Quantum Logic

Marek Perkowski

Sources

Origin of slides: John Hayes, Peter Shor, Martin Lukac, Mikhail Pivtoraiko, Alan Mishchenko, Pawel Kerntopf, Mosca, Ekert

Lee Spector

in collaboration with

Herbert J. Bernstein, Howard Barnum, Nikhil Swamy

{lspector, hbernstein, hbarnum, nikhil_swamy}@hampshire.edu}

School of Cognitive Science, School of Natural Science

Institute for Science and Interdisciplinary Studies (ISIS)

Hampshire College

Introduction

- Short-Term Objectives

Introduce Quantum Computing Basics to interested students at KAIST. Especially non-physics students

- Long-Term Objectives

Engage into AI/CS/Math Research projects benefiting from Quantum Computing. Continue our previous projects in quantum computing

- Prerequisite

- No linear algebra or quantum mechanics assumed
- A ECE, math, physics or CS background would be beneficial, <u>practically-oriented class.</u>

Introduction

- MainTextbook

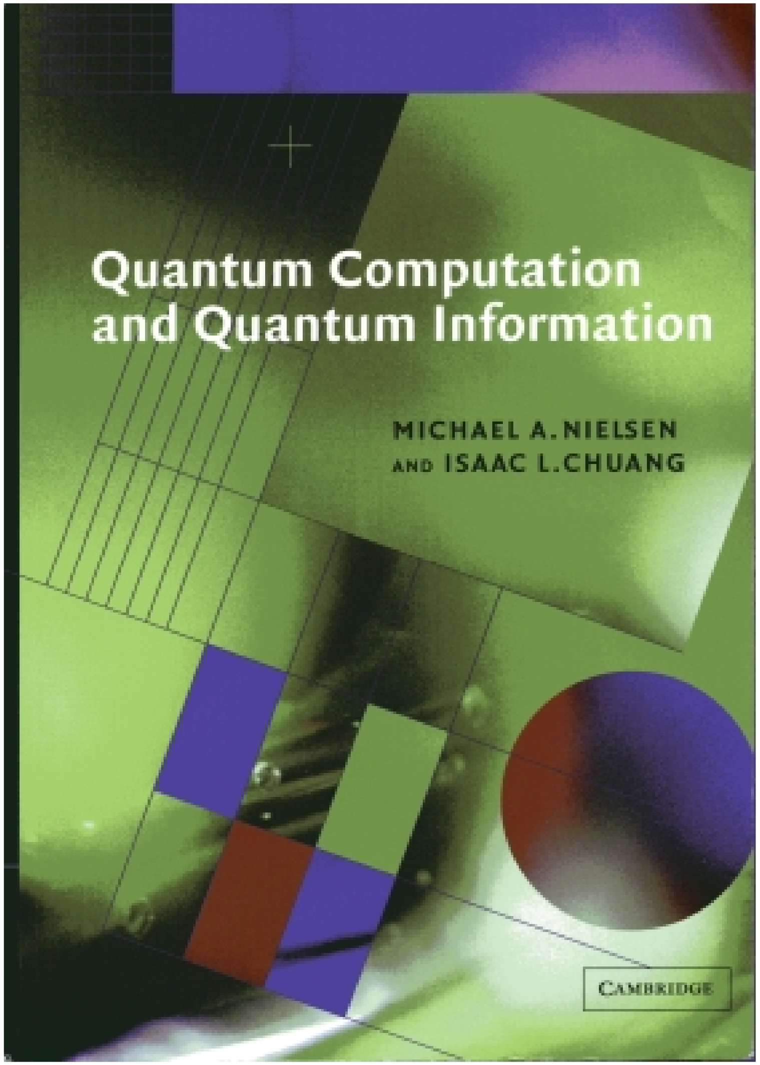
Quantum Computation
&
Quantum Information

Michael A. Nielsen
Isaac L. Chuang

ISBN: 0 521 63503 9 Paperback
ISBN: 0 521 63235 8 Hardback

Cost: $48.00 New Paperback
$35.45 Used Paperback

(http://www.amazon.com)
also in KAIST bookstore



Presentation Overview

Qubits

1 Qubit -> Bloch Sphere,
2 Qubits -> Bell States,
n Qubits

Quantum Computation

Gates: Single Qubit, Arbitrary Single Qubit -> Universal Quantum Gates, Multiple Qubit Gates -> CNOT
Other Computational Bases

Quantum Circuits

Qubit Swap Circuit
Qubit Copying Circuit
Bell State Circuit -> Quantum Teleportation

Quantum Algorithms

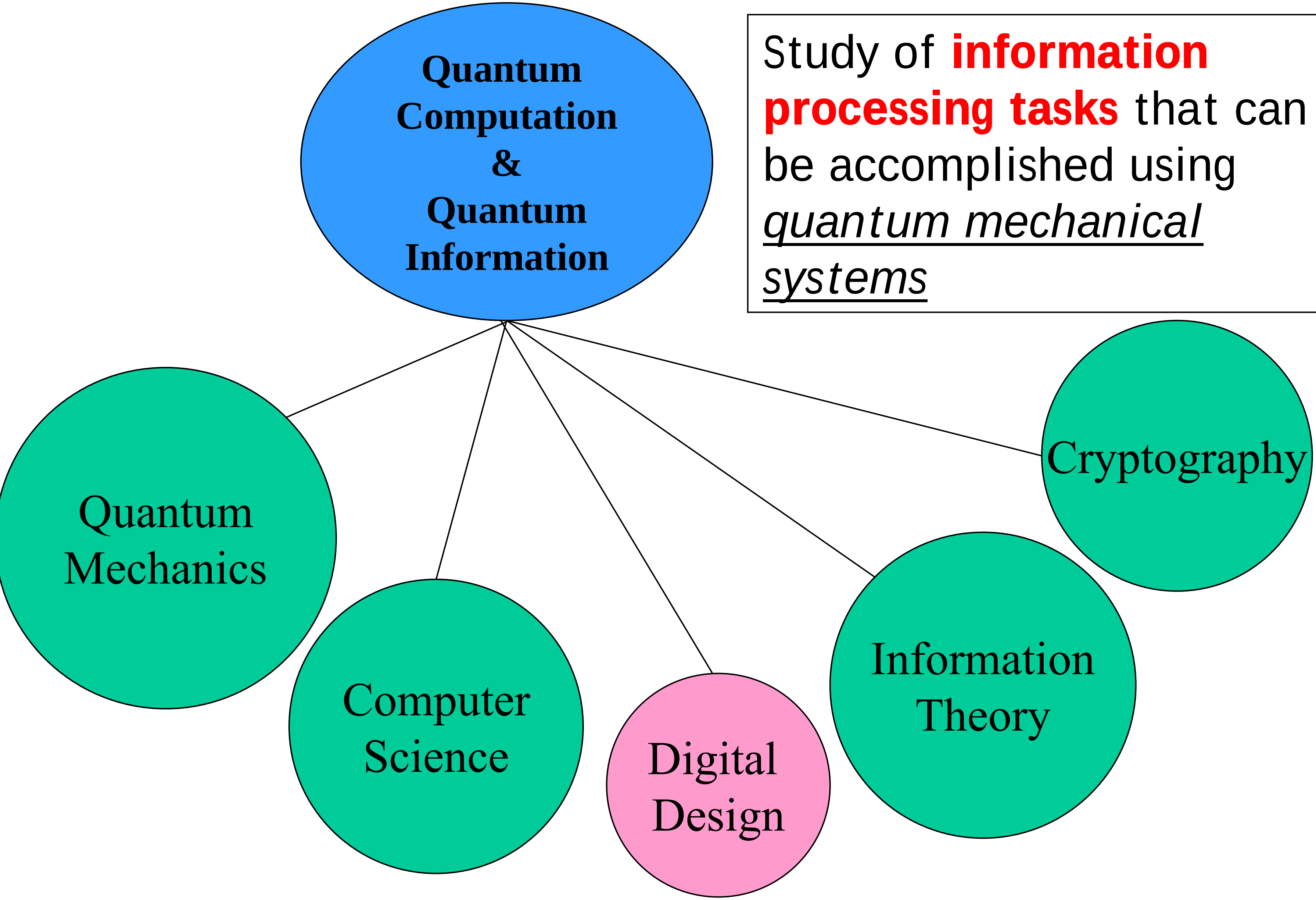
Toffoli Gate -> Quantum Parallelism -> Hadamard Transform
Deutsch's Algorithm, Deutsch-Josa Algorithm
Other Algorithms
- Fourier Transform, Quantum Search, Quantum Simulation

Quantum Information Processing

Stern-Gerlach, Optical Techniques, Traps, NMR, Quantum Dots

Historical Background and Links

Study of **information processing tasks** that can be accomplished using *quantum mechanical systems*



What will be discussed?

- Background
- Quantum circuits synthesis and algorithms
- Quantum circuits simulation
- Quantum Computation
- AI for quantum computation

- Quantum computation for AI
- Quantum logic emulation and evolvable hardware
- Quantum circuits verification
- Quantum-based robot control

What is quantum computation?

- **Computation with coherent atomic-scale dynamics.**
- **The behavior of a quantum computer is governed by the laws of quantum mechanics.**

Why bother with quantum computation?

- **Moore's Law:** We hit the quantum level 2010~2020.
- Quantum computation is more powerful than classical computation.
- More can be computed in less time—the complexity classes are different!

The power of quantum computation

- **In quantum systems possibilities count, even if they never happen!**
- **Each of exponentially many possibilities can be used to perform a part of a computation at the same time.**

Nobody understands quantum mechanics

"No, you're not going to be able to understand it. . . . You see, my physics students don't understand it either. That is because I don't understand it. Nobody does. ... The theory of quantum electrodynamics describes Nature as absurd from the point of view of common sense. And it agrees fully with an experiment. So I hope that you can accept Nature as She is -- absurd.

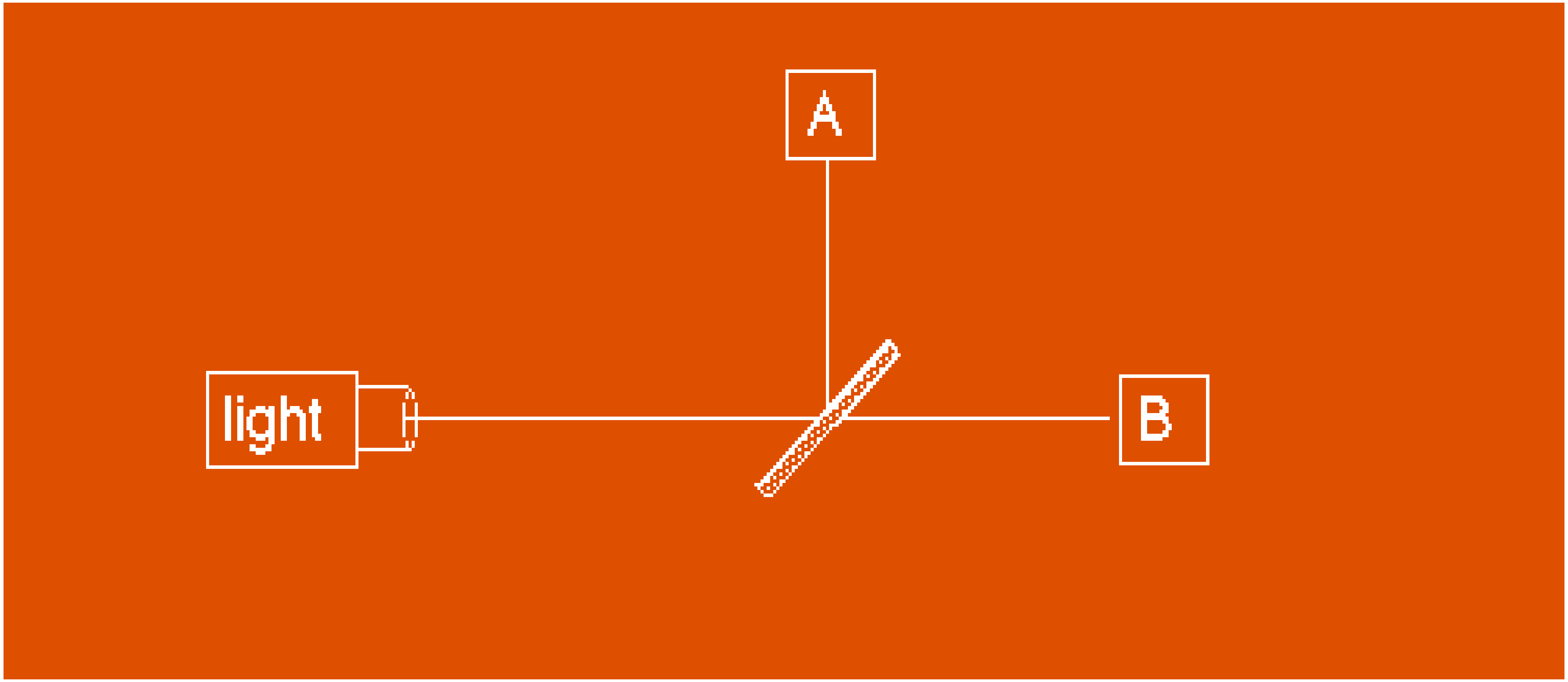
Richard Feynman

Absurd but taken seriously (not just quantum mechanics but also quantum computation)

- Under active investigation by many of the top physics labs around the world (including CalTech, MIT, AT&T, Stanford, Los Alamos, UCLA, Oxford, l’Université de Montréal, University of Innsbruck, IBM Research . . .)
- In the mass media (including The New York Times, The Economist, American Scientist, Scientific American, . . .)
- Here.

Quantum Logic Circuits

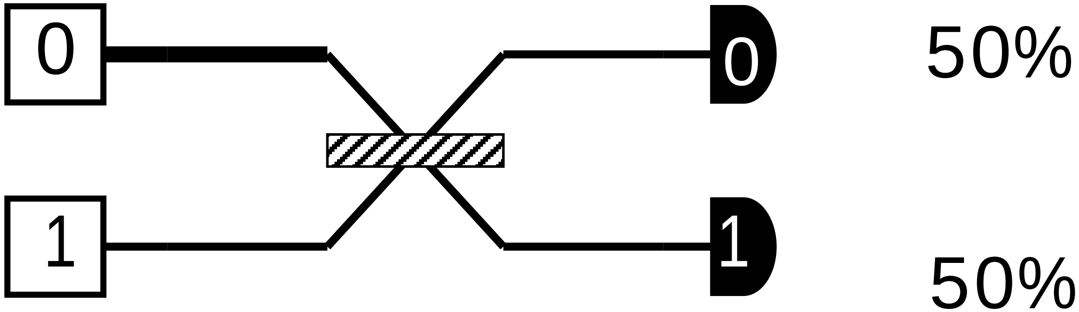
A beam splitter



Half of the photons leaving the light source arrive at detector A;

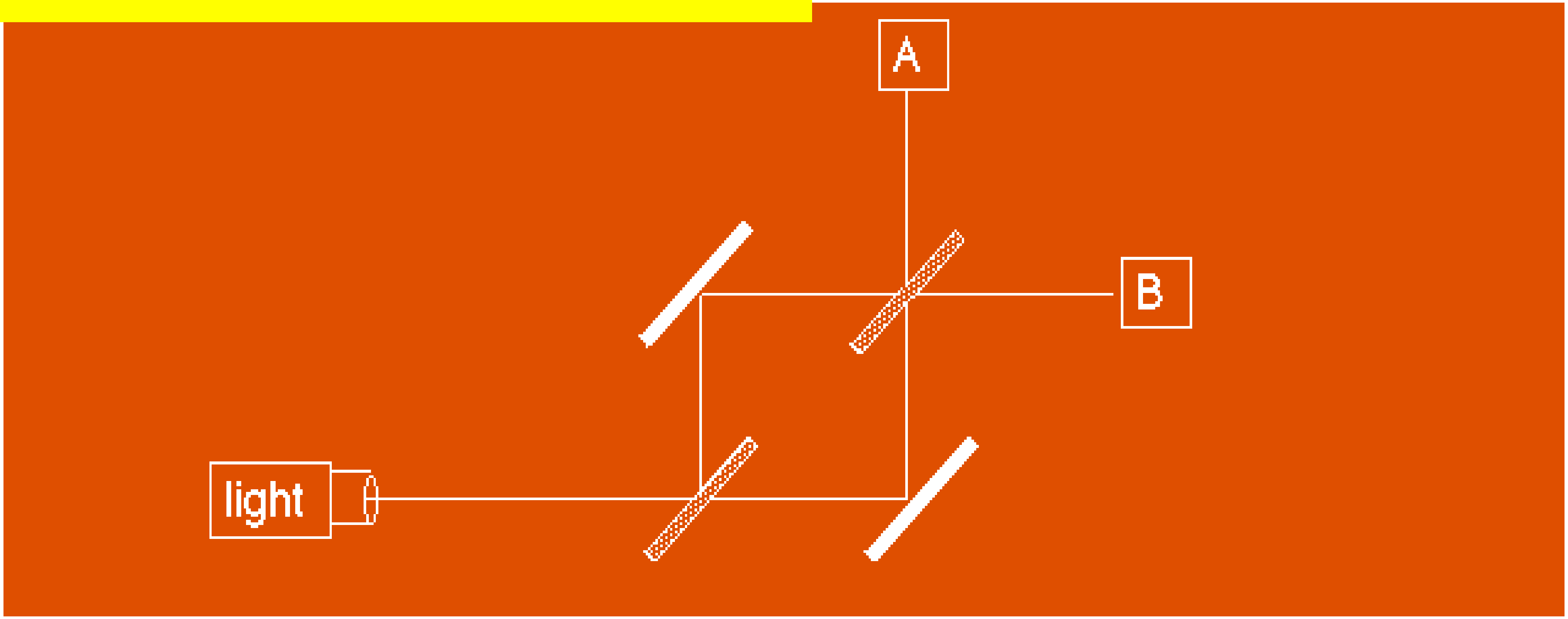
the other half arrive at detector B.

A beam-splitter



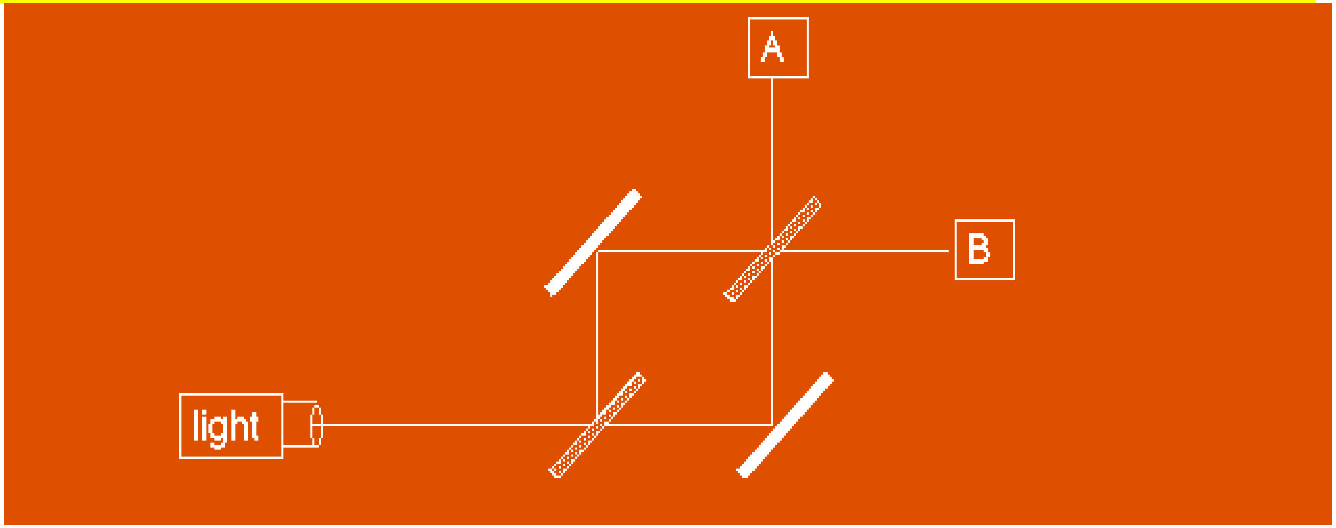
The simplest explanation is that the beam-splitter acts as a classical coin-flip, randomly sending each photon one way or the other.

An interferometer



- **Equal path lengths, rigid mirrors.**
- **Only one photon in the apparatus at a time.**
- **All photons leaving the source arrive at B.**
- **WHY?**

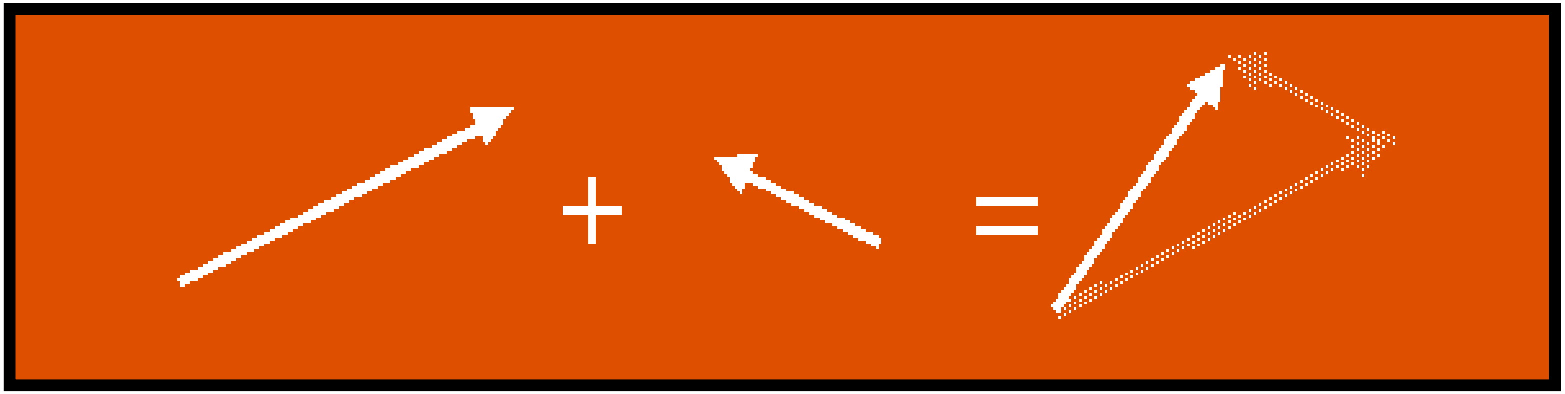
Possibilities count



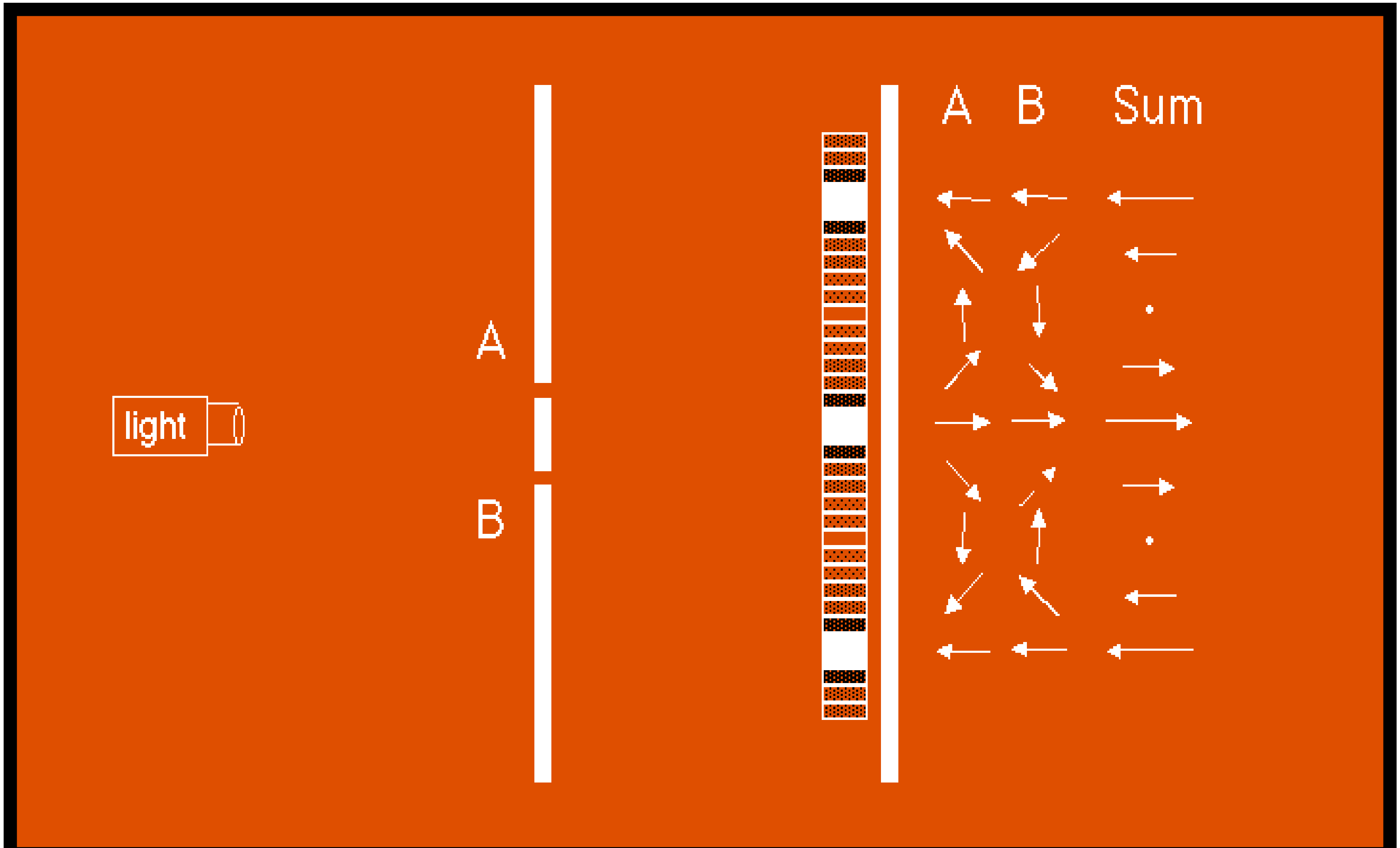
- There is a quantity that we'll call the "amplitude" for each possible path that a photon can take.
- The amplitudes can interfere constructively and destructively, even though each photon takes only one path.
- The amplitudes at detector A interfere destructively; those at detector B interfere constructively.

Calculating interference

- Arrows for each possibility.
- Arrows rotate; speed depends on frequency.
- Arrows flip 180º at mirrors, rotate 90º counter-clockwise when reflected from beam splitters.
- Add arrows and square the length of the result to determine the probability for any possibility.



Double slit interference



Quantum Interference : Amplitudes are added and not intensities !

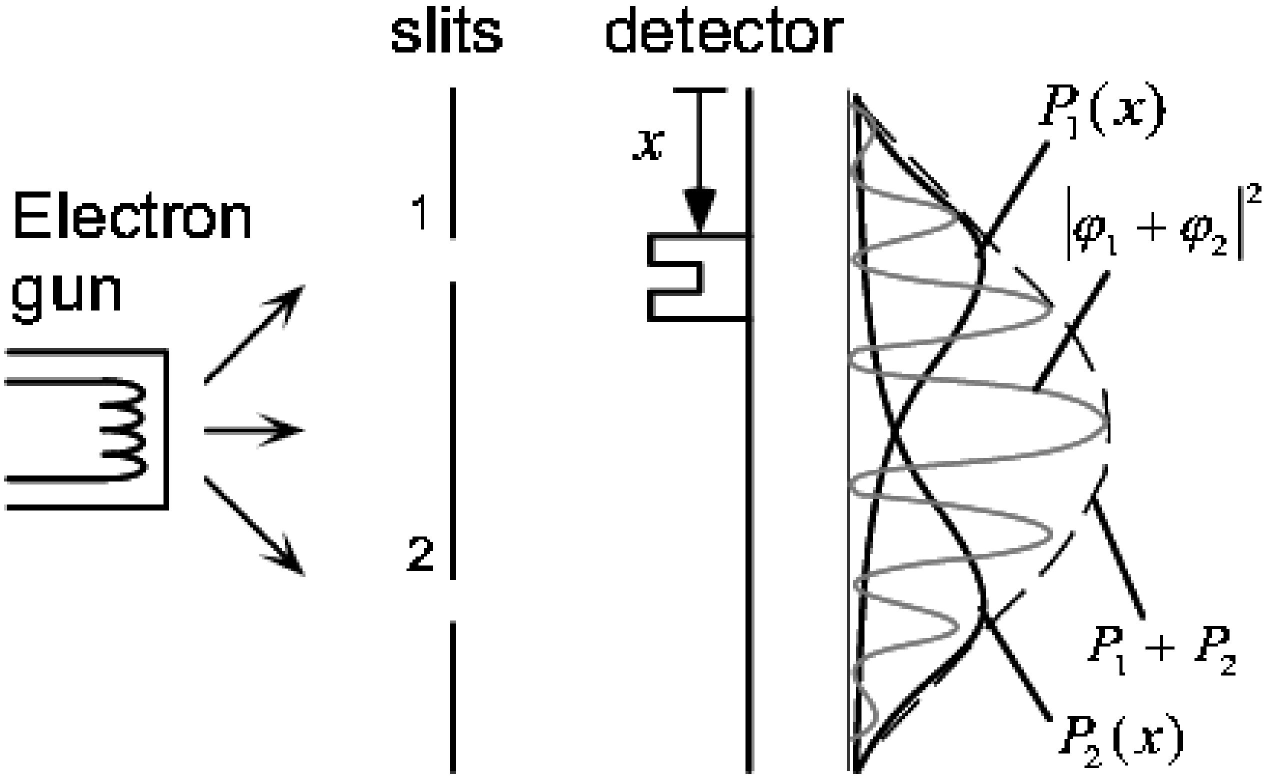
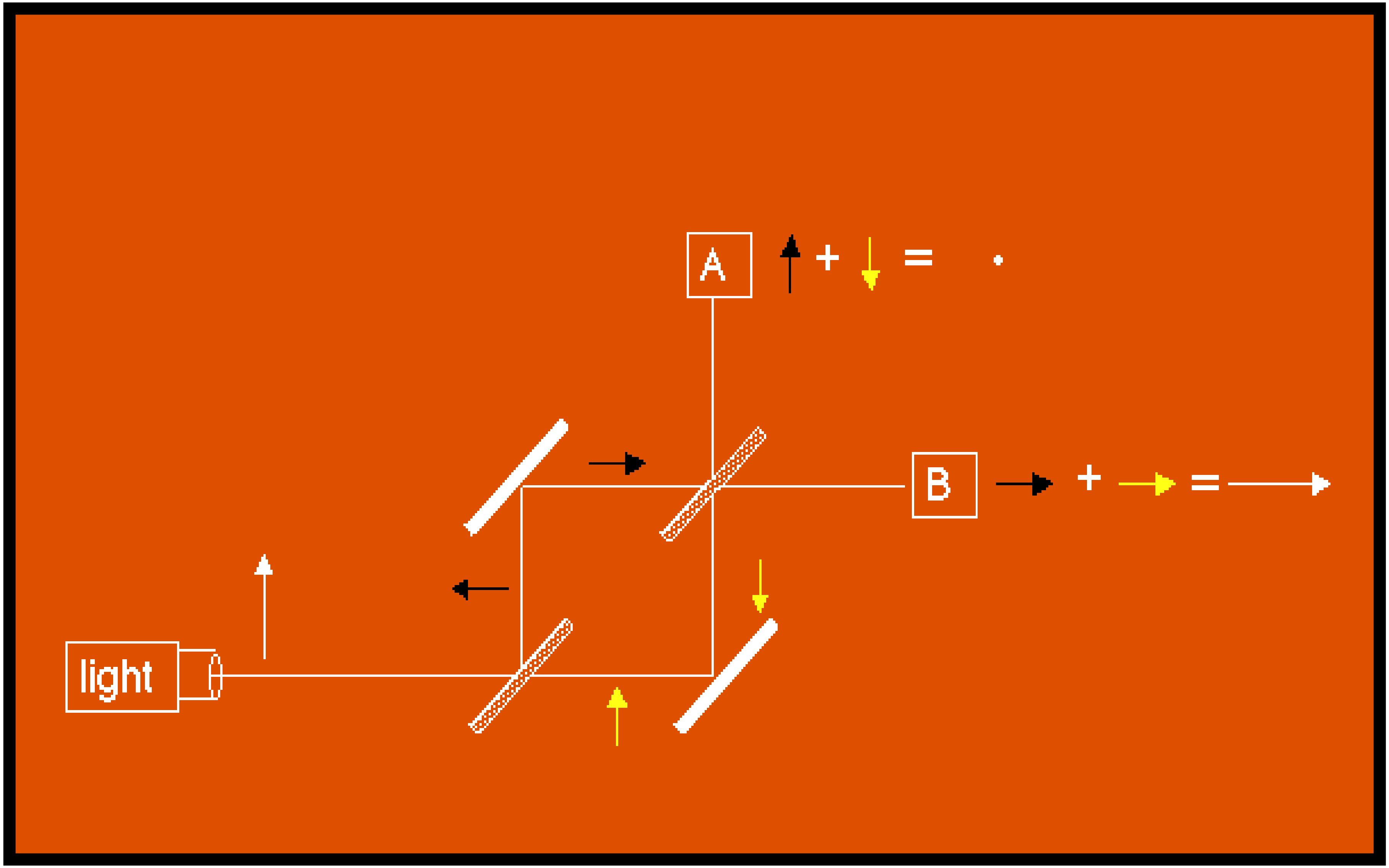
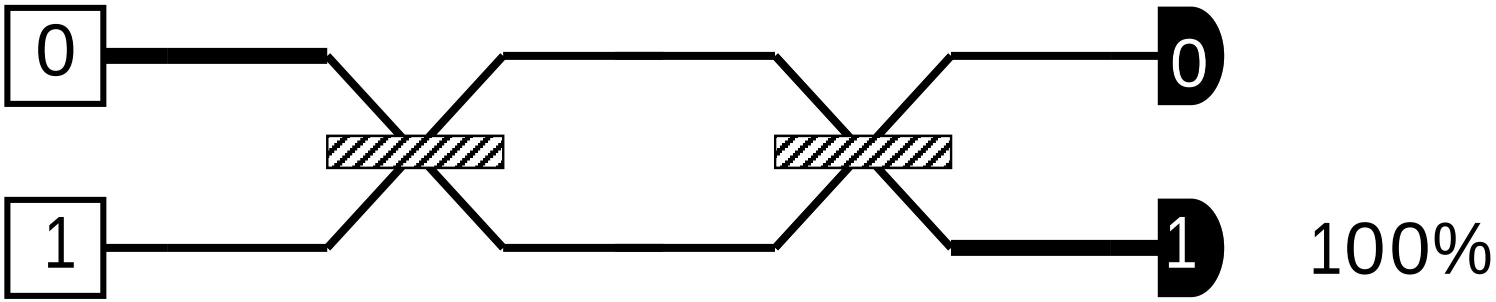


Figure 1: Two-slit experiment.

Interference in the interferometer

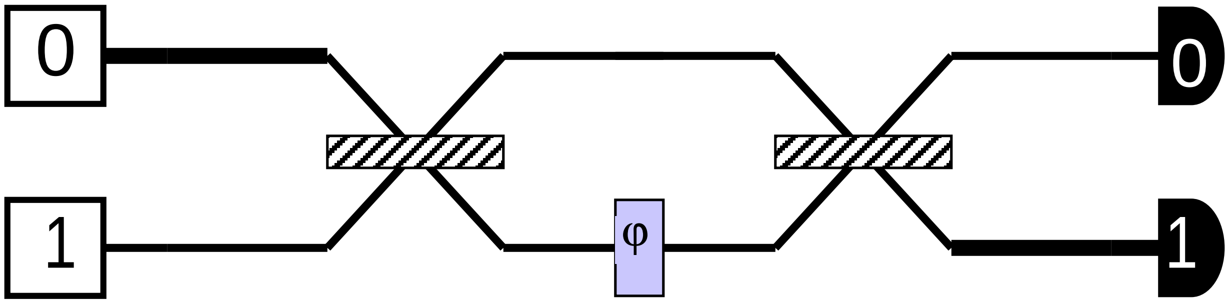


Quantum Interference



The simplest explanation must be wrong, since it would predict a 50-50 distribution.

More experimental data

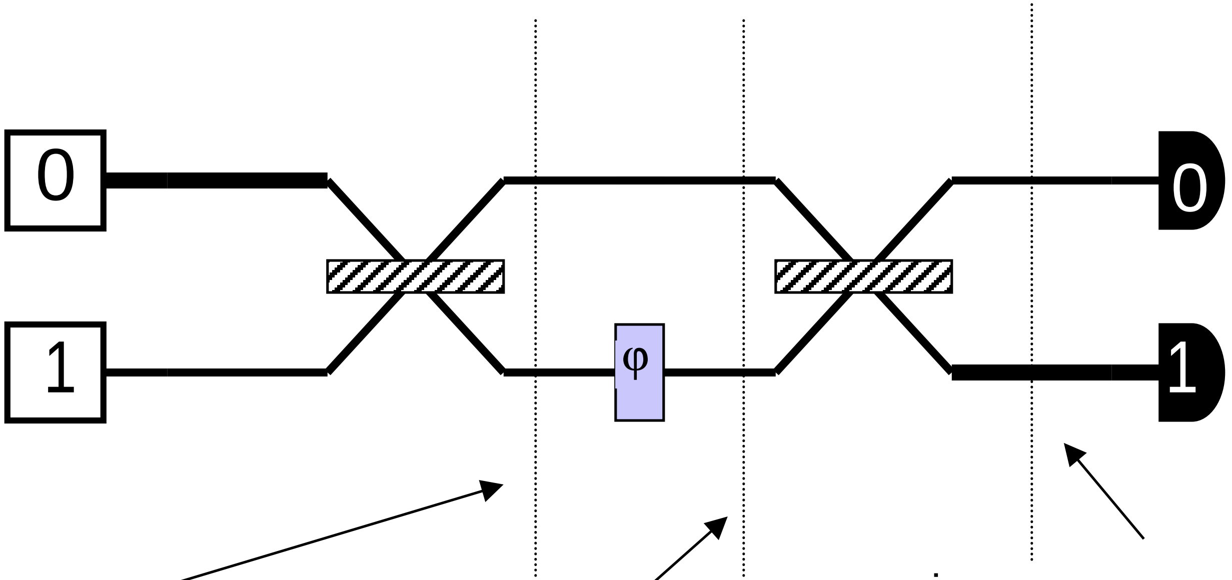


$\sin^2\left(\frac{\varphi}{2}\right)$

$\cos^2\left(\frac{\varphi}{2}\right)$

A new theory

The particle can exist in a linear combination or *superposition* of the two paths



$$\sin^2\left(\frac{\varphi}{2}\right)$$

$$\cos^2\left(\frac{\varphi}{2}\right)$$

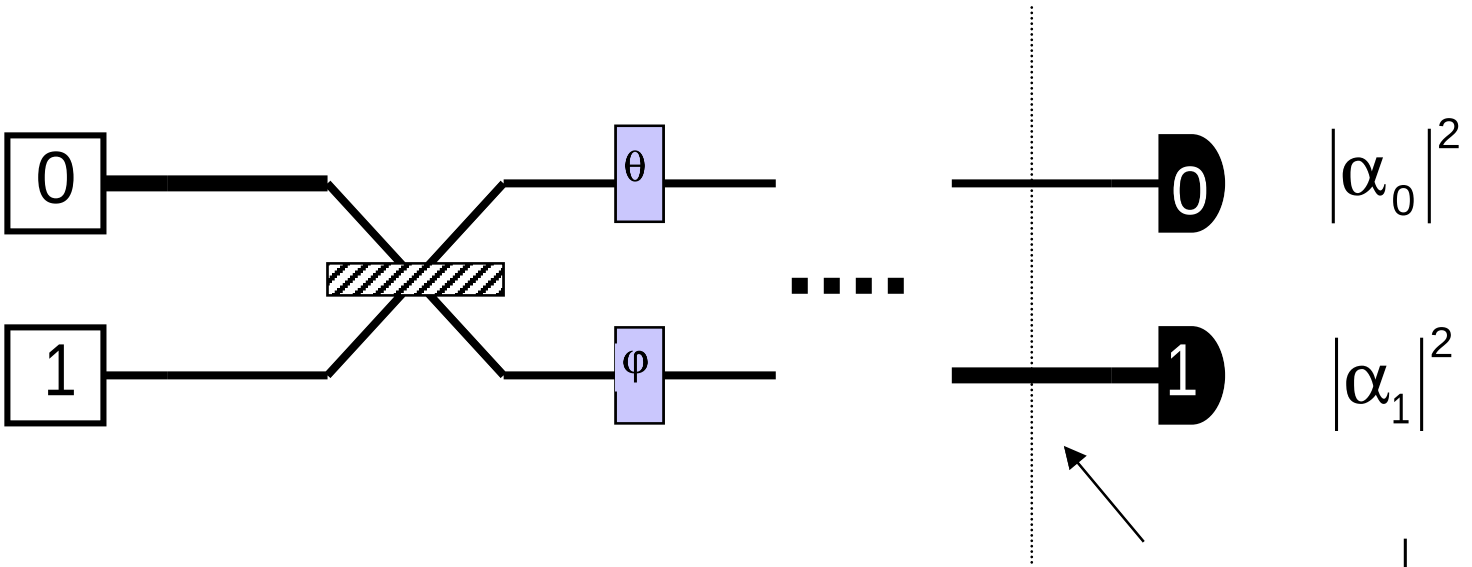
$$\frac{i}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$\frac{i}{\sqrt{2}}|0\rangle + \frac{e^{i\varphi}}{\sqrt{2}}|1\rangle$$

$$\frac{e^{i\varphi}-1}{2}|0\rangle + \frac{i(e^{i\varphi}+1)}{\sqrt{2}}|1\rangle$$

Probability Amplitude and Measurement

If the photon is measured when it is in the state $\alpha_0|0\rangle + \alpha_1|1\rangle$ then we get $|0\rangle$ with probability $|\alpha_0|^2$ and |1> with probability of $|a_1|^2$



$$|\alpha_0|^2 + |\alpha_1|^2 = 1$$

Quantum Operations

The operations are induced by the apparatus *linearly*, that is, if

$$|0\rangle \to \frac{i}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

and

$$|1\rangle \to \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$$

then

$$\alpha_0|0\rangle + \alpha_1|1\rangle \to \alpha_0\left(\frac{i}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) + \alpha_1\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle\right)$$

$$= \left(\alpha_0\frac{i}{\sqrt{2}} + \alpha_1\frac{1}{\sqrt{2}}\right)|0\rangle + \left(\alpha_0\frac{1}{\sqrt{2}} + \alpha_1\frac{i}{\sqrt{2}}\right)|1\rangle$$

Quantum Operations

Any linear operation that takes states

$\alpha_0|0\rangle + \alpha_1|1\rangle$ satisfying $|\alpha_0|^2 + |\alpha_1|^2 = 1$

and maps them to states

$\alpha_0^{'}|0\rangle + \alpha_1^{'}|1\rangle$ satisfying $|\alpha_0^{'}|^2 + |\alpha_1^{'}|^2 = 1$

must be UNITARY

Linear Algebra

$$U = \begin{bmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{bmatrix}$$

is unitary if and only if

$$UU^t = \begin{bmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{bmatrix} \begin{bmatrix} u^*_{00} & u^*_{10} \\ u^*_{01} & u^*_{11} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

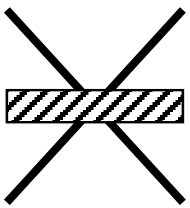
Linear Algebra

$|0\rangle$ corresponds to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

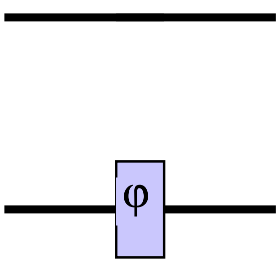
$|1\rangle$ corresponds to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\alpha_0|0\rangle + \alpha_1|1\rangle$ corresponds to $\alpha_0\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_1\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}$

Linear Algebra

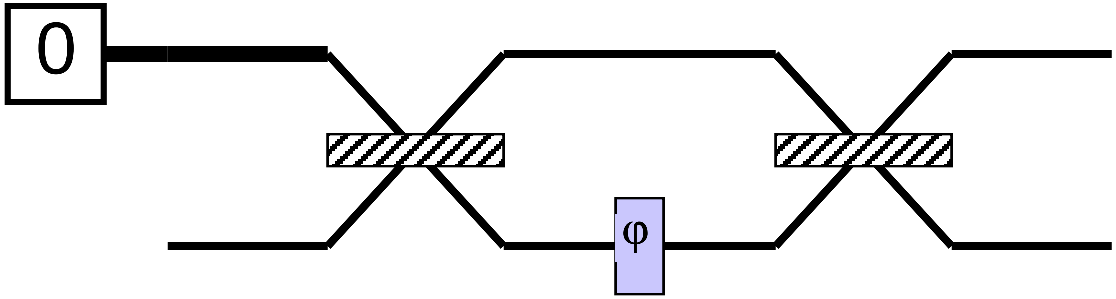


corresponds to $\begin{pmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}$



corresponds to $\begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}$

Linear Algebra



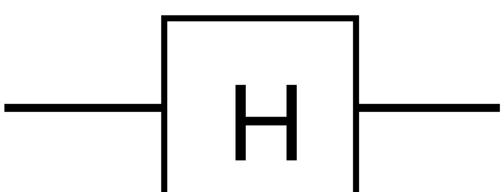
corresponds to

$$\begin{pmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix} \begin{pmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

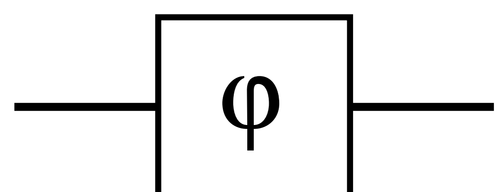
Abstraction

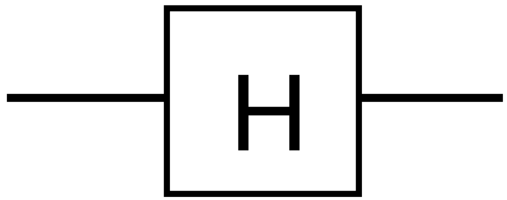
The two position states of a photon in a Mach-Zehnder apparatus is just one example of a quantum bit or ***qubit***

Except when addressing a particular physical implementation, we will simply talk about "basis" states $|0\rangle$ and $|1\rangle$
and **unitary operations** like

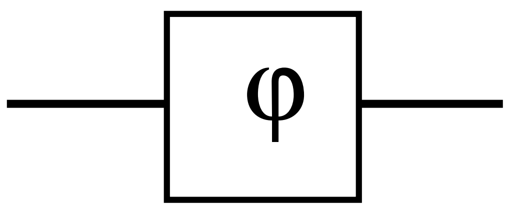


and



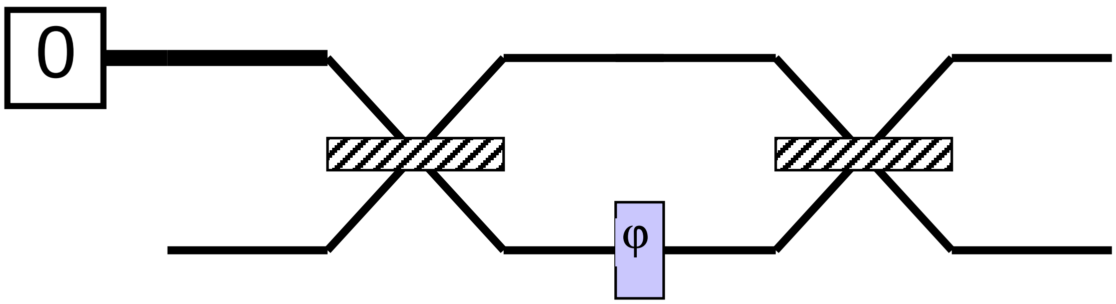
where 

corresponds to $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$

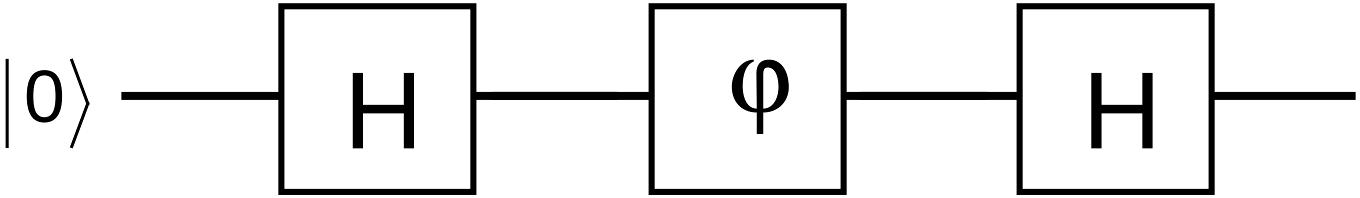
and 

corresponds to $\begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}$

An arrangement like



is represented with a network *like*



More than one qubit

If we concatenate two qubits

$$\left(\alpha_0|0\rangle + \alpha_1|1\rangle\right)\ \left(\beta_0|0\rangle + \beta_1|1\rangle\right)$$

we have a 2-qubit system with 4 basis states

$$|0\rangle|0\rangle = |00\rangle \qquad |0\rangle|1\rangle = |01\rangle \qquad |1\rangle|0\rangle = |10\rangle \qquad |1\rangle|1\rangle = |11\rangle$$

and we can also describe the state as

$$\alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle$$

or by the vector

$$\begin{pmatrix} \alpha_0\beta_0 \\ \alpha_0\beta \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \otimes \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$$

More than one qubit

In general we can have arbitrary superpositions

$$\alpha_{00}|0\rangle|0\rangle + \alpha_{01}|0\rangle|1\rangle + \alpha_{10}|1\rangle|0\rangle + \alpha_{11}|1\rangle|1\rangle$$

$$|\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1$$

where there is no factorization into the tensor product of two independent qubits.
These states are called *entangled.*

Entanglement

- Qubits in a multi-qubit system are not independent—they can become "entangled."
- To represent the state of n qubits we use 2^n complex number amplitudes.

Measuring multi-qubit systems

If we measure both bits of

$$\alpha_{00}|0\rangle|0\rangle + \alpha_{01}|0\rangle|1\rangle + \alpha_{10}|1\rangle|0\rangle + \alpha_{11}|1\rangle|1\rangle$$

we get $|x\rangle|y\rangle$ with probability $|\alpha_{xy}|^2$

Measurement

- $\sum|\alpha|^2$, **for amplitudes of all states matching an output bit-pattern, gives the probability that it will be read.**
- **Example:**

  $$0.316|00\rangle + 0.447|01\rangle + 0.548|10\rangle + 0.632|11\rangle$$

  – **The probability to read the rightmost bit as 0 is** $|0.316|^2 + |0.548|^2 = 0.4$
- **Measurement during a computation changes the state of the system but can be used in some cases to increase efficiency (measure and halt or continue).**

Classical Versus Quantum

Classical vs. Quantum Circuits

- Goal: Fast, low-cost implementation of useful algorithms using standard components (gates) and design techniques
- Classical Logic Circuits
  - Circuit behavior is governed implicitly by classical physics
  - Signal states are simple bit vectors, e.g. $X = 01010111$
  - Operations are defined by Boolean Algebra
  - No restrictions exist on copying or measuring signals
  - Small well-defined sets of universal gate types, e.g. {NAND}, {AND,OR,NOT}, {AND,NOT}, etc.
  - Well developed CAD methodologies exist
  - Circuits are easily implemented in fast, scalable and macroscopic technologies such as CMOS

Classical vs. Quantum Circuits

- <u>Quantum Logic Circuits</u>
  - Circuit behavior is governed explicitly by quantum mechanics
  - Signal states are vectors interpreted as a superposition of binary "qubit" vectors with complex-number coefficients

$$|\Psi\rangle = \sum_{i=0}^{2^n-1} c_i \left| i_{n-1} i_{n-1} \ldots i_0 \right\rangle$$

  - Operations are defined by linear algebra over Hilbert Space and can be represented by unitary matrices with complex elements
  - <u>Severe restrictions</u> exist on copying and measuring signals
  - Many universal gate sets exist but the <u>best types are not obvious</u>
  - Circuits must use microscopic technologies that are slow, fragile, and not yet scalable, e.g., NMR

Quantum Circuit Characteristics

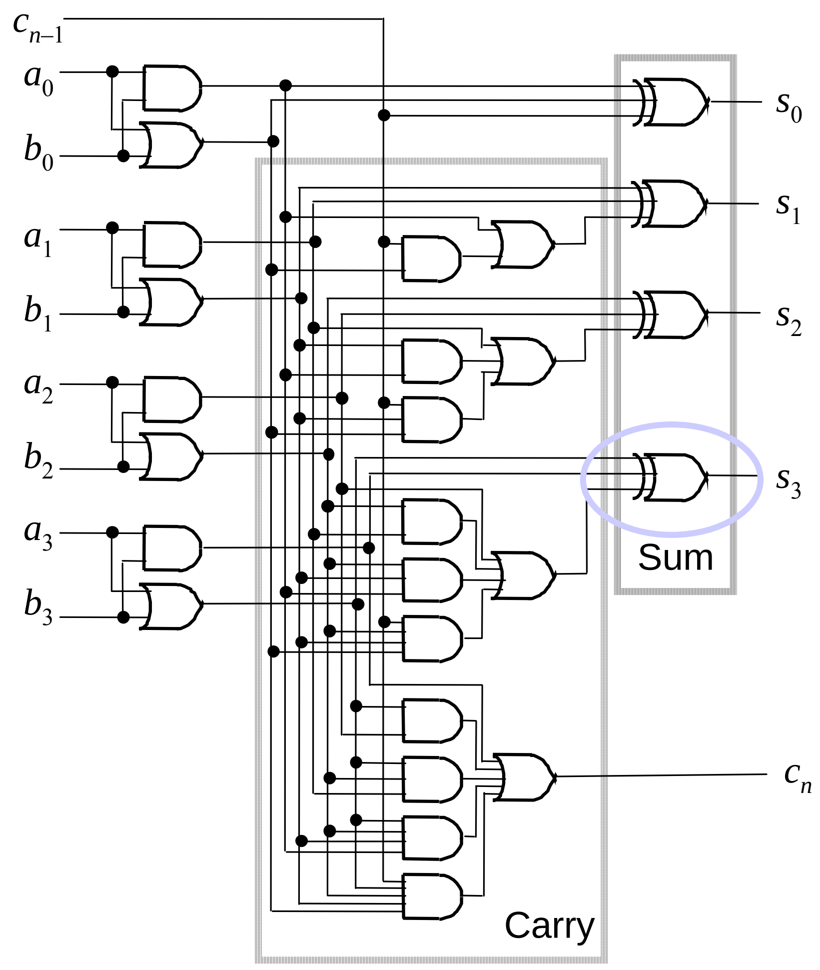
- Unitary Operations
  - Gates and circuits must be reversible (information-lossless)
    - Number of output signal lines = Number of input signal lines
    - The circuit function must be a bijection, implying that output vectors are a permutation of the input vectors
  - **Classical** logic behavior can be represented by permutation matrices
  - **Non-classical** logic behavior can be represented including state sign (phase) and entanglement

Quantum Circuit Characteristics

- Quantum Measurement
  - Measurement yields only one state *X* of the superposed states
  - Measurement also makes *X* the new state and so ***interferes with computational processes***
  - *X* is determined with some **probability**, implying uncertainty in the result
  - States cannot be copied ("cloned"), implying that **signal fanout is not permitted**
  - Environmental interference can cause a measurement-like state collapse (**decoherence**)

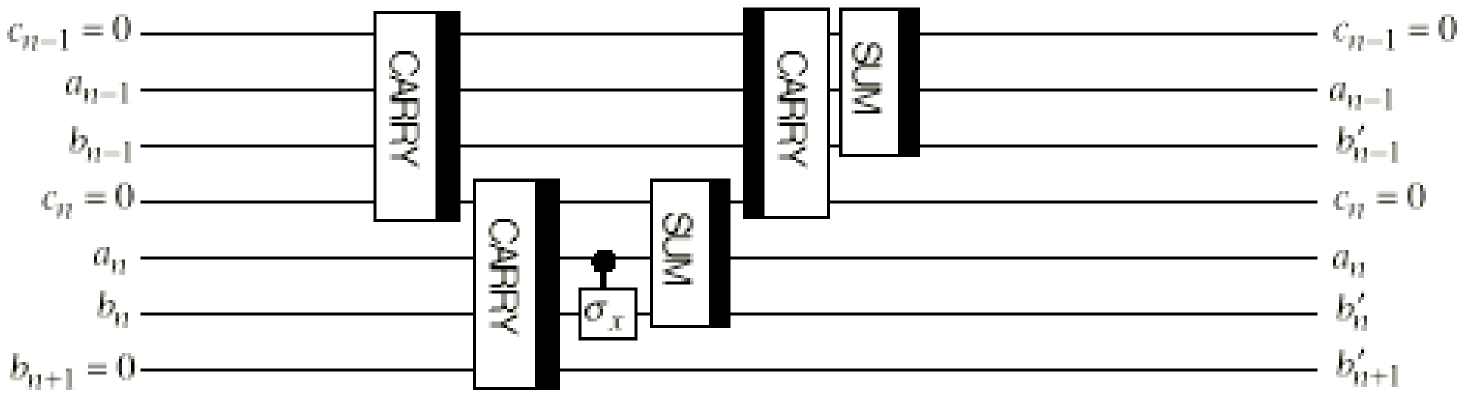
Classical vs. Quantum Circuits

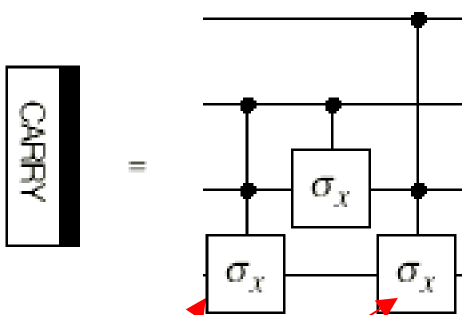
Classical adder

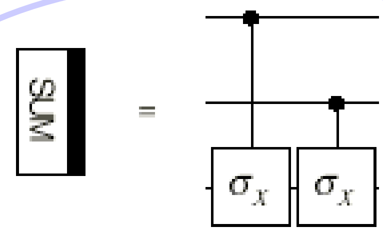


Classical vs. Quantum Circuits

Quantum adder

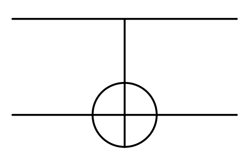






- Here we use Pauli rotations notation.
- Controlled σ_x is the same as controlled NOT

Controlled-controlled σ_x is the same as Toffoli

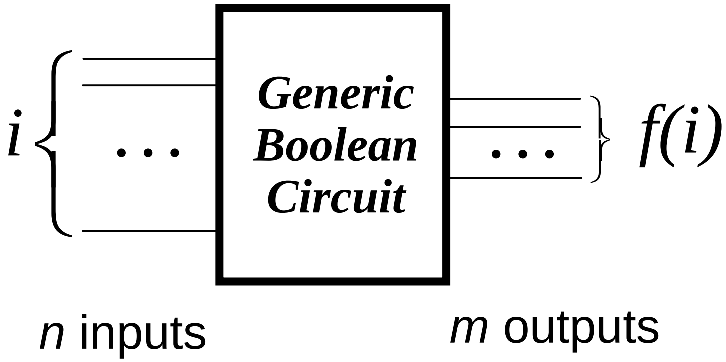


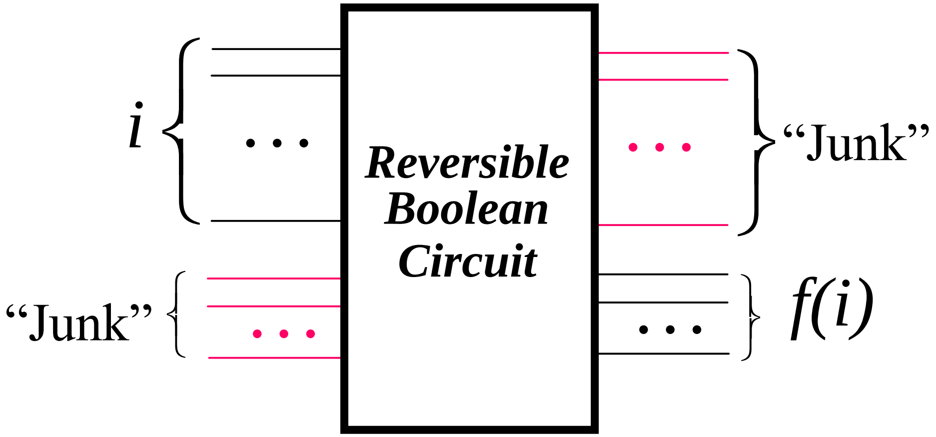
Controlled σ_x is the same as Feynman

Reversible Circuits

Reversible Circuits

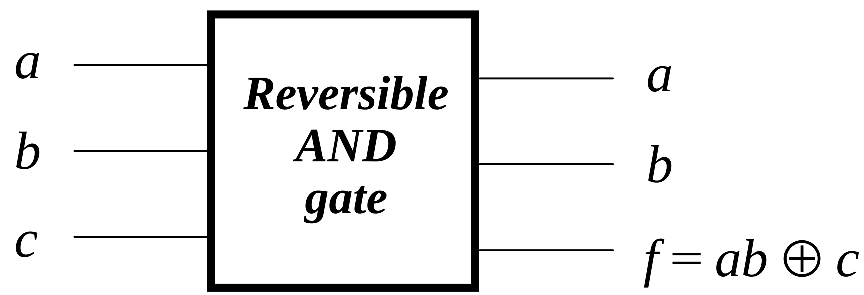
- Reversibility was studied around 1980 motivated by power minimization considerations
- Bennett, Toffoli et al. showed that any classical logic circuit *C* can be made reversible with modest overhead





Reversible Circuits

- How to make a given f reversible
  - Suppose $f: i \rightarrow f(i)$ has n inputs m outputs
  - Introduce n extra outputs and m extra inputs
  - Replace f by $f_{\text{rev}}: i, j \rightarrow i, f(i) \oplus j$ where $\oplus$ is XOR
- Example 1: $f(a,b) = \text{AND}(a,b)$

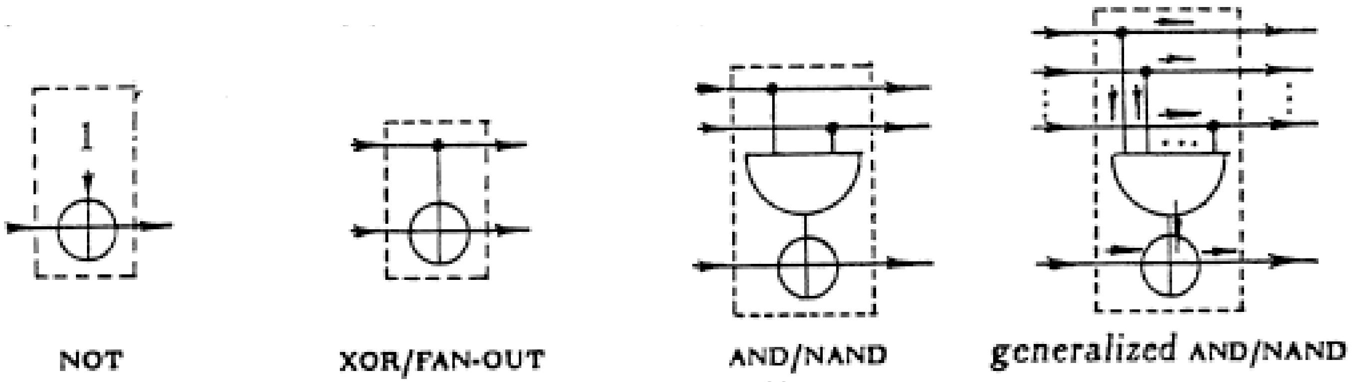


a	b	c	a	b	f
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	0	1
1	1	0	1	1	1
1	1	1	1	1	0

- This is the well-known Toffoli gate, which realizes AND when $c = 0$, and NAND when $c = 1$.

Reversible Circuits

- Reversible gate family [Toffoli 1980]



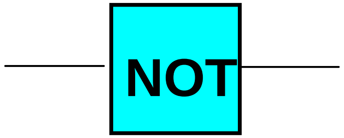
(Toffoli gate)

- Every Boolean function has a reversible implementation using Toffoli gates.
- There is no universal reversible gate with fewer than three inputs

Quantum Gates

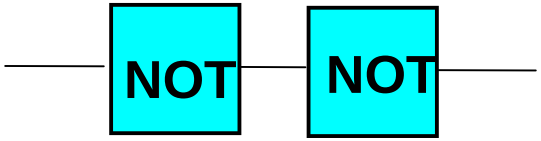
Quantum Gates

- **One-Input gate:** NOT
  - Input state: $c_0|0\rangle + c_1|1\rangle$
  - Output state: $c_1|0\rangle + c_0|1\rangle$



  - Pure states are mapped thus: $|0\rangle \rightarrow |1\rangle$ and $|1\rangle \rightarrow |0\rangle$
  - Gate operator (matrix) is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
  - As expected:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



Quantum Gates

- **One-Input gate:** "Square root of NOT"
  - Some matrix elements are imaginary
  - Gate operator (matrix):
  - We find:

$$\begin{pmatrix} i/\sqrt{1/2} & 1/\sqrt{1/2} \\ 1/\sqrt{1/2} & i/\sqrt{1/2} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$$

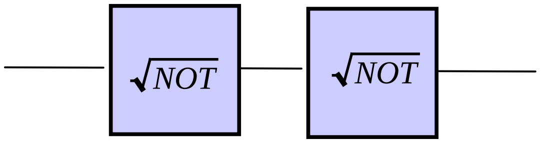
so $|0\rangle \rightarrow |0\rangle$ with probability $|i/\sqrt{2}|^2 = 1/2$

and $|0\rangle \rightarrow |1\rangle$ with probability $|1/\sqrt{2}|^2 = 1/2$

Similarly, this gate randomizes input $|1\rangle$

  - But concatenation of two gates eliminates the randomness!

$$\frac{1}{2}\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$



Other variant of square root of not - we do not use complex numbers - only real numbers

A square-root-of-NOT (SRN) gate

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

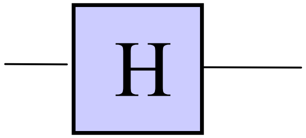
- Applied once to a classical state, this ~randomizes the value of the qubit.
- Applied twice in a row, this is ~equivalent to NOT:

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} * \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Quantum Gates

- **One-Input gate:** Hadamard

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

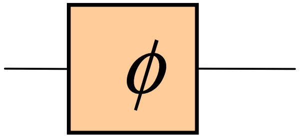


  - Maps $|0\rangle \to 1/\sqrt{2}\,|0\rangle + 1/\sqrt{2}\,|1\rangle$ and $|1\rangle \to 1/\sqrt{2}\,|0\rangle - 1/\sqrt{2}\,|1\rangle$.
  - Ignoring the normalization factor $1/\sqrt{2}$, we can write

    $|x\rangle \to (-1)^x\,|x\rangle - \,|1 - x\rangle$

- **One-Input gate:** Phase shift

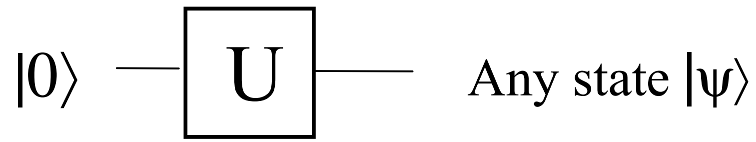
$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}$$



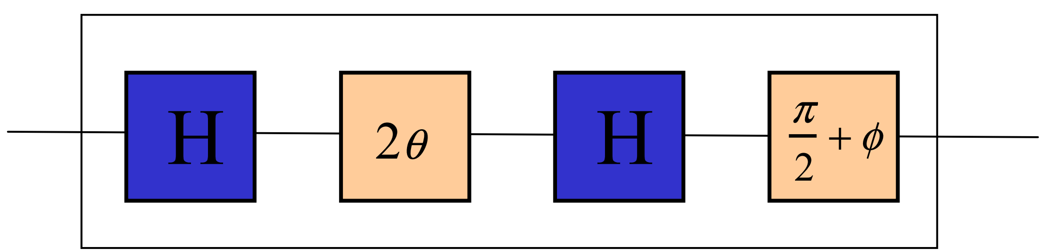
Quantum Gates

Universal One-Input Gate Sets

- Requirement:



- **Hadamard** and **phase-shift** gates form a universal gate set of 1-qubit gates, every 1-qubit gate can be built from them.
- *Example*: The following circuit generates
  $|\psi\rangle = cos\ \theta\ |0\rangle + e^{i\phi} sin\ \theta\ |1\rangle$ up to a global factor



Other Quantum Gates

Rotation $(U\theta)$: $\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$

Hadamard (H): $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

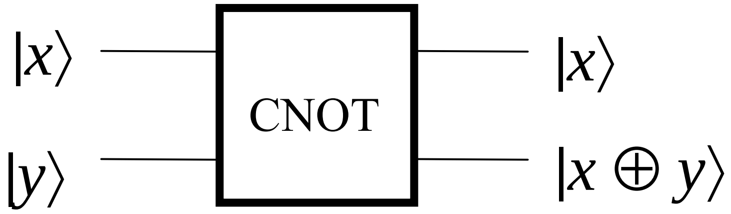
$CNOT$: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ $CPHASE$: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\alpha} \end{bmatrix}$

There are many small "universal" sets of gates.

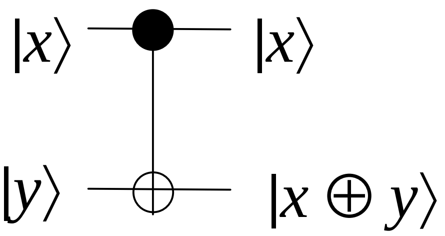
Gates must be unitary: $U^{\dagger}U = UU^{\dagger}$, where $U^{\dagger}$ is the Hermitean adjoint of U.

Quantum Gates

- **Two-Input Gate:** Controlled NOT (CNOT)



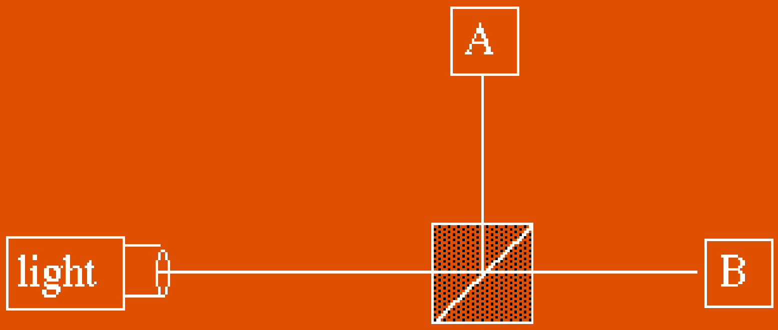
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



  - CNOT maps $|x\rangle|0\rangle \rightarrow |x\rangle||x\rangle$ and $|x\rangle|1\rangle \rightarrow |x\rangle||\text{NOT } x\rangle$

    $|x\rangle|0\rangle \rightarrow |x\rangle||x\rangle$ *looks like cloning*, <u>but it's not</u>. These mappings are **<u>valid only for the pure states</u>** $|0\rangle$ and $|1\rangle$
  - Serves as a "non-demolition" measurement gate

Polarizing Beam-Splitter CNOT gate from [Cerf,Adami, Kwiat]

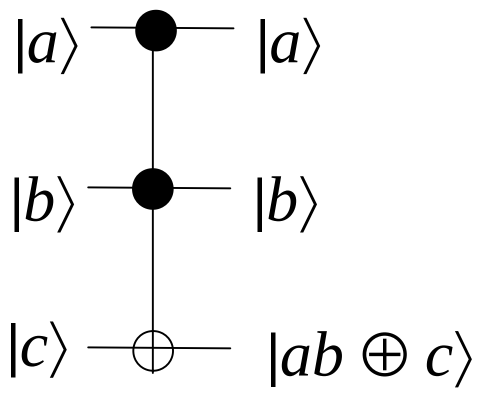


- Two qubits encoded in one photon, one in momentum (direction) and one in polarization.
- Polarization controls change in momentum.
- Cannot be scaled up directly, but demonstrates an implementation of a 2-qubit gate.

Quantum Gates

- **3-Input gate:** Controlled CNOT (C^2NOT or Toffoli gate)

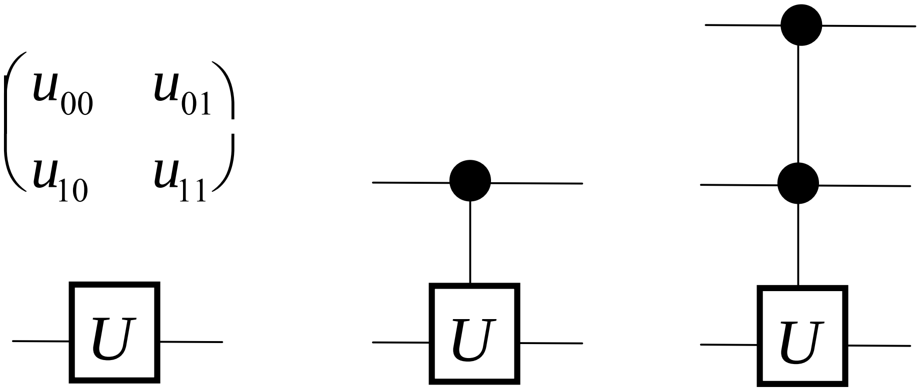
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$



Quantum Gates

- General controlled gates that control some 1-qubit unitary operation U are useful

$$\begin{pmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{pmatrix}$$



etc.

U $\quad$ $C(U)$ $\quad$ $C^2(U)$

Quantum Gates

Universal Gate Sets

- To implement any unitary operation on *n* qubits exactly requires an infinite number of gate types
- The (infinite) set of all 2-input gates is universal
  - Any *n*-qubit unitary operation can be implemented using $\Theta(n^3 4^n)$ gates [Reck et al. 1994]
- CNOT and the (infinite) set of all 1-qubit gates is universal

Quantum Gates

Discrete Universal Gate Sets

- The **error** on implementing U by V is defined as

$$E(U,V) = \max_{|\Psi\rangle} \left\| (U - V)|\Psi\rangle \right\|$$

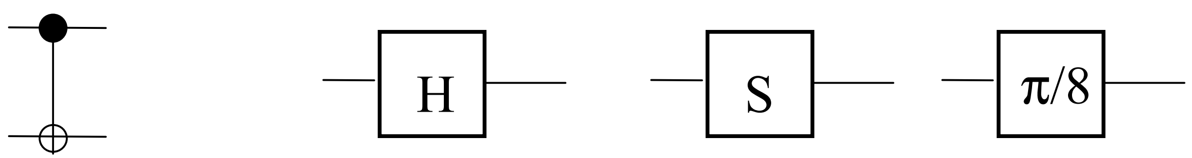
- If U can be implemented by K gates, we can simulate U with a total error less than ε with a gate overhead that is polynomial in log(K/ε)
- A discrete set of gate types ***G*** is universal, if we can approximate any U to within any $\varepsilon > 0$ using a sequence of gates from ***G***

Quantum Gates

Discrete Universal Gate Set

- **Example 1**: Four-member "standard" gate set

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$



CNOT Hadamard Phase π/8 (T) gate

- **Example** 2: {CNOT, Hadamard, Phase, Toffoli}

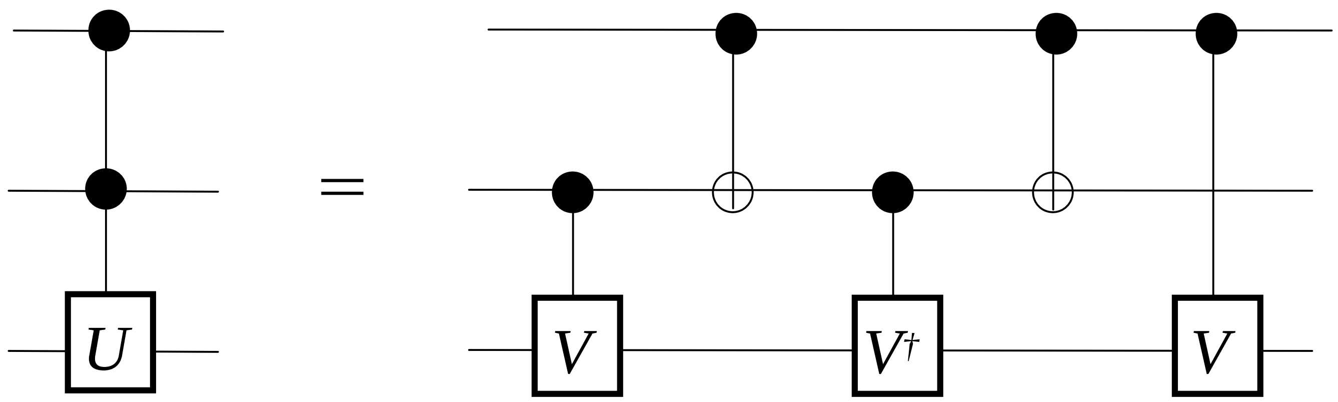
Quantum Circuits

Quantum Circuits

- A quantum (combinational) circuit is a sequence of quantum gates, linked by "wires"
- The circuit has fixed "width" corresponding to the number of qubits being processed
- Logic design (classical and quantum) attempts to find circuit structures for needed operations that are
  - Functionally correct
  - Independent of physical technology
  - Low-cost, e.g., use the minimum number of qubits or gates
- Quantum logic design is not well developed!

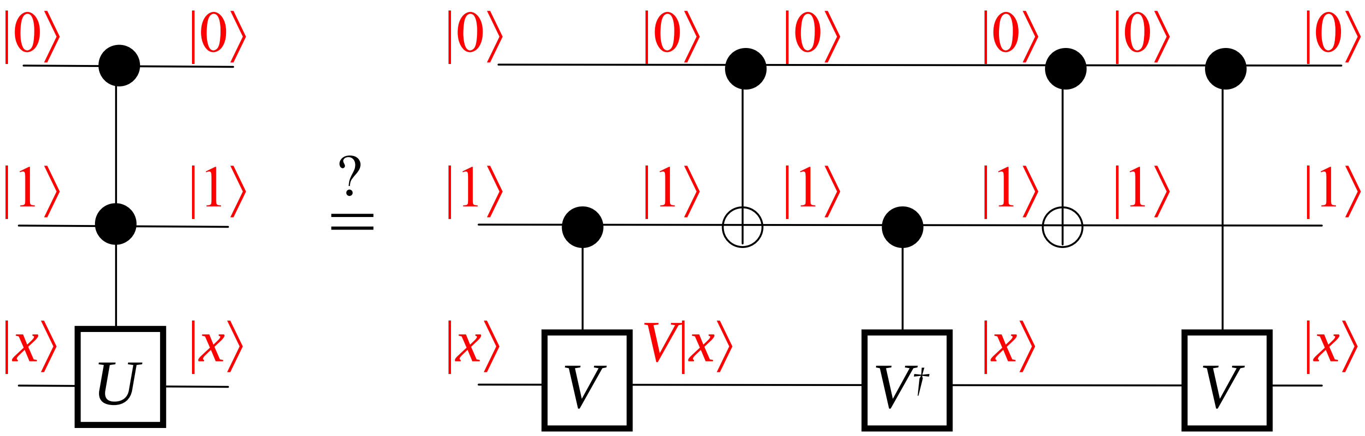
Quantum Circuits

- Ad hoc designs known for many specific functions and gates
- **Example 1** illustrating a theorem by [Barenco et al. 1995]: Any $C^2(U)$ gate can be built from CNOTs, $C(V)$, and $C(V^\dagger)$ gates, where $V^2 = U$



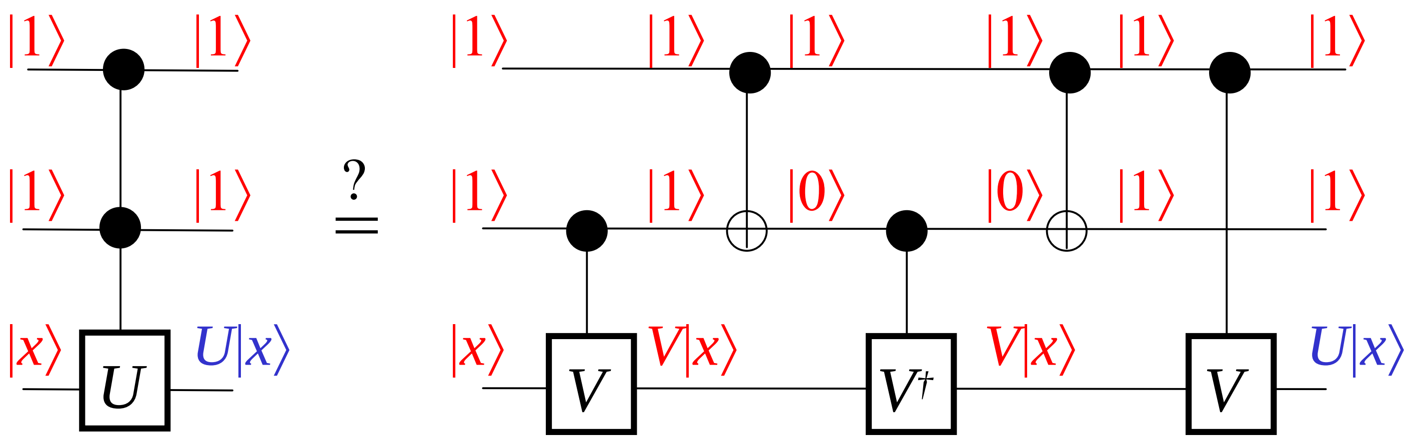
Quantum Circuits

Example 1: Simulation



Quantum Circuits

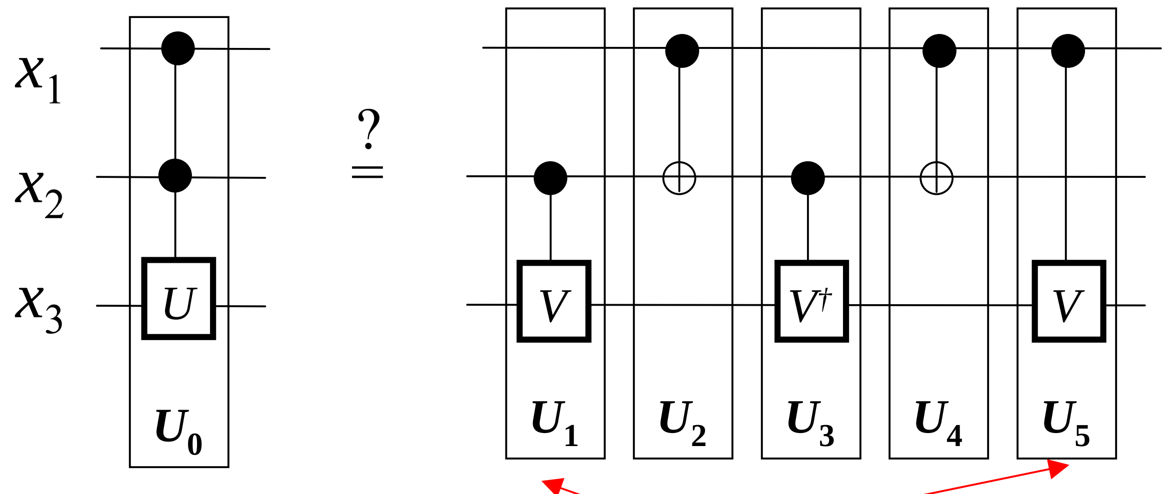
Example 1: Simulation (contd.)



- *Exercise*: Simulate the two remaining cases

Quantum Circuits

Example 1: Algebraic analysis



- Is $U_0(x_1, x_2, x_3) = U_5U_4U_3U_2U_1(x_1, x_2, x_3)$

$= (x_1, x_2, x_1x_2 \oplus U(x_3))$?

We will verify unitary matrix of Toffoli gate

Observe that the order of matrices U_i is inverted.

Quantum Circuits

Example 1 (contd);

We calculate the Unitary Matrix U_1 of the first block from left.

$$U_1 = I_1 \otimes C(V)$$

$$= \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{Unitary matrix of a wire}} \underbrace{\otimes}_{\text{Kronecker}} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & v_{00} & v_{01} \\ 0 & 0 & v_{10} & v_{11} \end{pmatrix}}_{\text{Unitary matrix of a controlled V gate}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & v_{00} & v_{01} & 0 & 0 & 0 & 0 \\ 0 & 0 & v_{10} & v_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & v_{00} & v_{01} \\ 0 & 0 & 0 & 0 & 0 & 0 & v_{10} & v_{11} \end{pmatrix}$$

Unitary matrix of a wire

Kronecker since this is a parallel connection

Unitary matrix of a controlled V gate (from definition)

Quantum Circuits

Example 1 (contd);

We calculate the Unitary Matrix U_2 of the second block from left.

$$U_2 = U_4 = CNOT(x_1, x_2) \otimes I_1$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Unitary matrix of CNOT or Feynman gate with EXOR down

As we can check in the schematics, the Unitary Matrices U_2 and U_4 are the same

Quantum Circuits

Example 1 (contd);

$$U_2 = U_4 = CNOT(x_1, x_2) \otimes I_1$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Quantum Circuits

Example 1 (contd);

- U_5 is the same as U_1 but has $x_1$and x_2 permuted (tricky!)
- It remains to evaluate the product of five 8 x 8 matrices $U_5U_4U_3U_2U_1$ using the fact that $VV^{\dagger} = I$ and $VV = U$

$$\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&v_{00}&v_{01}&0&0\\0&0&0&0&v_{10}&v_{11}&0&0\\0&0&0&0&0&0&v_{00}&v_{01}\\0&0&0&0&0&0&v_{10}&v_{11}\end{pmatrix}\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\end{pmatrix}\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&\mathbf{v}_{00}&\mathbf{v}_{10}&0&0&0&0\\0&0&\mathbf{v}_{01}&\mathbf{v}_{11}&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&\mathbf{v}_{00}&\mathbf{v}_{10}\\0&0&0&0&0&0&\mathbf{v}_{01}&\mathbf{v}_{11}\end{pmatrix}\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\end{pmatrix}\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&v_{00}&v_{01}&0&0&0&0\\0&0&v_{10}&v_{11}&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&v_{00}&v_{01}\\0&0&0&0&0&0&v_{10}&v_{11}\end{pmatrix}$$

$$= \begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&\mathbf{v}_{00}v_{00}+\mathbf{v}_{10}v_{10}&\mathbf{v}_{00}v_{01}+\mathbf{v}_{10}v_{11}\\0&0&0&0&0&0&\mathbf{v}_{01}v_{00}+\mathbf{v}_{11}v_{10}&\mathbf{v}_{01}v_{01}+\mathbf{v}_{11}v_{11}\end{pmatrix} = U_0$$

Quantum Circuits

Example 1 (contd);

- *We calculate matrix U_3*

This is a hermitian matrix, so we transpose and next calculate complex conjugates, we denote complex conjugates by bold symbols

$$
\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \otimes \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mathbf{v_{00}} & \mathbf{v_{10}} \\ 0 & 0 & \mathbf{v_{01}} & \mathbf{v_{11}} \end{vmatrix}
$$

$$
= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbf{v}_{00} & \mathbf{v}_{10} & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbf{v}_{01} & \mathbf{v}_{11} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mathbf{v}_{00} & \mathbf{v}_{10} \\
0 & 0 & 0 & 0 & 0 & 0 & \mathbf{v}_{01} & \mathbf{v}_{11}
\end{pmatrix}
$$

Quantum Circuits

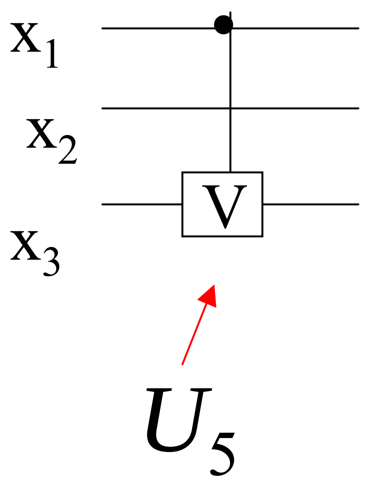
Example 1 (contd);

- U_5 is the same as U_1 but has x_1 and x_2 permuted because in U_1 black dot is in variable x_2 and in U_5 black dot is in variable x_1
- This can be also checked by definition, see next slide.

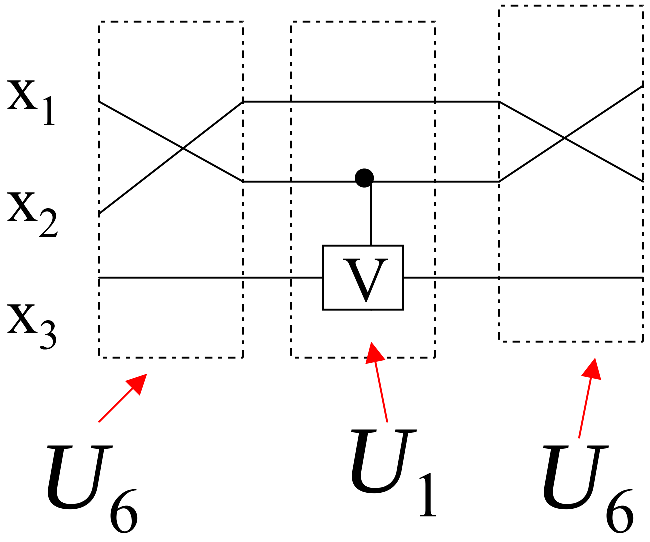
$$U_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v_{00} & v_{01} & 0 & 0 \\ 0 & 0 & 0 & 0 & v_{10} & v_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & v_{00} & v_{01} \\ 0 & 0 & 0 & 0 & 0 & 0 & v_{10} & v_{11} \end{pmatrix}$$

Quantum Circuits

Example 1 (here we explain in detail how to calculate U_5)



$=$



U_6 is calculated as a Kronecker product of U_7 and I_1

U_7 is a unitary matrix of a swap gate

$$U_5 = U_6\, U_1\, U_6$$

Quantum Circuits

Example 1 (contd);

– It remains to evaluate the product of five 8 x 8 matrices $U_5U_4U_3U_2U_1$ using the fact that $VV^{\dagger} = I$ and $VV = U$

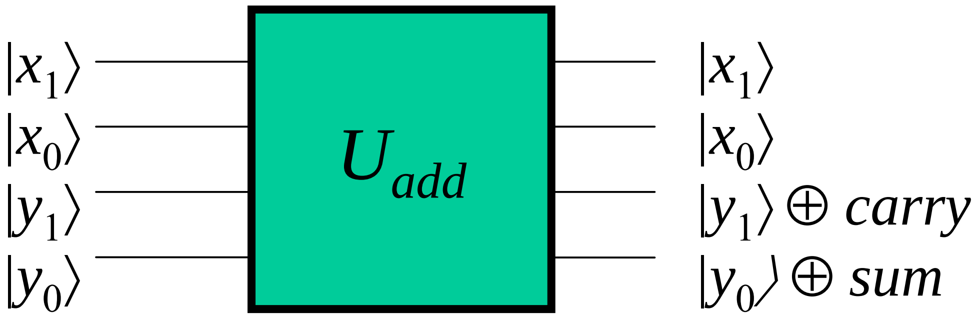
$$\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&v_{00}&v_{01}&0&0\\0&0&0&0&v_{10}&v_{11}&0&0\\0&0&0&0&0&0&v_{00}&v_{01}\\0&0&0&0&0&0&v_{10}&v_{11}\end{pmatrix}\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\end{pmatrix}\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&\mathbf{v}_{00}&\mathbf{v}_{10}&0&0&0&0\\0&0&\mathbf{v}_{01}&\mathbf{v}_{11}&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&\mathbf{v}_{00}&\mathbf{v}_{10}\\0&0&0&0&0&0&\mathbf{v}_{01}&\mathbf{v}_{11}\end{pmatrix}\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\end{pmatrix}\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&v_{00}&v_{01}&0&0&0&0\\0&0&v_{10}&v_{11}&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&v_{00}&v_{01}\\0&0&0&0&0&0&v_{10}&v_{11}\end{pmatrix}$$

U_1

$$= \begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&\mathbf{v}_{00}v_{00}+\mathbf{v}_{10}v_{10}&\mathbf{v}_{00}v_{01}+\mathbf{v}_{10}v_{11}\\0&0&0&0&0&0&\mathbf{v}_{01}v_{00}+\mathbf{v}_{11}v_{10}&\mathbf{v}_{01}v_{01}+\mathbf{v}_{11}v_{11}\end{pmatrix} = U_0$$

Quantum Circuits

- **Implementing a Half Adder**
  - *Problem:* Implement the classical functions *sum* = $x_1 \oplus x_0$ and *carry* = $x_1 x_0$
- Generic design:



Quantum Circuits

- **Half Adder**: Generic design (contd.)

$$U_{ADD} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}$$

Quantum Circuits

- **Half Adder**: Specific (reduced) design