

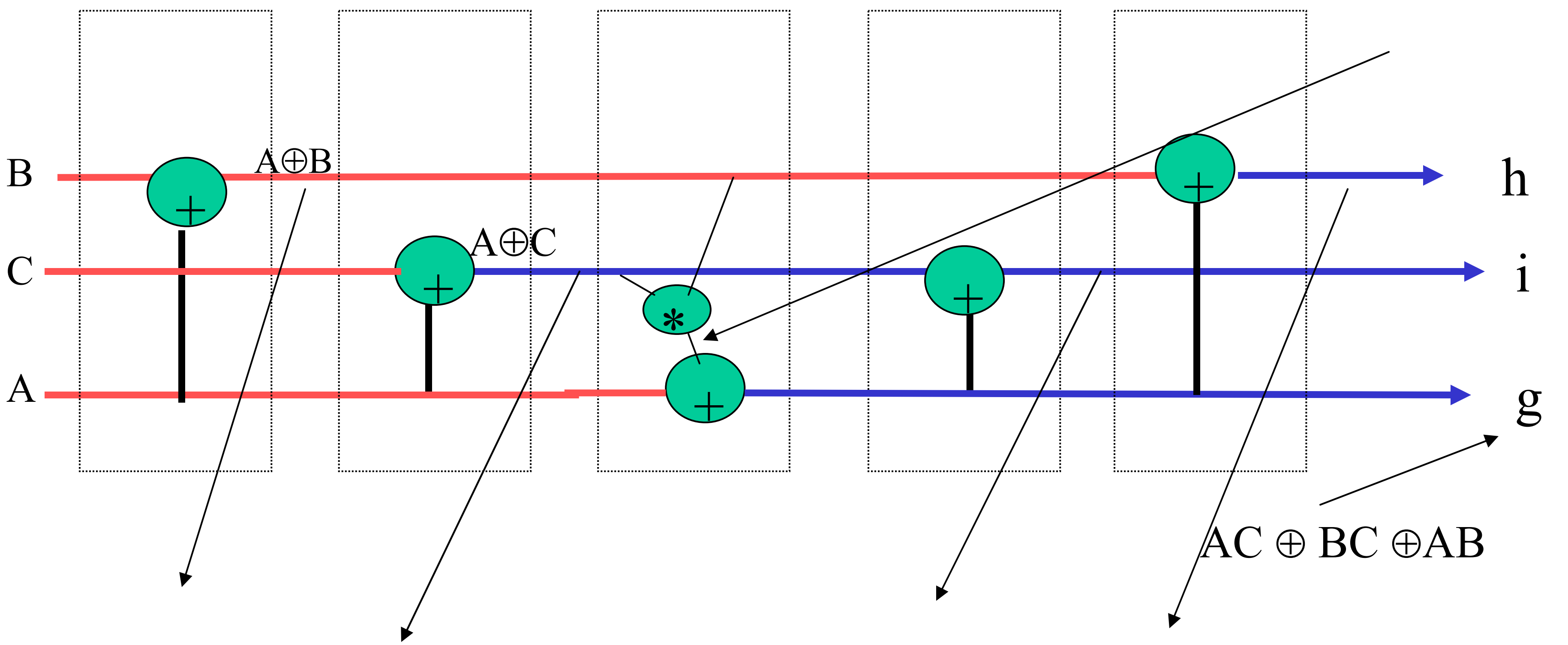


(d) Self-Single-Shift



(e) Self-Dual-Shift

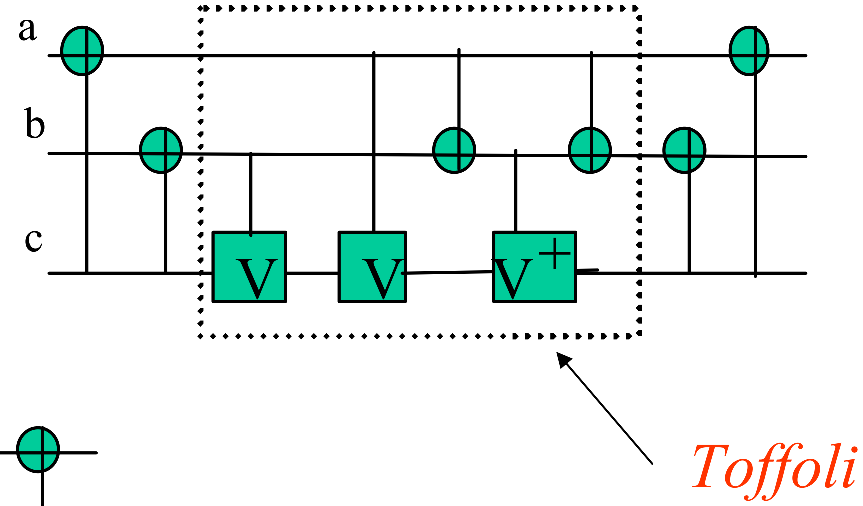
Optimal Solution to Ternary Miller Function



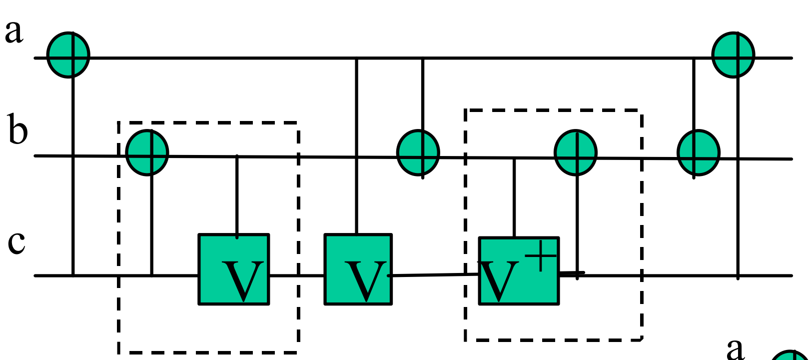
Check ternary maps

2-qubit quantum realization of Miller Gate

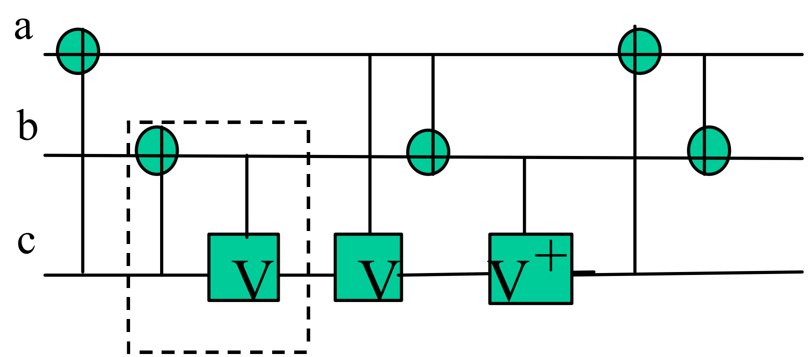
a)



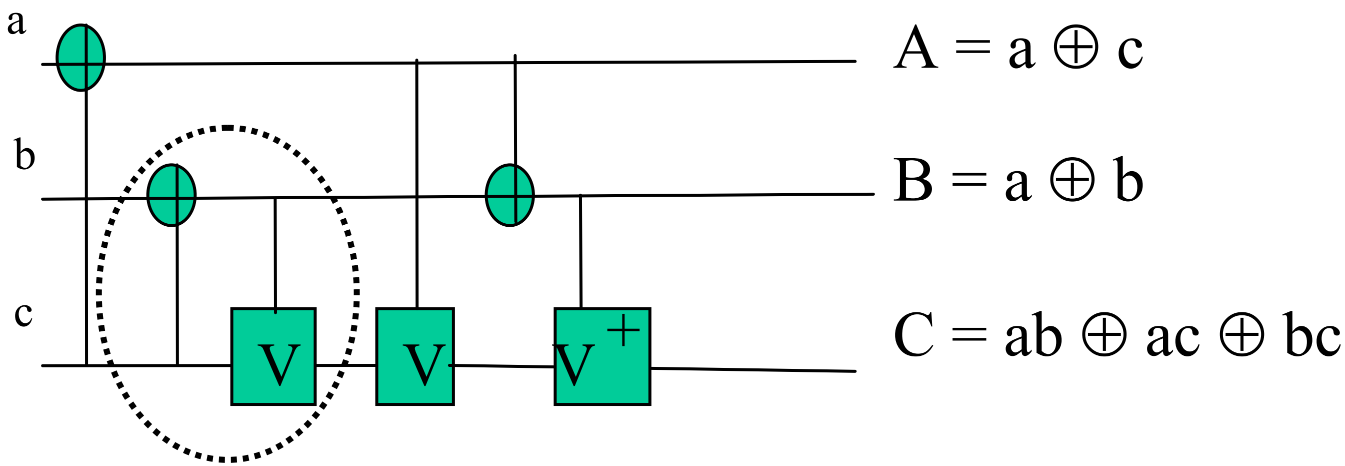
b)



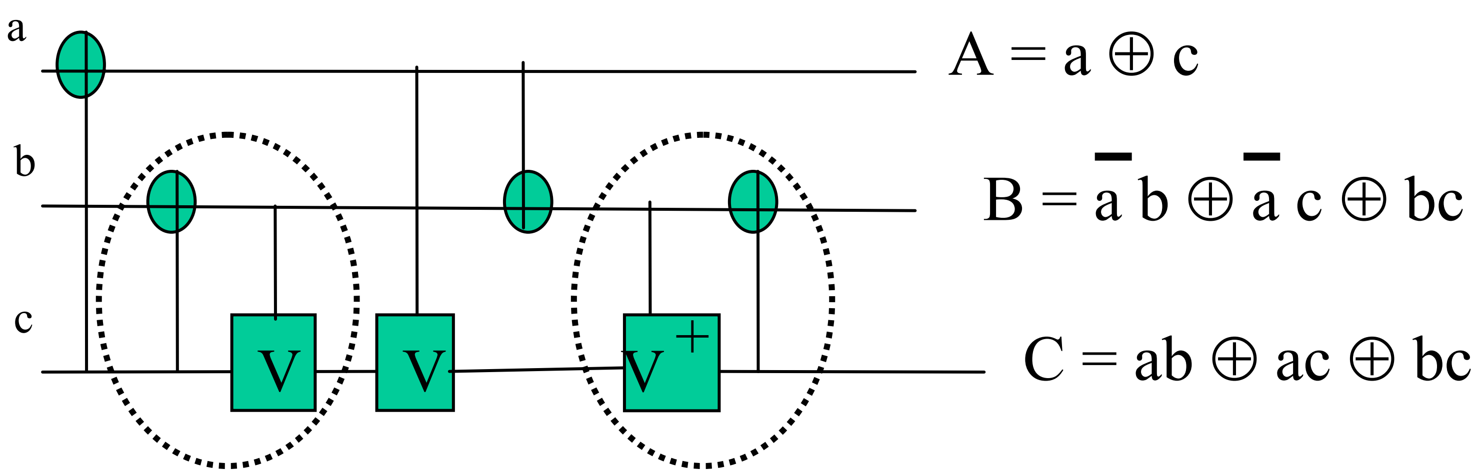
c)



These techniques can be also applied for Multiple-Valued Quantum Logic a)

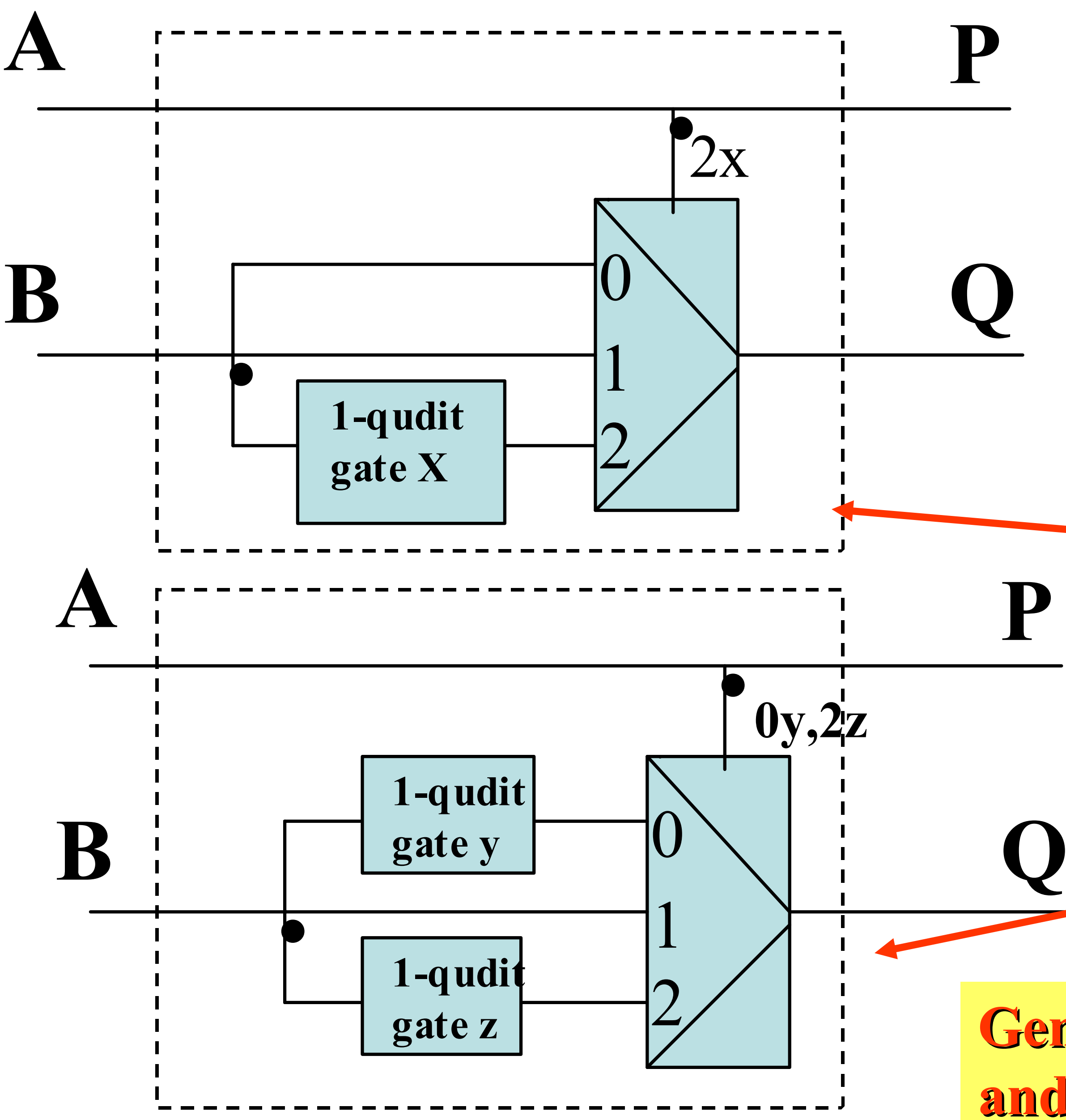


b)



Design a Ternary Toffoli Gate from 2-qubit quantum primitives

Ternary controlled gates

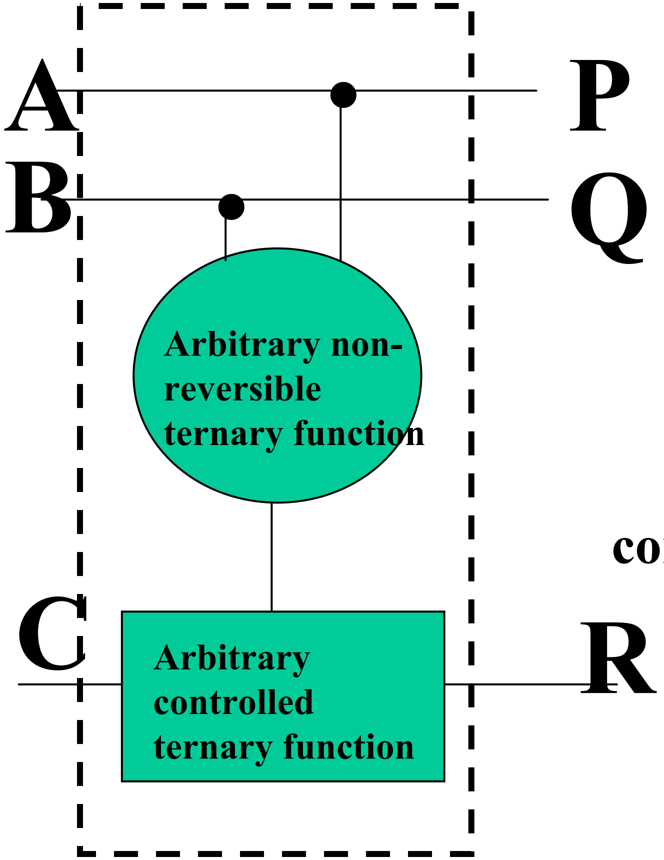


First task is to demonstrate that a universal 3-qubit gate can be built from MV quantum primitives

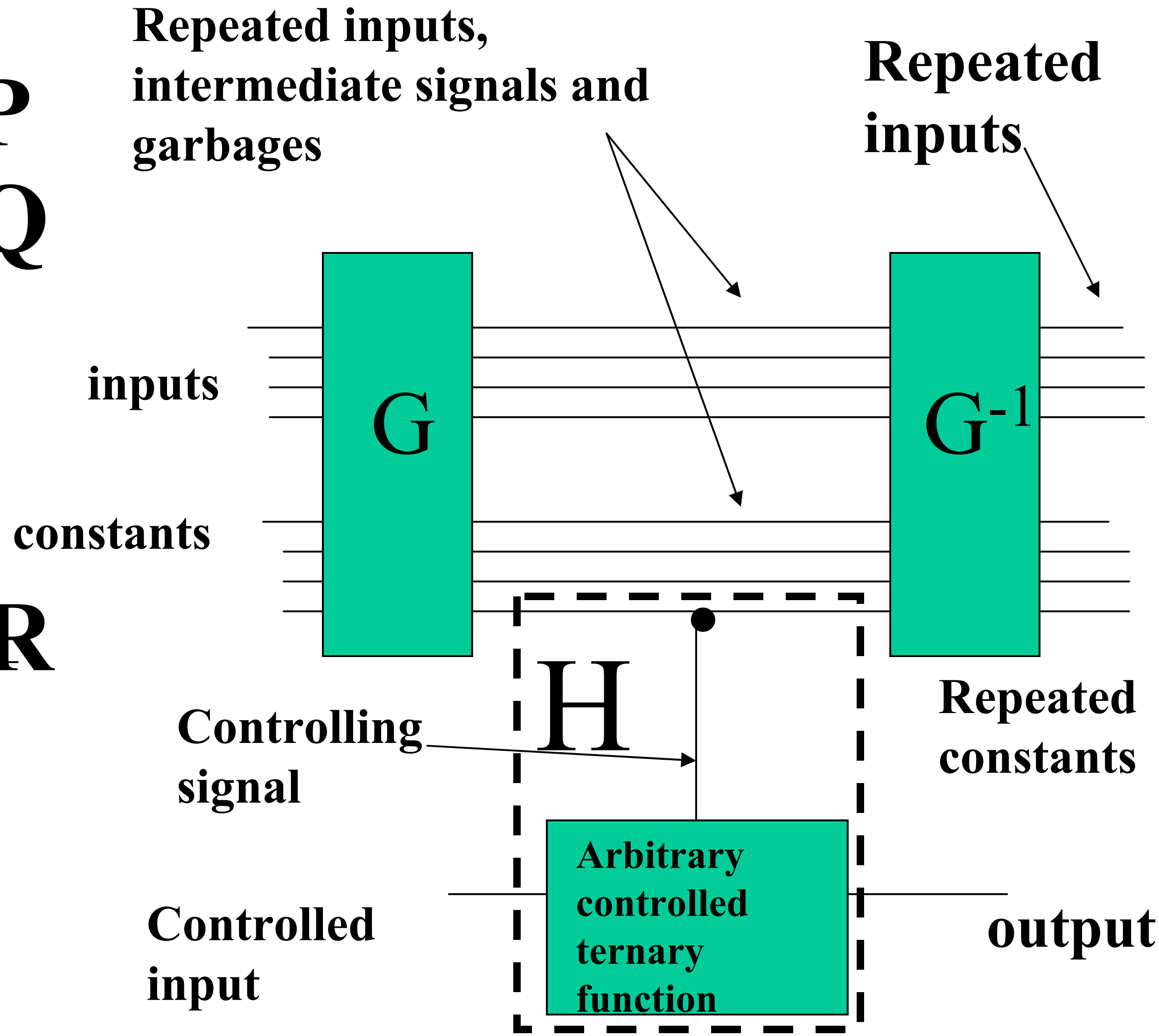
2-qubit controlled gate with controlling value d-1 = 2

2-qubit controlled gate with controlling values 0 and 2

Generalization of Stroud and De Vos gates

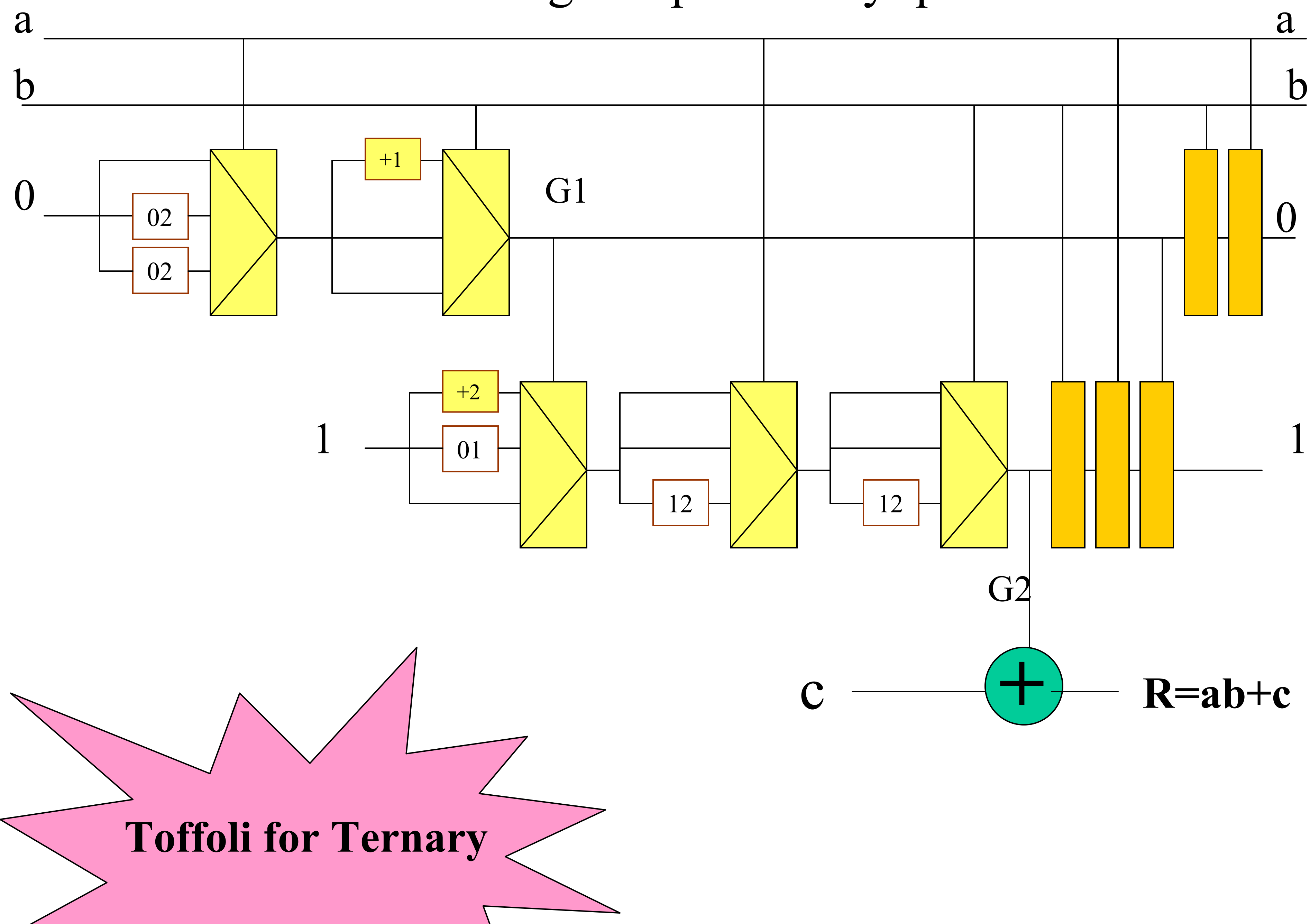


Ternary generalized Toffoli

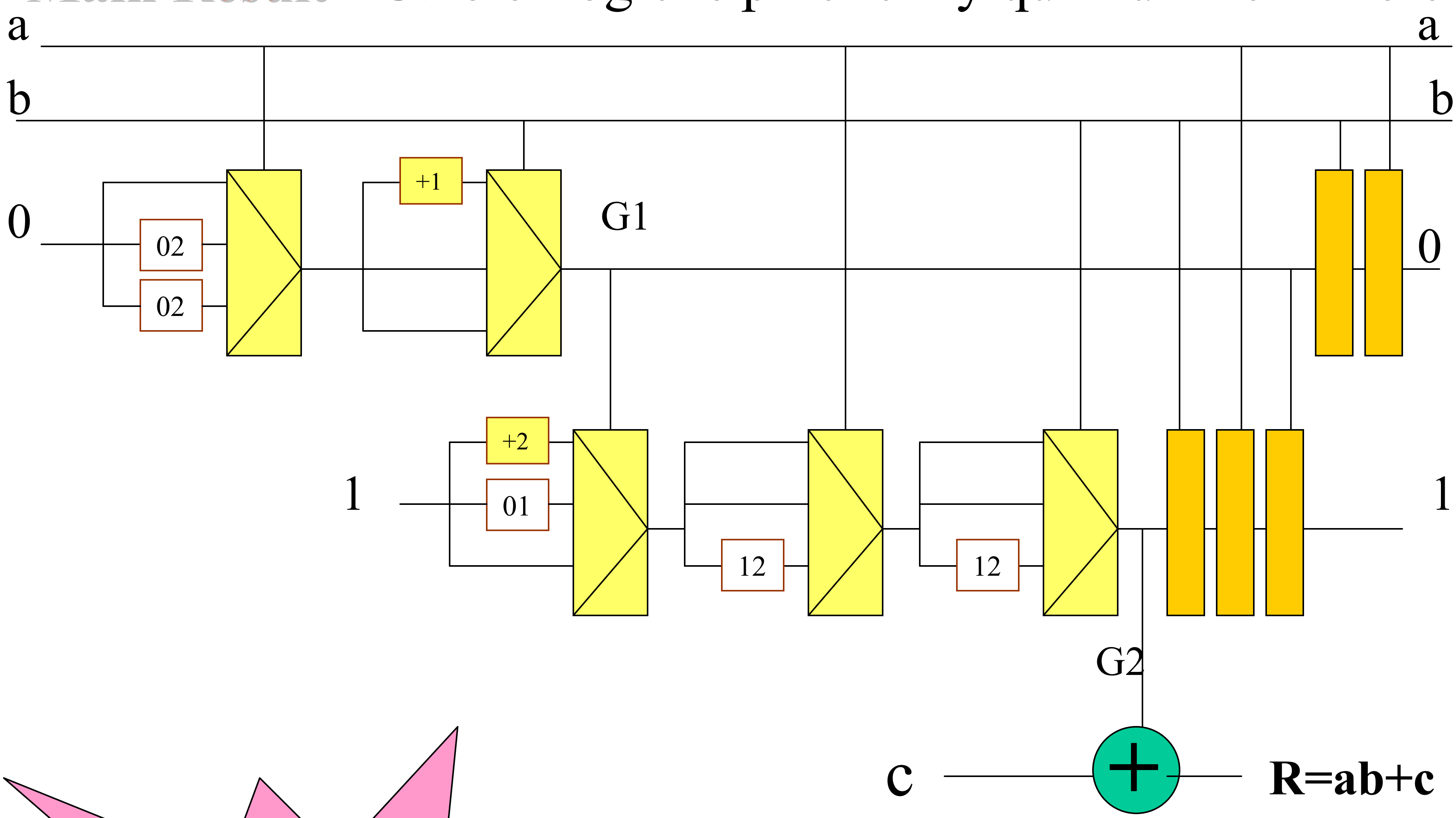


Principle of creating arbitrary reversible gates

Main Result - Galois Logic is practically quantum-realizable



Main Result - Galois Logic is practically quantum-realizable



But is it worthy?

This structure realizes also a very huge family of ternary Toffoli-like gates

Complete ternary systems

- System 1. Post literal, min, max
- System 2. Power of variable, shifts of variable (two of them for ternary – these are optional), Galois ADD, Galois MUL
- System 3. Post Literals, MIN, MODSUM.
- These three are most popular, but there are many other.

Are they good for quantum?

Ternary Operator Kmaps

A \ B	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Galois Multiplication. Also has latin square for non-zero columns and rows

A \ B	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

MODSUM which for primary number 3 is the same as Galois Addition. Observe latin square property, very important

Example : Ternary Kmaps of ternary adder

C

A \ B	0	1	2
0	0	0	0
1	0	0	1
2	0	1	1

$C = {}^{1}A * {}^{2}B + {}^{2}A * {}^{1}B + {}^{2}A * {}^{2}B$

Step1: write from Kmap the formula for mv minterms

S

A \ B	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

This is modsum3 by inspection, so $S = A +_3 B$. But you can also calculate is with much formula writing the same as I show for C

Step 2. Algebraic Simplifications using rules of ternary Galois Field Algebra

$$C = {}^{1}A * {}^{2}B + {}^{2}A * {}^{1}B + {}^{2}A * {}^{2}B = (2A^2+2A) * (2B^2+B) + (2A^2+A) * (2B^2+2B) + (2A^2+A) * (2B^2+B)$$

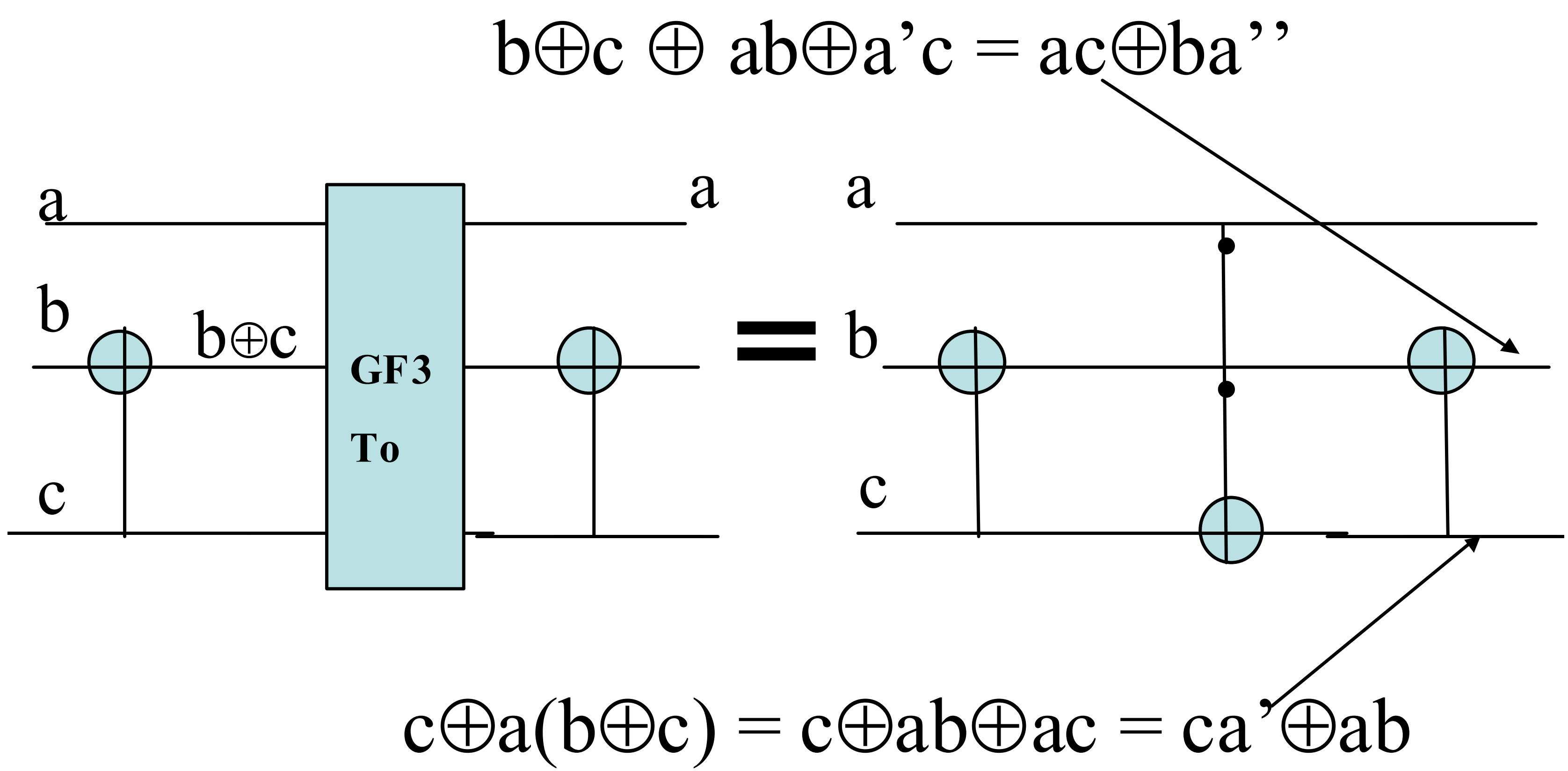
$$= 2(A^2+A) * (2B^2+B) + 2*2A^2B^2 + 2*2A^2B + 2AB^2 + 2AB + 2*2A^2B^2 + 2A^2B + 2AB^2 + AB = \mathbf{2A^2B + 2AB^2 + 2AB}$$

Example of Post literal, it has value 1 for argument value 1 and 0 otherwise

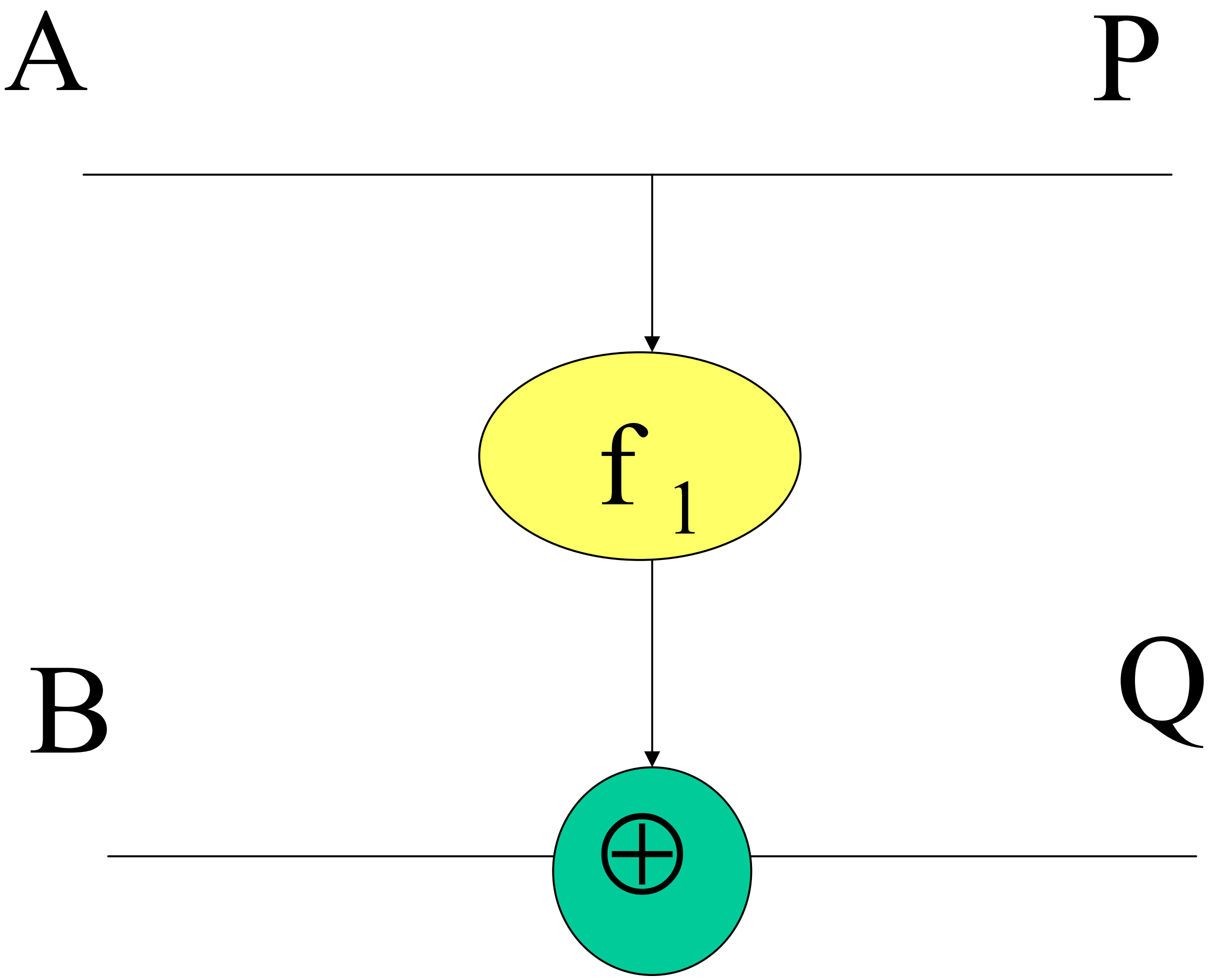
Here Post literals are next replaced by tautological polynomials in Galois Field

Complex Ternary Quantum Gates

Ternary Fredkin Gate build from Ternary Toffoli and Ternary Feynman gates

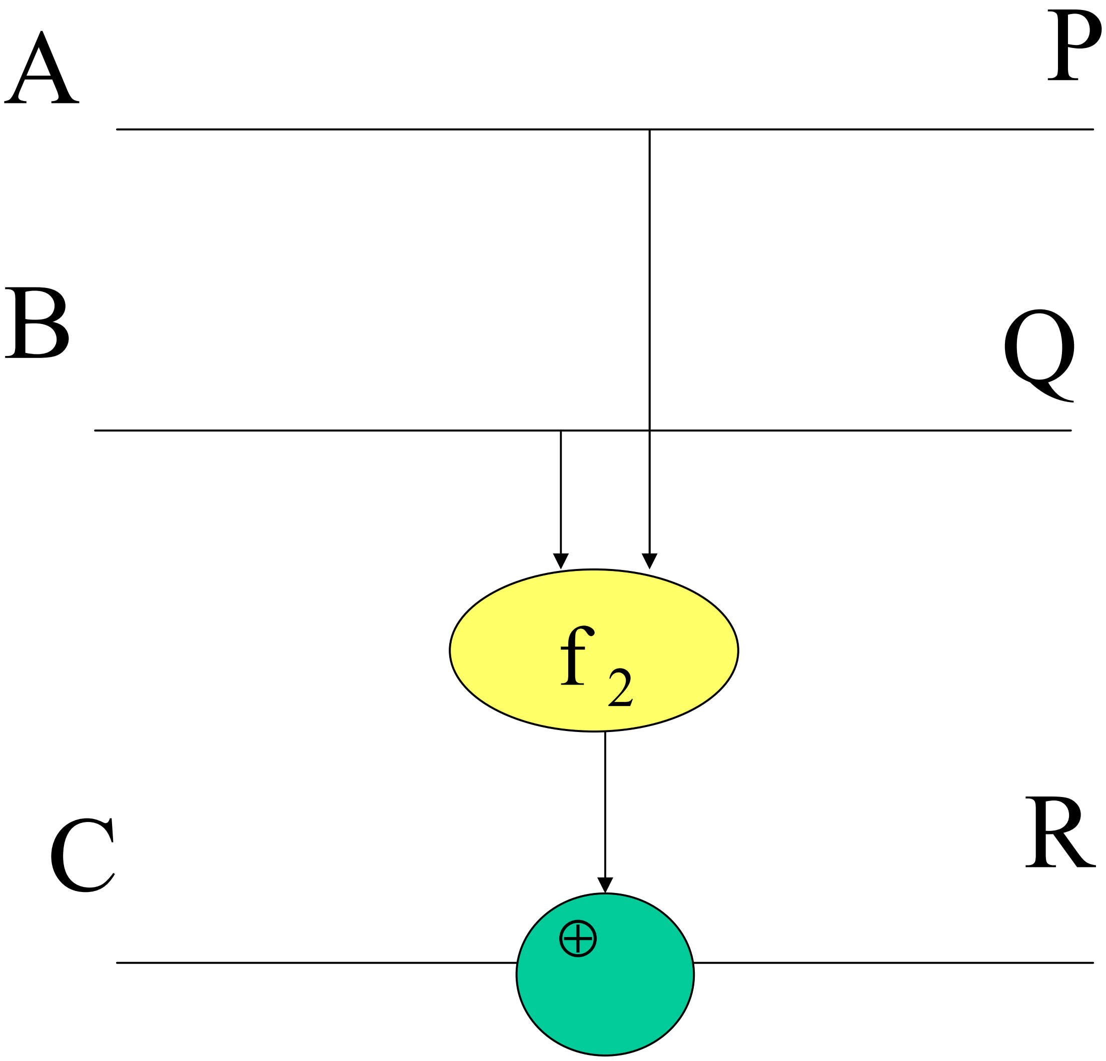


Generalized Ternary Feynman Gate



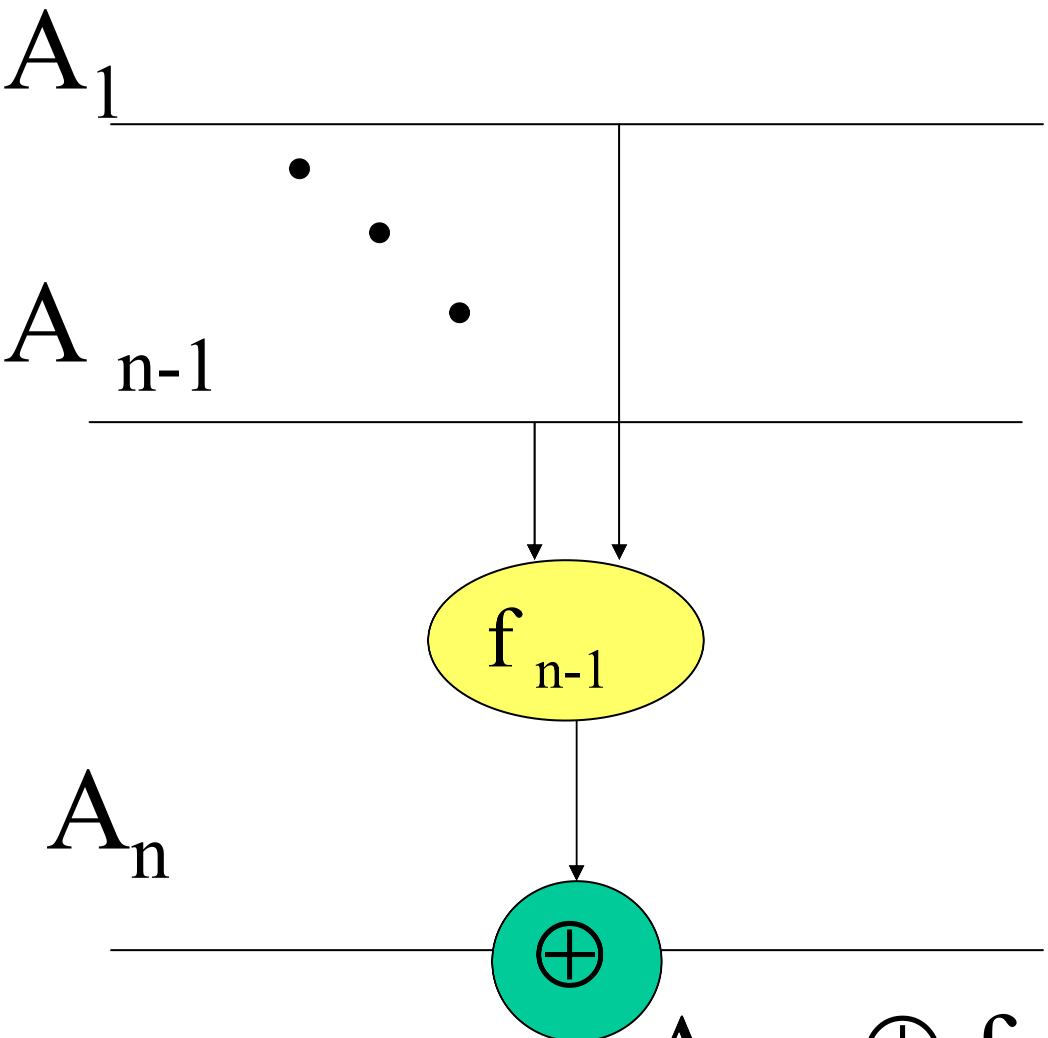
If f1 is reversible , gate is correct, what about non-reversible f1, please check if the gate is still reversible

Generalized Ternary 3*3 Toffoli Gate



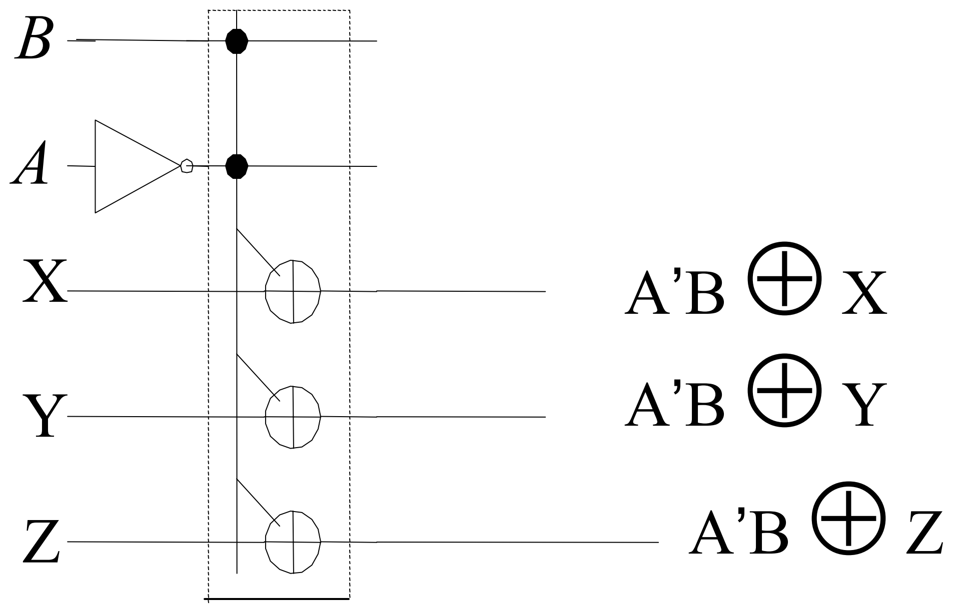
Do the same exercise as in previous slide, this will help you get intuition in MV logic.

Generalized Ternary n*n Toffoli Gate



$$A_{n-1} \oplus f_{n-1}(A_1, \ldots, A_{n-1})$$

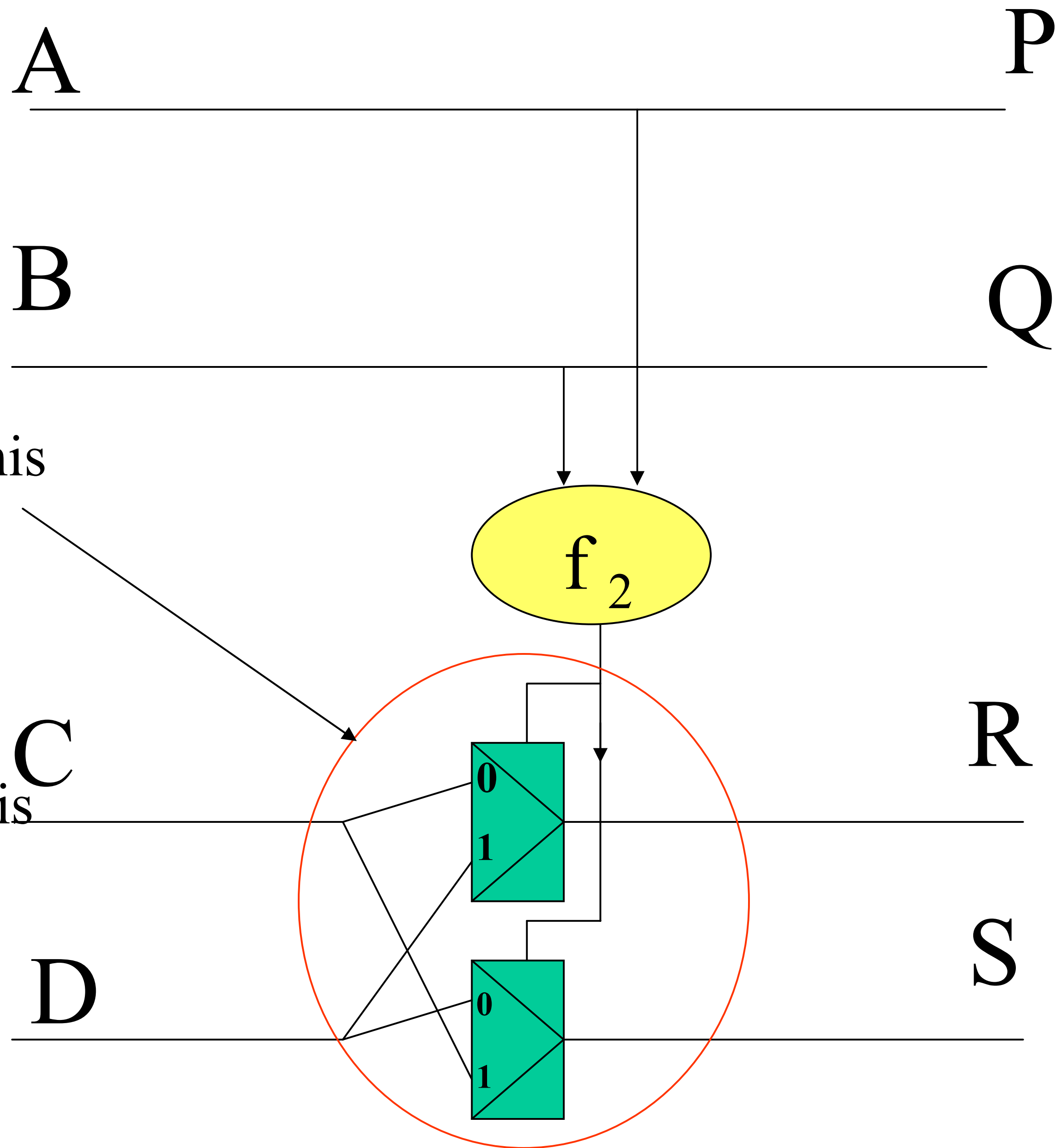
Do the same exercise as in previous slide, this will help you get intuition in MV logic.



Is this a realizable quantum gate ? -yes

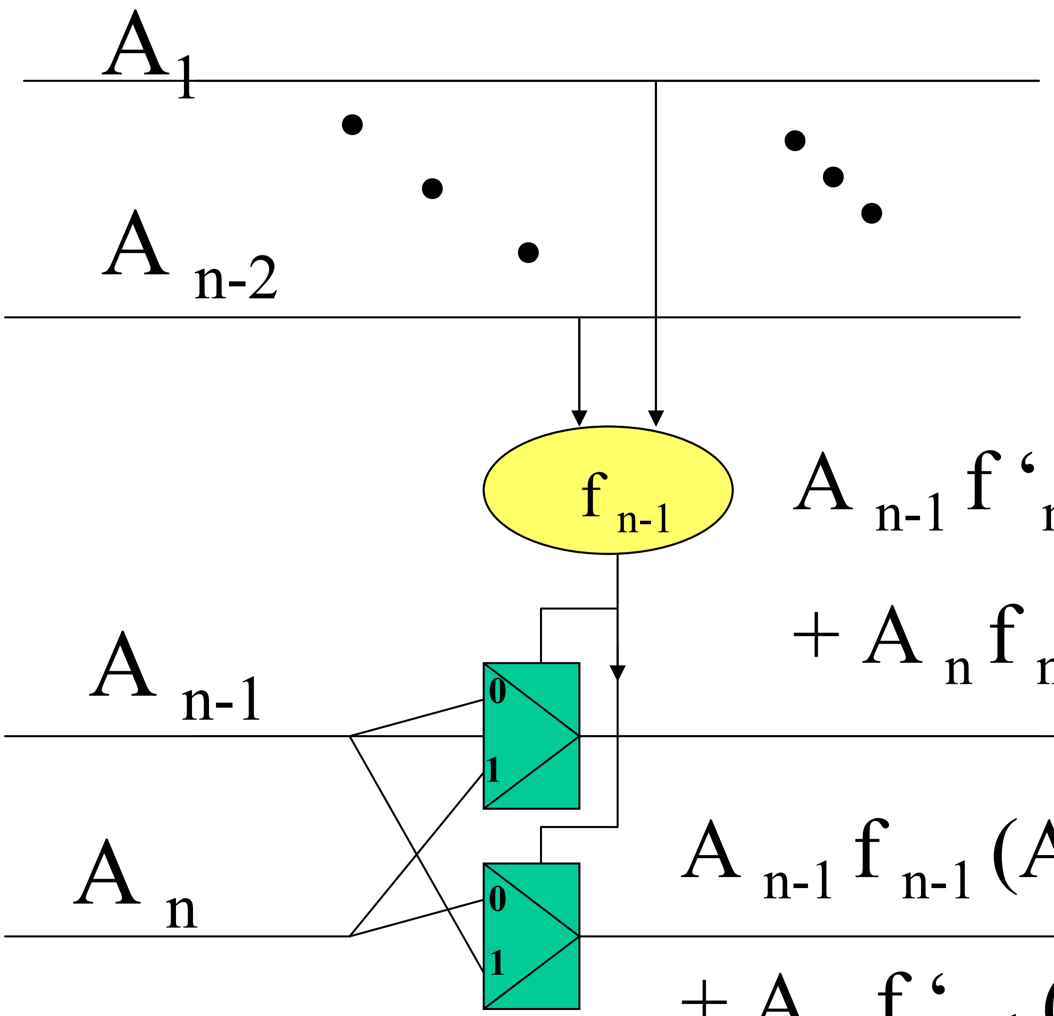
Generalized Ternary 4*4 Fredkin Gate

Rewrite this part to quantum ternary notation with Galois Logic



Generalized Ternary n*n Fredkin Gate

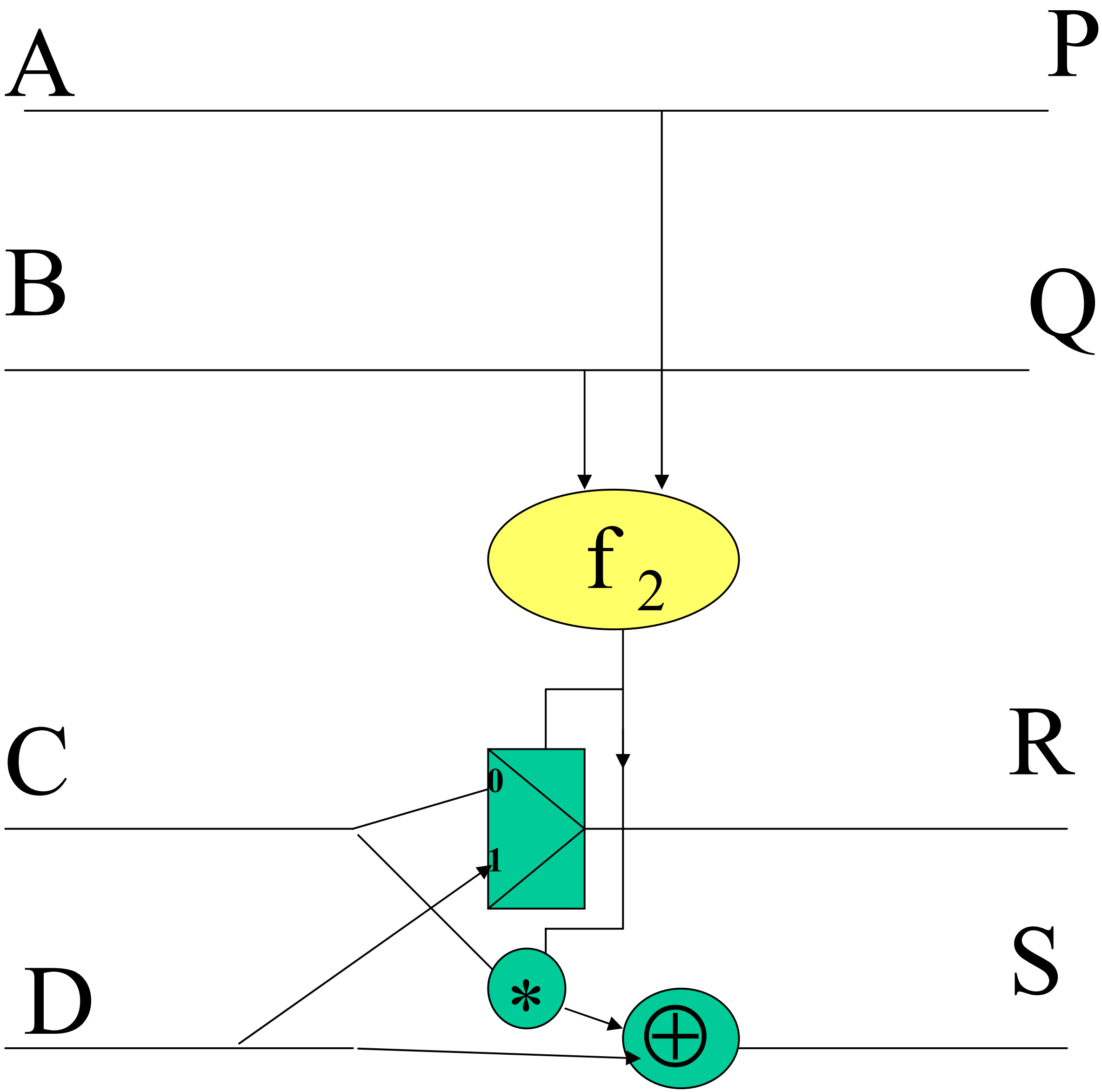
Do the same exercise as in previous slide, this will help you get intuition in MV logic.



$$A_{n-1}\, f'_{n-1}(A_1, \ldots, A_{n-1}) + A_n\, f_{n-1}(A_1, \ldots, A_{n-1})$$

$$A_{n-1}\, f_{n-1}(A_1, \ldots, A_{n-1}) + A_n\, f'_{n-1}(A_1, \ldots, A_{n-1})$$

Generalized Ternary 4*4 Kerntopf Gate

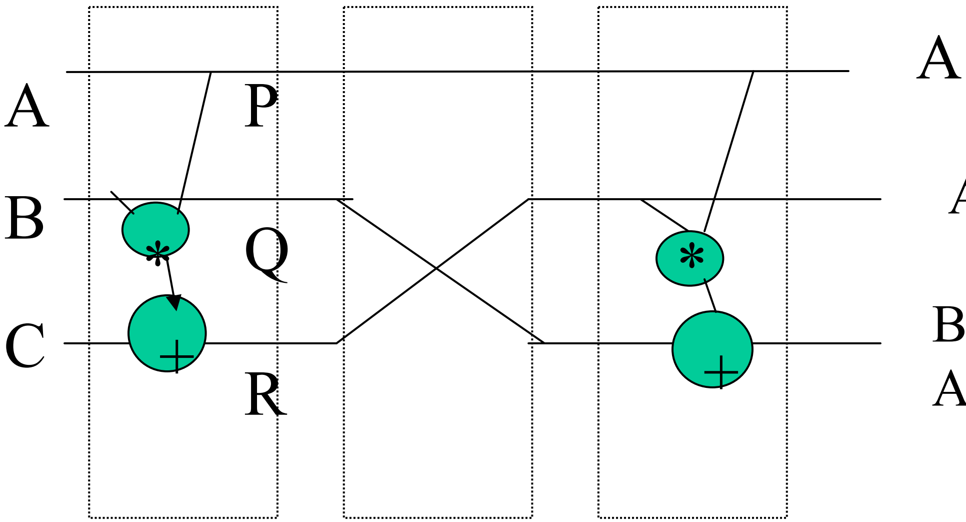


Do the same exercise as in previous slide, this will help you get intuition in MV logic.

Ternary Quantum Circuits

Ternary GFSOP Cascade (non-optimal)

All operations are Galois



AB ⊕ C

B ⊕ A (AB ⊕ C) = B⊕ AB ⊕ AC = A'B ⊕ AC

<u>Notation for EACH gate:</u>

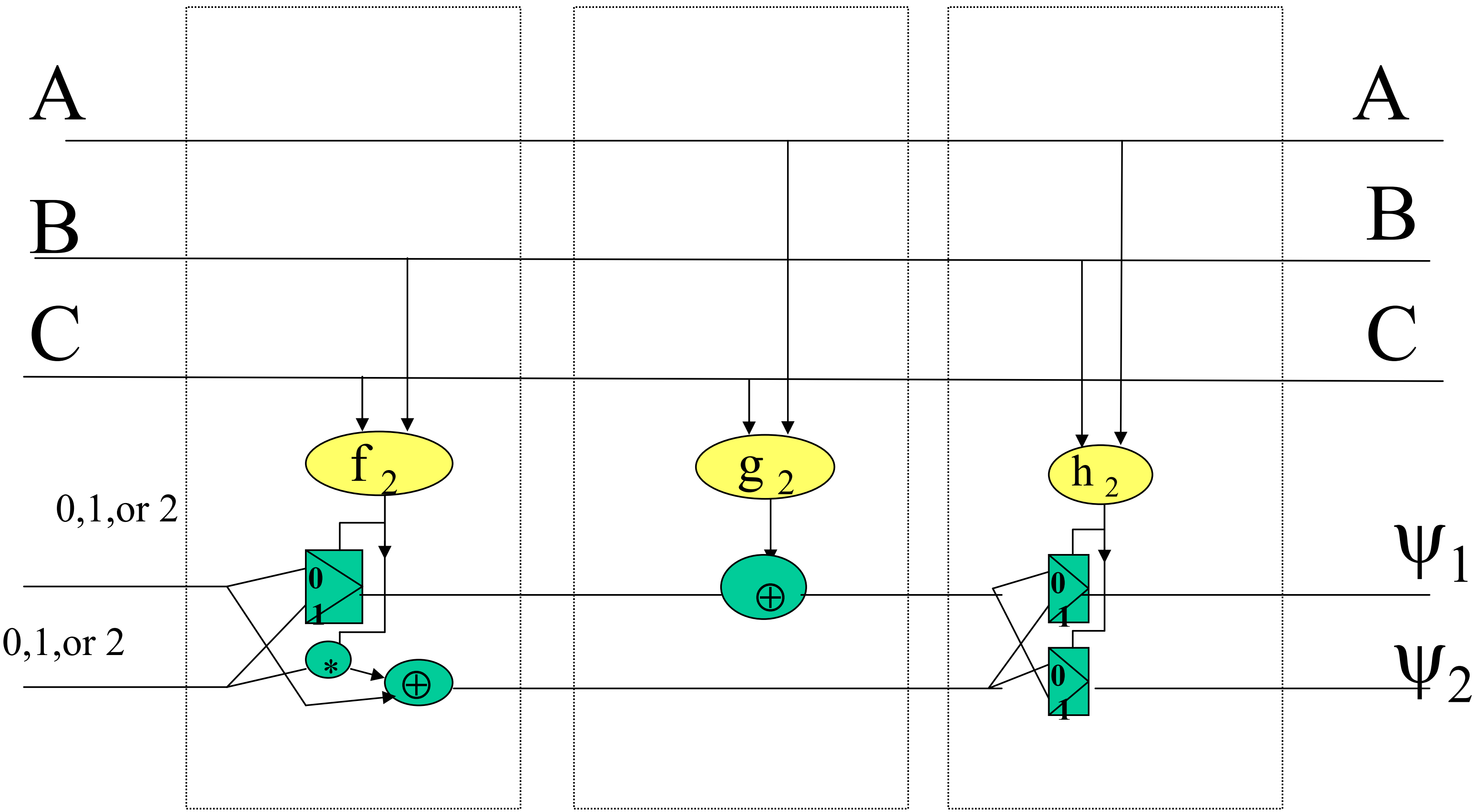
Inputs: A,B,C

Outputs: P,Q,R

How to realize ternary swap gate?

In any case, this is very costly!

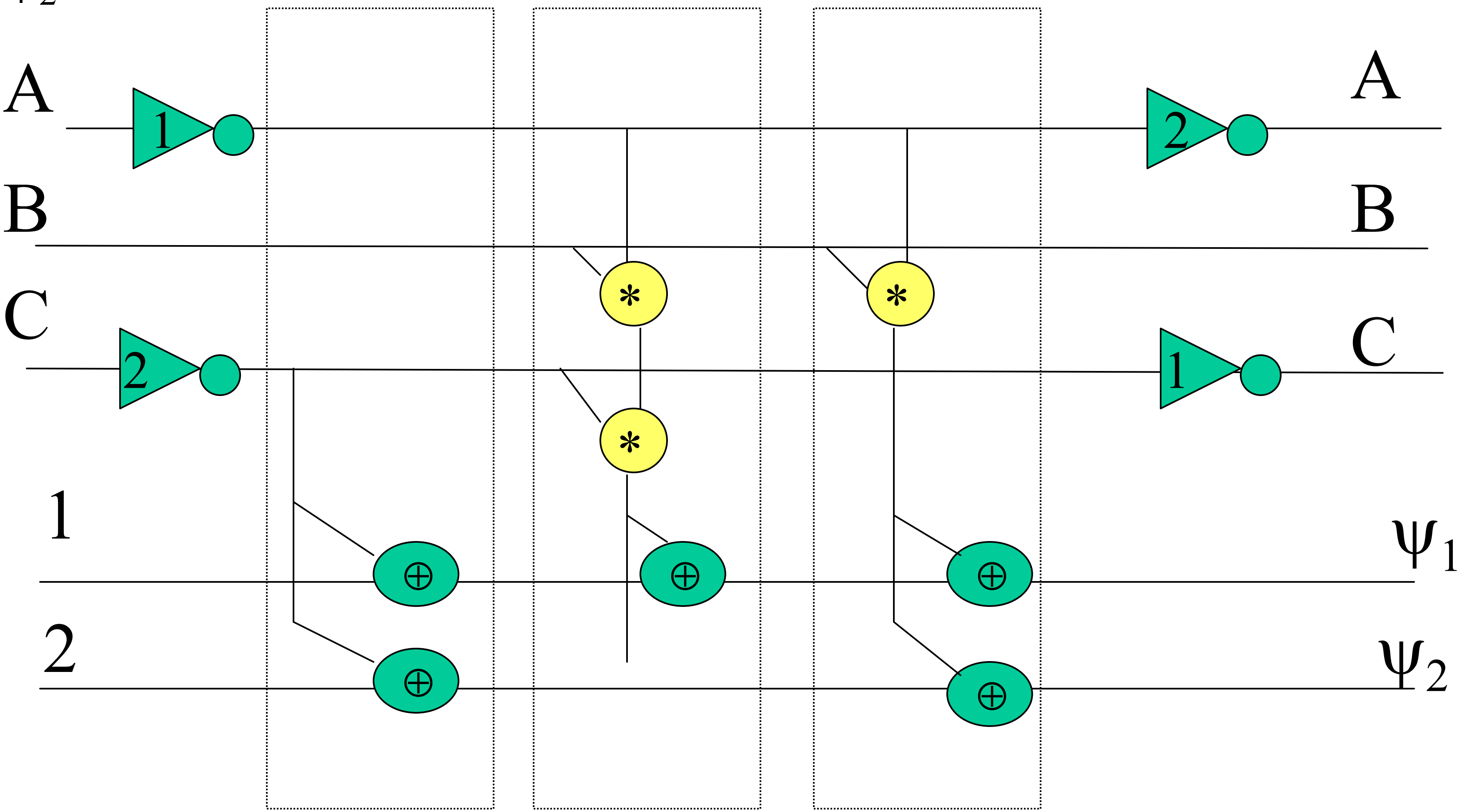
General Ternary Cascade of Kerntopf, Toffoli and Fredkin Family Gates



Example of multi-output FPRM-like GFSOP cascade of Toffoli family gates

$\psi_1 = 1 \oplus C'' \oplus A'BC \oplus A' B$

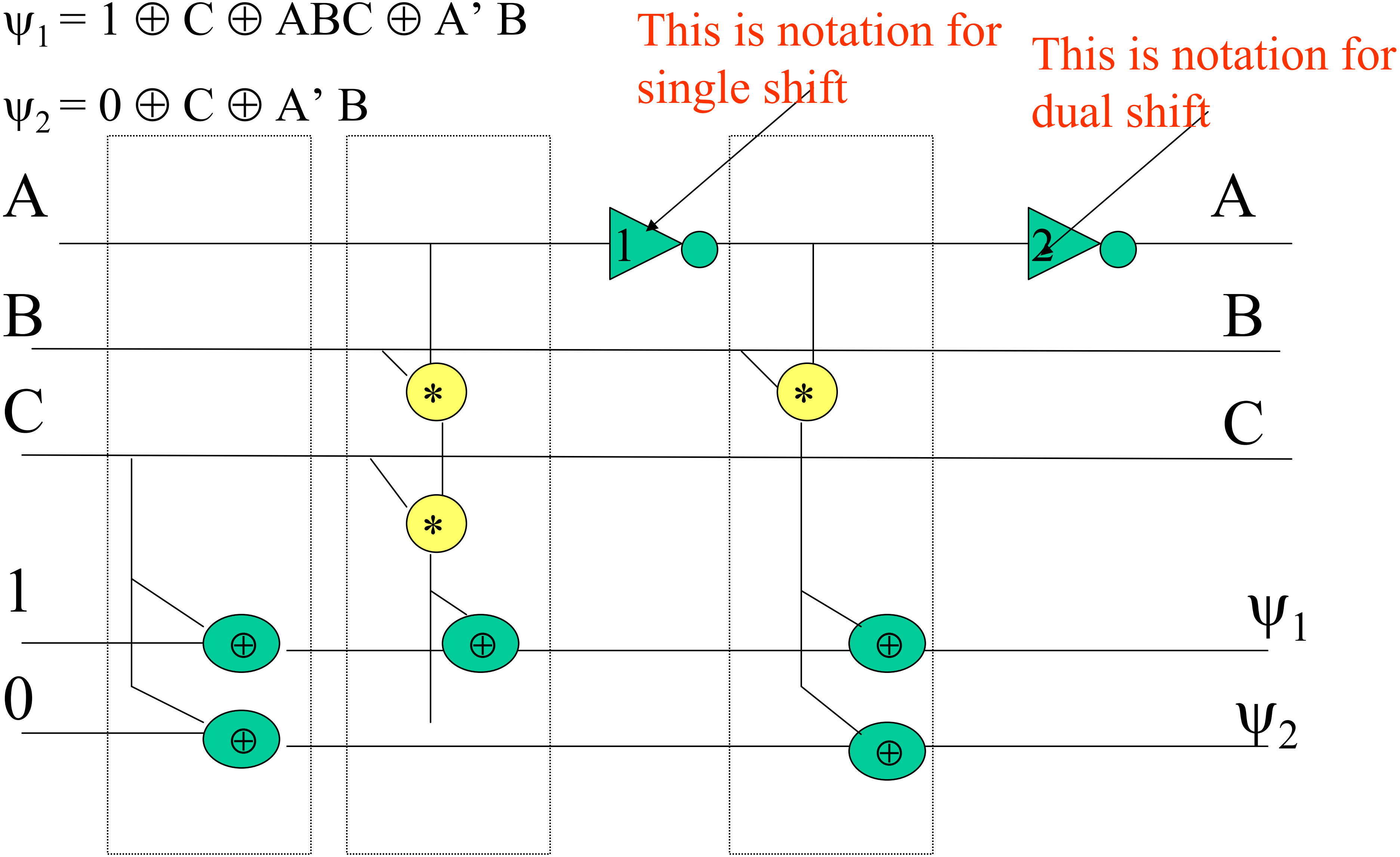
$\psi_2 = 2 \oplus C'' \oplus A' B$



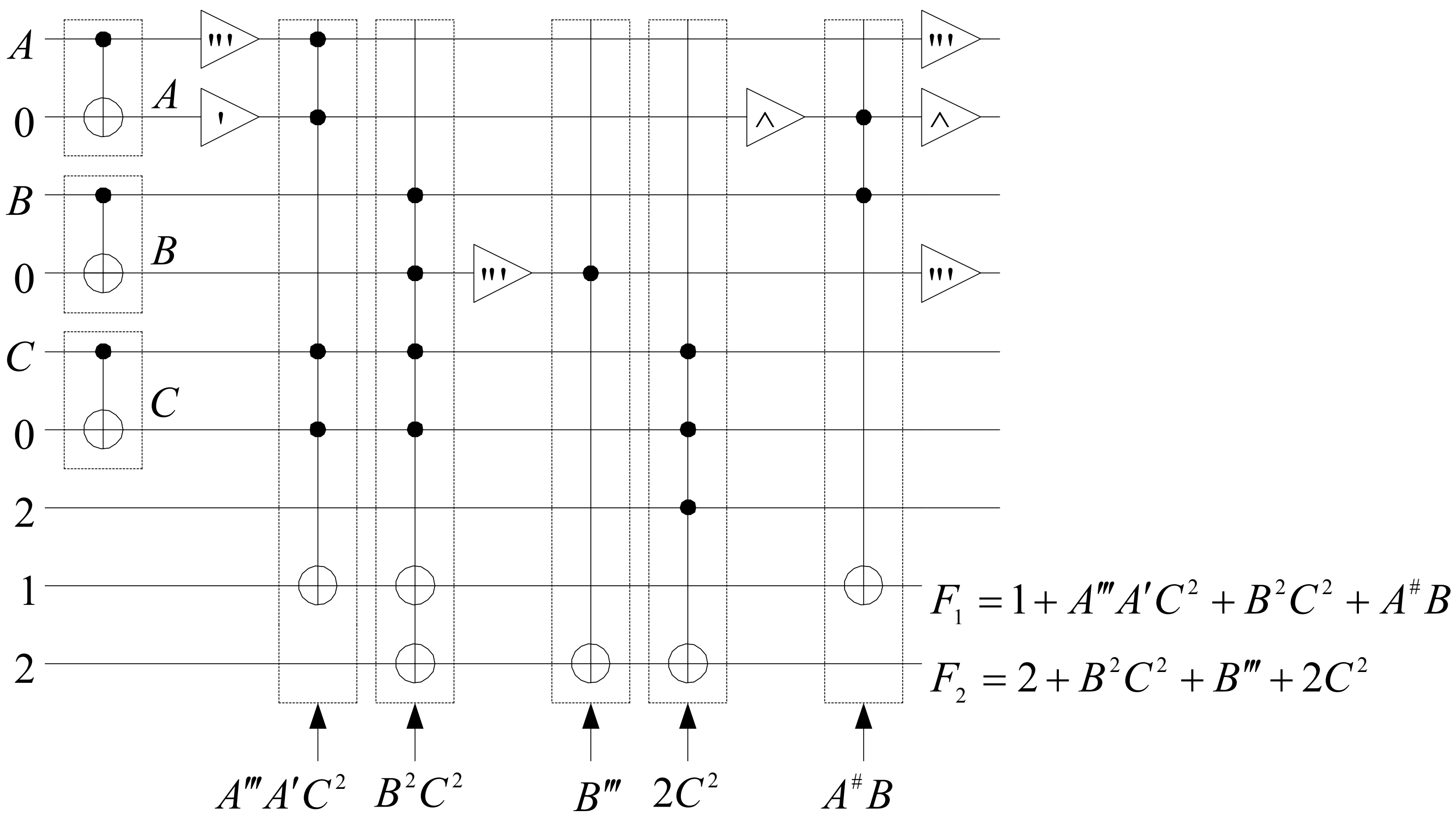
Example of ternary multi-output GFSOP cascade of Toffoli family gates

$\psi_1 = 1 \oplus C \oplus ABC \oplus A' B$

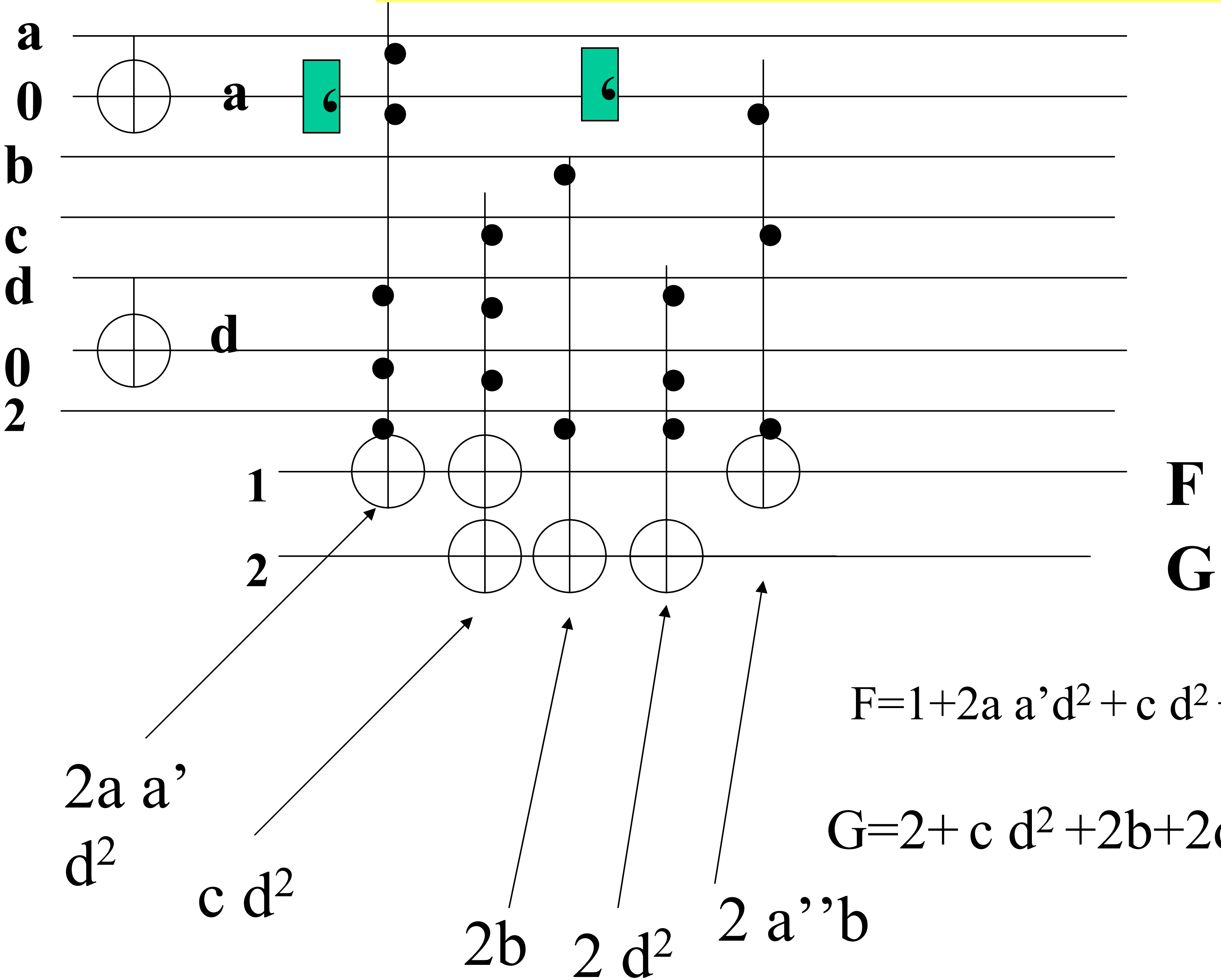
$\psi_2 = 0 \oplus C \oplus A' B$



The general pattern of a cascade to implement any ternary function using ternary Toffoli gates

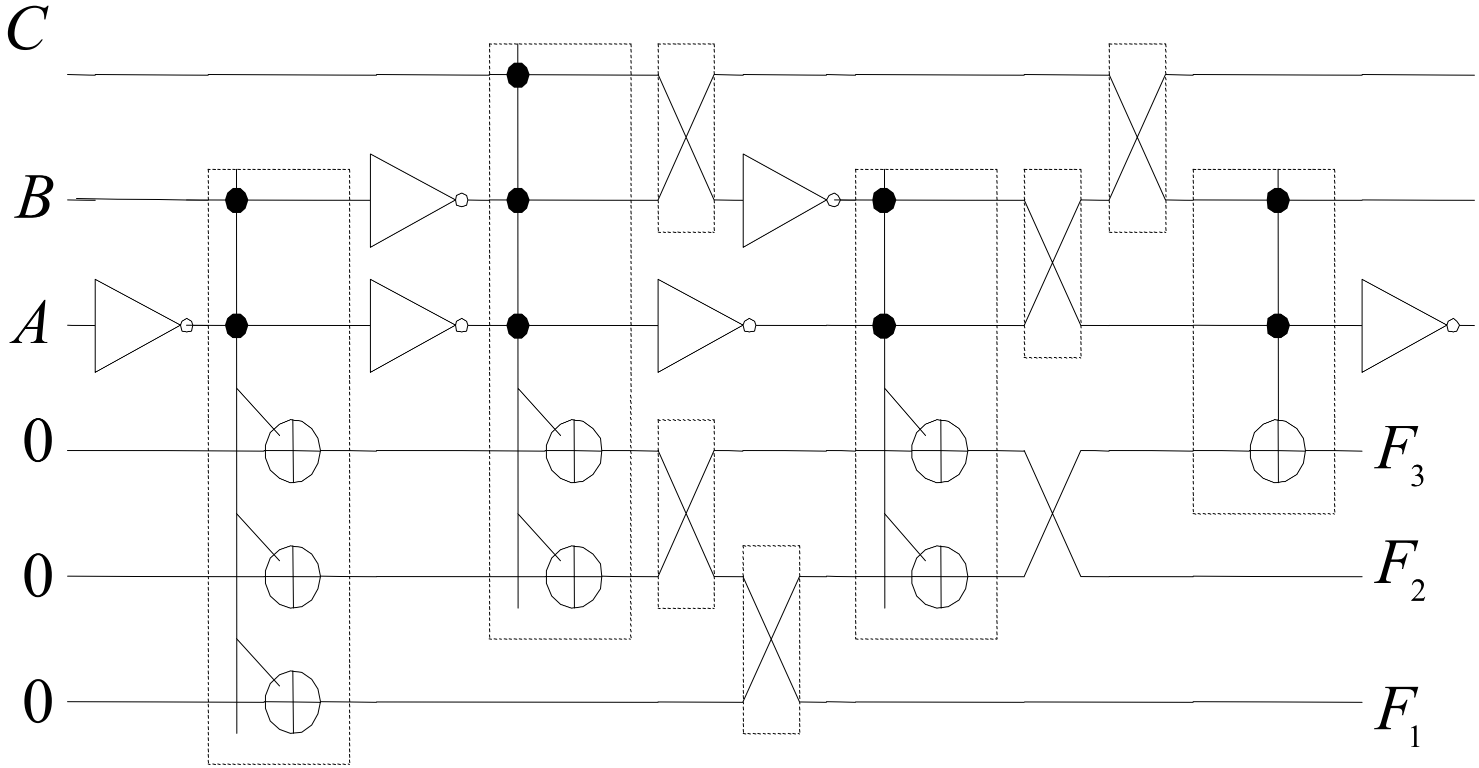


Simplified GFSOP array when powers are not used for some variables. Function of four variables



$F = 1 + 2a\,a'd^2 + c\,d^2 + 2\,a''c$

$G = 2 + c\,d^2 + 2b + 2d^2$



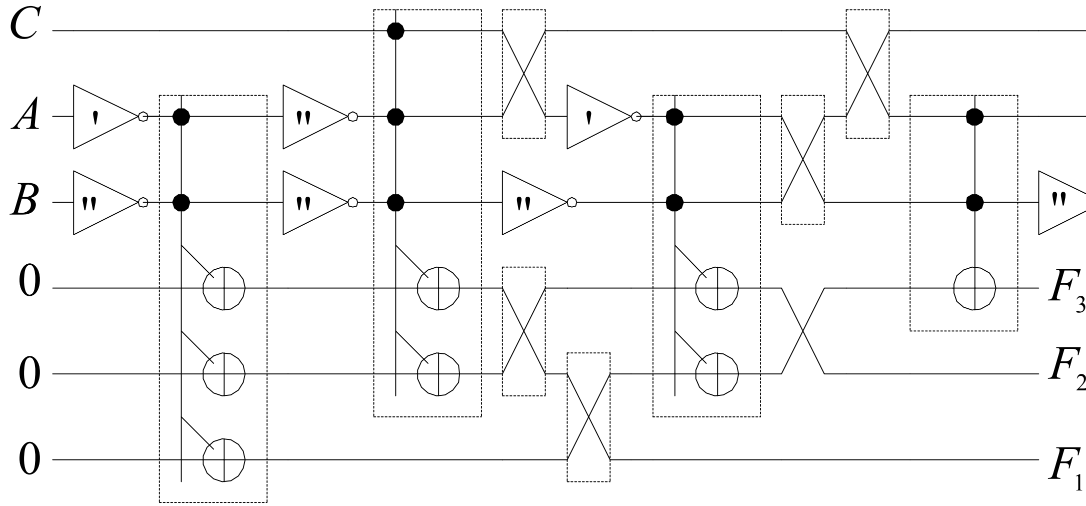
$$F_1 = A'B \oplus A\,B'\,C$$

$$F_2 = A'B \oplus A'C \oplus B'C'$$

$$F_3 = A'B \oplus B'C \oplus AC'$$

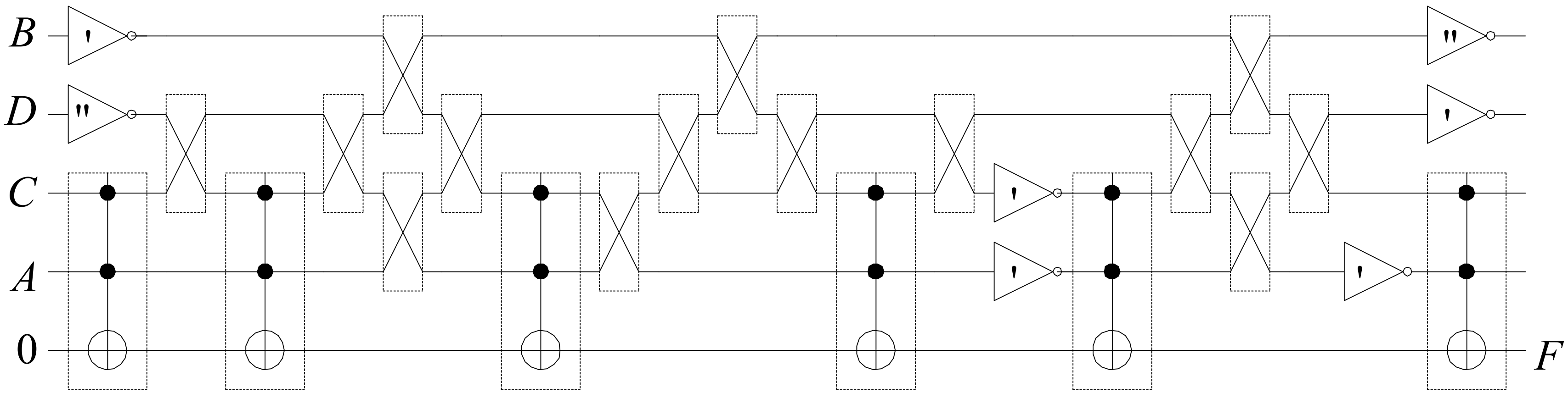
(a) Realization of multi-output ESOP

Macrogeneration introduces many Feynman gates that originate from swaps



$$F_1 = A'B'' \oplus AB'C$$
$$F_2 = A'B'' \oplus AB'C \oplus BC'$$
$$F_3 = A'B'' \oplus BC' \oplus AC'$$

(a) Realization of multi-output GFSOP



$$F = AC \oplus AD'' \oplus B'C \oplus B'D'' \oplus CD \oplus A'B''$$

(b) Realization of single-output GFSOP

MV Quantum Design Structures and Approaches

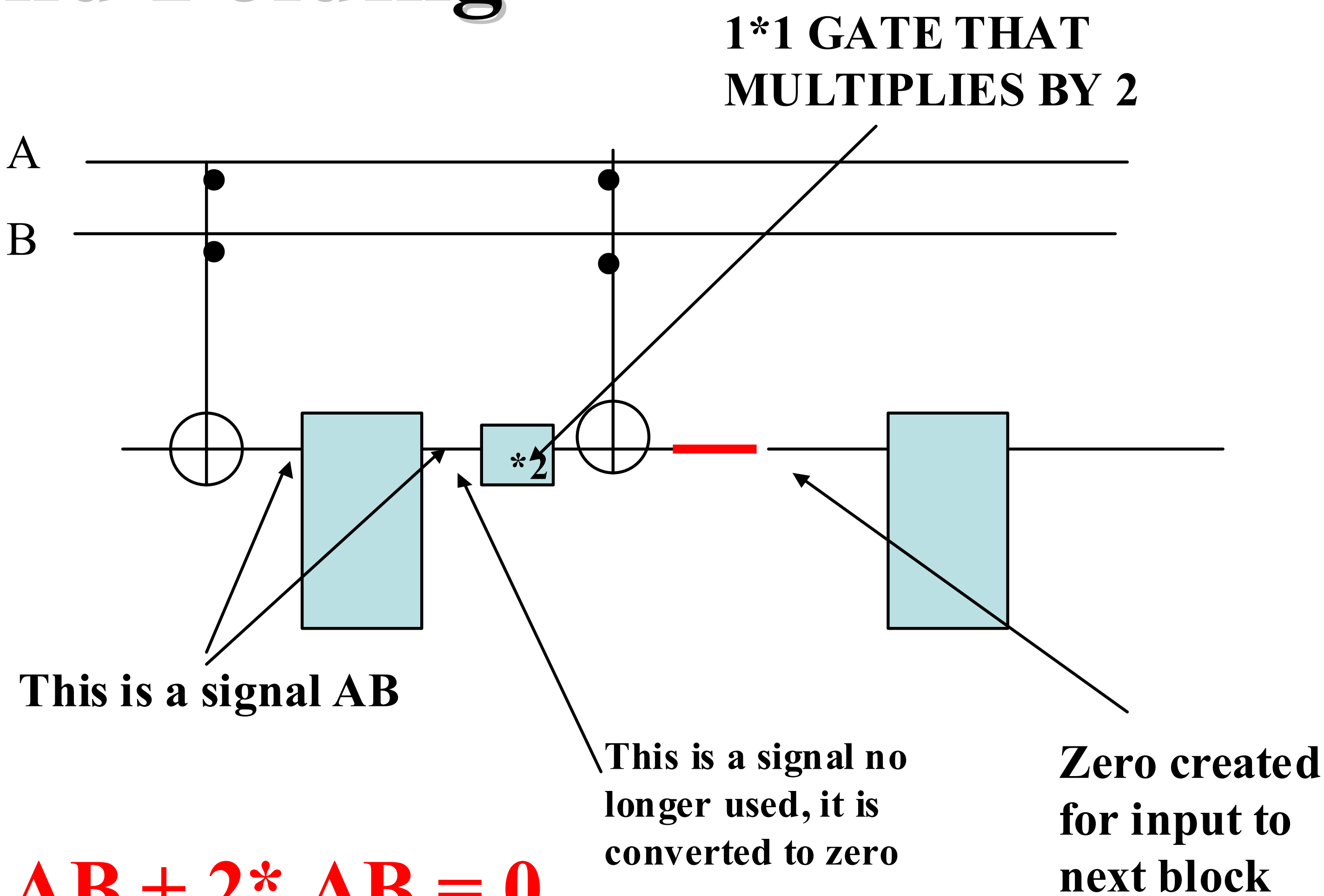
- **1.** GFSOP
- **2.** Multiple-Valued Reed-Muller
- **3.** Canonical Forms over Galois Logic (equivalents of PPRM, FPRM, GRM, etc)
- **4.** Multiple-Valued Maitra Cascades and Wave Cascades.
- **5.** Other cascades of specific type of elements
- **6.** Cascades of general gates

Design Issues

- **1.** Local mirroring
- **2.** Variable ordering versus gate ordering
- **3.** Return to zero and folding
- **4.** Realization of complex multiple-valued reversible gates (permutation gates) using directly 1-qubit and 2-qubit quantum primitives

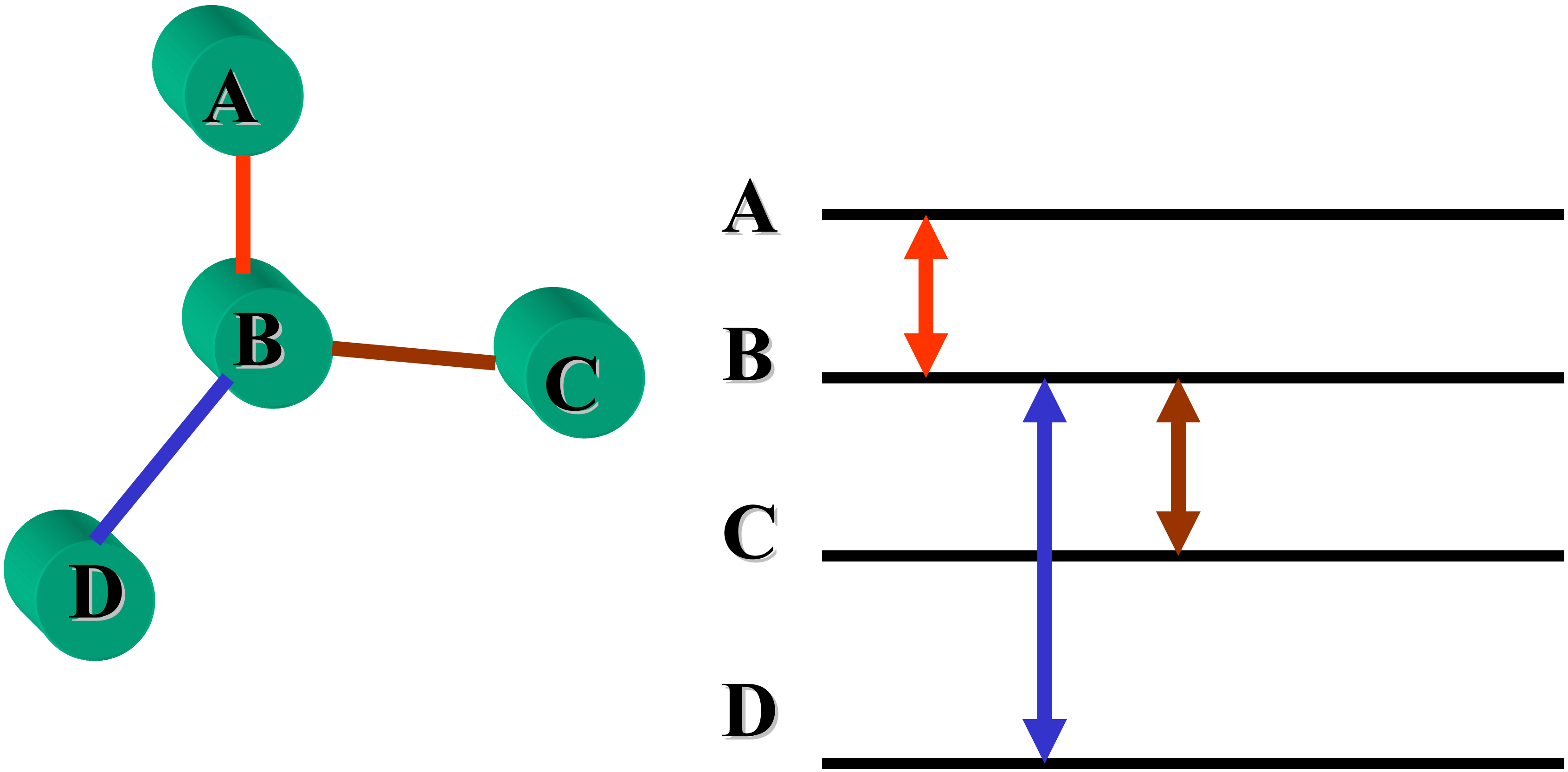
“Return to Zero” and Folding

Technique of local mirror can improve your ternary circuits, reduce the number of zeros in inputs. Here is the explanation for **ternary** logic



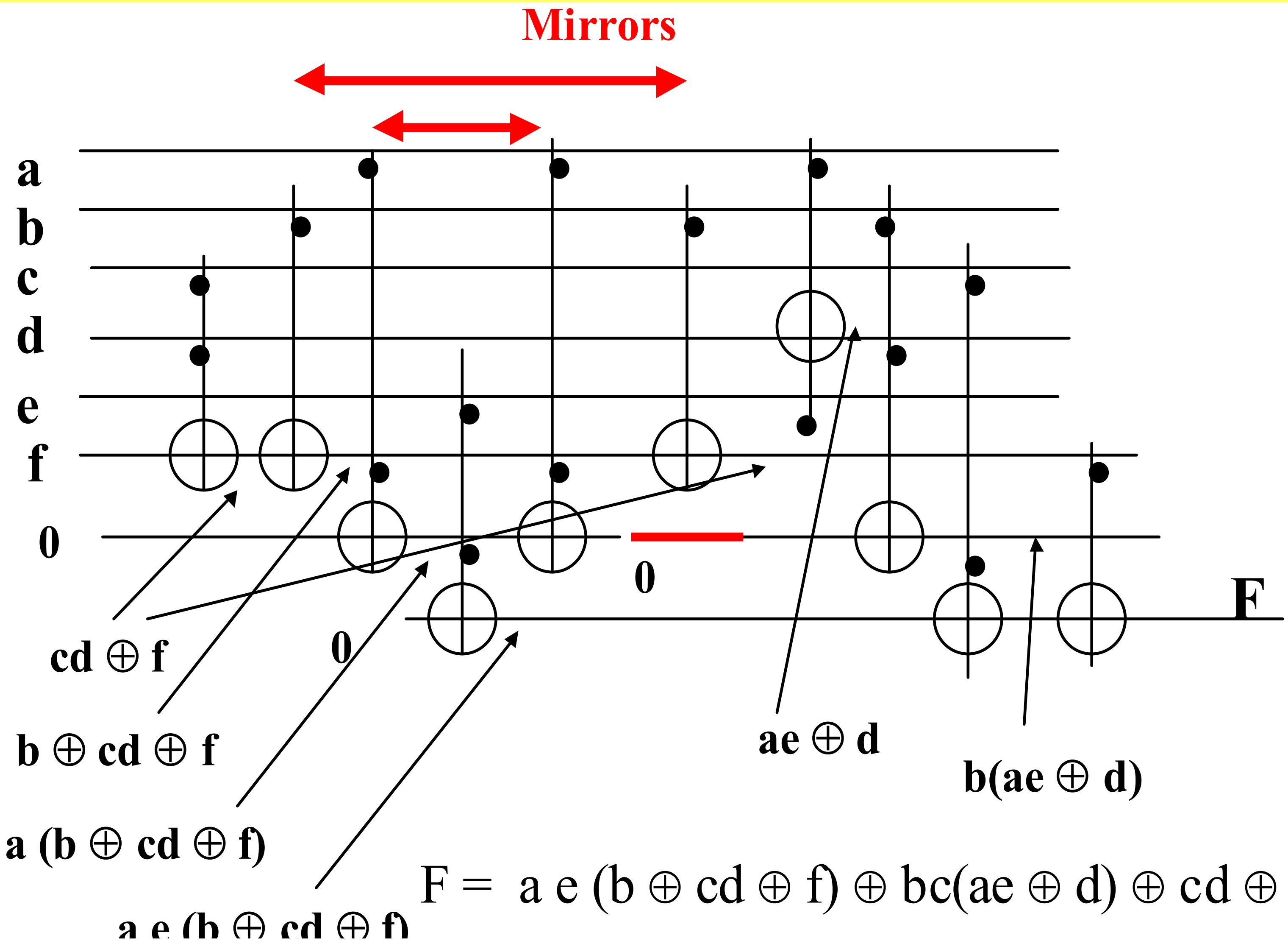
AB + 2* AB = 0

Molecule - Driven Layout and Logic Synthesis

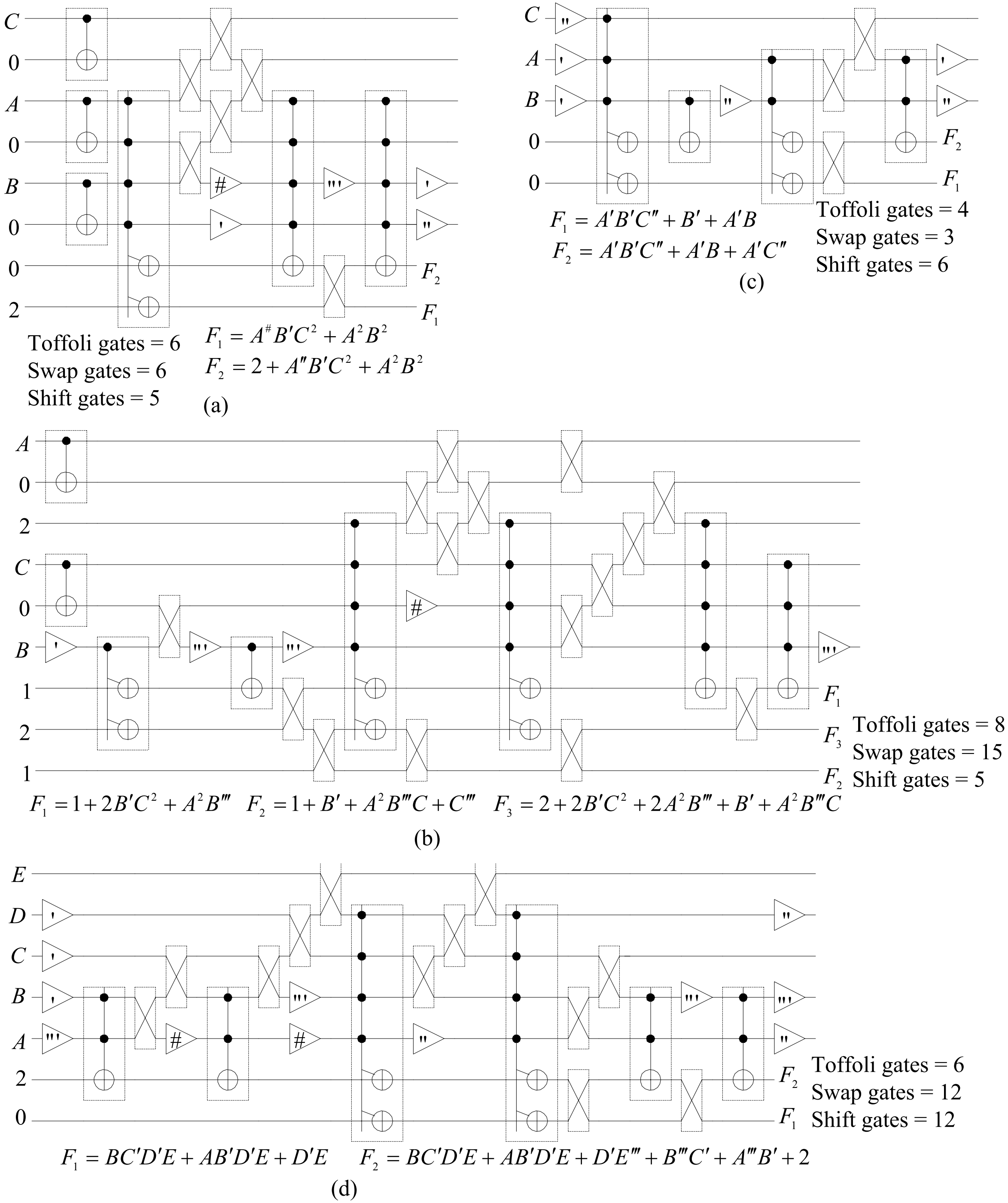


Allowed gate neighborhood for 2 qubit gates

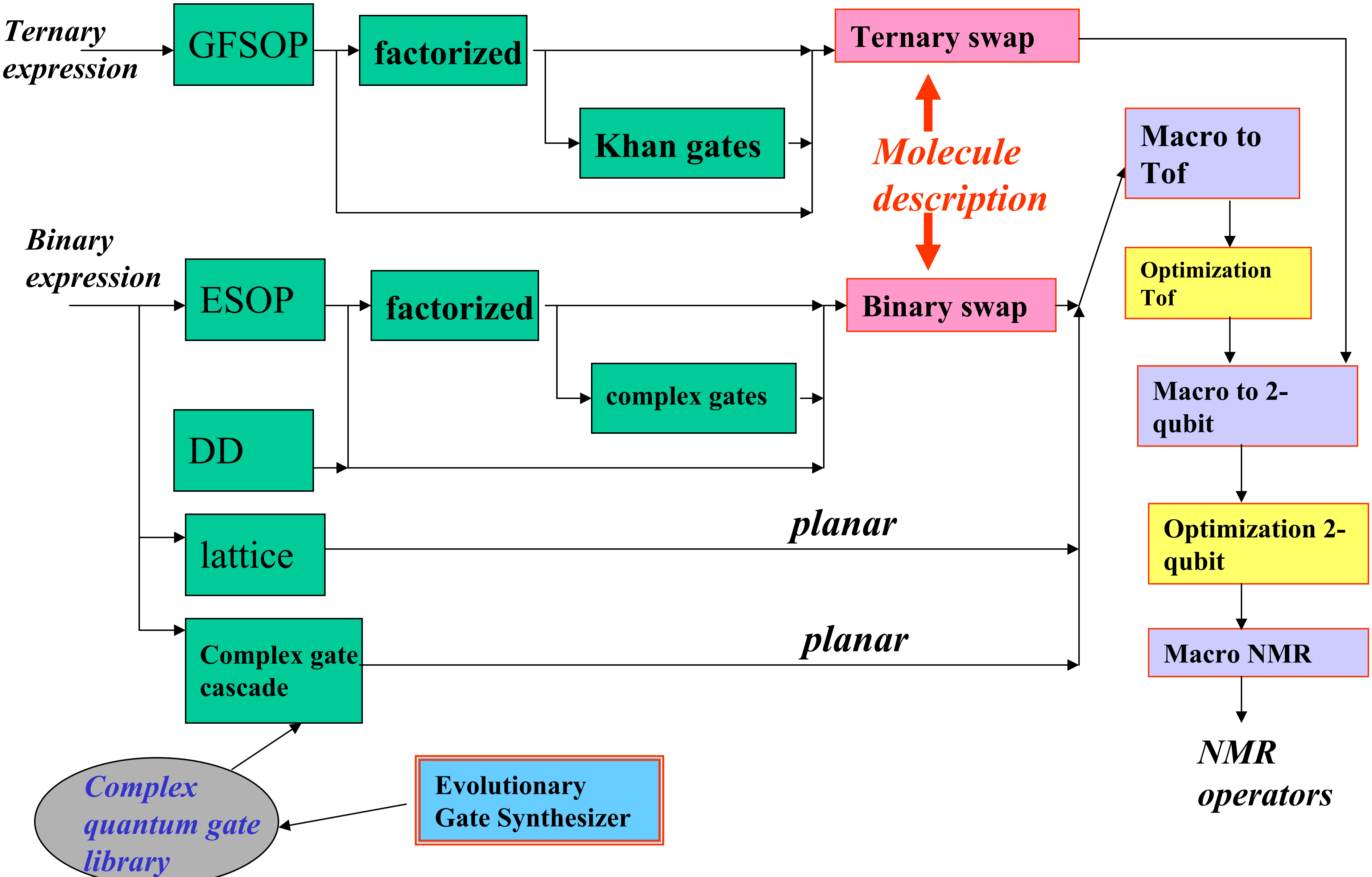
Using Local Mirrors and Return-to-zero factorization



$F = a e (b \oplus cd \oplus f) \oplus bc(ae \oplus d) \oplus cd \oplus f$


C
0
A
0
B
0
0
2
F2
F1
Toffoli gates = 6
Swap gates = 6
Shift gates = 5
$F_1 = A^{\#}B'C^2 + A^2B^2$
$F_2 = 2 + A''B'C^2 + A^2B^2$
(a)
C
A
B
0
0
F2
F1
$F_1 = A'B'C'' + B' + A'B$
$F_2 = A'B'C'' + A'B + A'C''$
Toffoli gates = 4
Swap gates = 3
Shift gates = 6
(c)
A
0
2
C
0
B
1
2
1
F1
F3
F2
Toffoli gates = 8
Swap gates = 15
Shift gates = 5
$F_1 = 1 + 2B'C^2 + A^2B'''$
$F_2 = 1 + B' + A^2B'''C + C'''$
$F_3 = 2 + 2B'C^2 + 2A^2B''' + B' + A^2B'''C$
(b)
E
D
C
B
A
2
0
F2
F1
Toffoli gates = 6
Swap gates = 12
Shift gates = 12
$F_1 = BC'D'E + AB'D'E + D'E$
$F_2 = BC'D'E + AB'D'E + D'E''' + B'''C' + A'''B' + 2$
(d)

System for mixed quantum logic NMR



Open Problems

1. How to select the best gates for permutation circuit synthesis.

2. Simplest practical realization of a ternary Toffoli-like gate

3. Best realization, in quantum circuit sense (simplicity and ease of realization), of other Galois gates and non-Galois standard MV operators such as minimum, maximum, truncated sum and others.

4. Synthesis algorithms for MV reversible circuit families:
   - GFSOP ,
   - nets,
   - lattices,
   - PLAs
   - MV counterparts of Maitra cascades and wave cascades
   - other reversible cascades

Conclusion

- Practical algorithms for MV quantum circuits. Quantum permutation circuits design (for NMR) is **not** the same as standard reversible logic.
- **CAD Tools** for quantum physicists: *link levels of design*.
- Evolutionary Approaches versus GFSOP-like approaches
- MV Quantum Simulation
- MV Quantum Circuits Verification
- Designing MV counterparts of Deutch, Shorr, Grover and other original MV quantum algorithms
- Generalization to MV Of efficient Garbage-less quantum gates by Barenco, DiVincenzo, etc.
- NMR realization of ternary logic.
- MV Quantum Computational Intelligence