Postulates of Quantum Mechanics

SOURCES
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Readings & homeworks are now posted

Basic quantum theory
  - Background concepts: Systems & states
  - Distinguishable states, state vectors, Hilbert spaces
  - Ket Notation, measurement, wavefunctions
  - Operators, observables, entanglement
  - Unitary transformations & time-evolution
Linear Operators

- $V, W$: Vector spaces.
- A **linear operator** $A$ from $V$ to $W$ is a linear function $A: V \rightarrow W$. An operator **on** $V$ is an operator from $V$ to itself.
- Given bases for $V$ and $W$, we can represent linear operators as matrices.
- An operator $A$ on $V$ is **Hermitian** iff it is self-adjoint ($A = A^\dagger$). Its **diagonal elements** are **real**.
Eigenvalues & Eigenvectors

• $v$ is called an *eigenvector* of linear operator $A$ iff $A$ just multiplies $v$ by a scalar $x$, *i.e.* $Av = xv$
  – “eigen” (German) = “characteristic”.

• $x$, the *eigenvalue* corresponding to eigenvector $v$, is just the scalar that $A$ multiplies $v$ by.

• $x$ is *degenerate* if it is shared by 2 eigenvectors that are not scalar multiples of each other.

• Any Hermitian operator has all *real-valued eigenvectors*, which are *orthogonal* (for distinct eigenvalues).
Unitary Transformations

- A matrix (or linear operator) $U$ is unitary iff its inverse equals its adjoint: $U^{-1} = U^\dagger$

- Some properties of unitary transformations:
  - Invertible, bijective, one-to-one.
  - The set of row vectors is orthonormal.
  - Ditto for the set of column vectors.
  - Preserves vector length: $|U\Psi| = |\Psi|$
    - Therefore also preserves total probability over all states:
      $$|\Psi|^2 = \sum |\Psi(s_i)|^2$$
  - Corresponds to a change of basis, from one orthonormal basis to another.
  - Or, a generalized rotation of $\Psi$ in Hilbert space
The Solvay Congress of 1927

A great breakthrough
Postulates of Quantum Mechanics

Lecture objectives

• Why are postulates important?
  – … they provide the connections between the physical, real, world and the quantum mechanics mathematics used to model these systems

• Lecture Objectives
  – Description of connections
  – Introduce the postulates
  – Learn how to use them
  – …and when to use them
# Physical Systems - Quantum Mechanics Connections

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Postulate 1: State Space
Intuitively speaking, a physical system consists of a region of spacetime & all the entities (e.g. particles & fields) contained within it.
- The universe (over all time) is a physical system
- Transistors, computers, people: also physical systems.

One physical system A is a subsystem of another system B (write $A \subseteq B$) iff A is completely contained within B.

Later, we may try to make these definitions more formal & precise.
Closed vs. Open Systems

• A subsystem is closed to the extent that no particles, information, energy, or entropy enter or leave the system.
  – The universe is (presumably) a closed system.
  – Subsystems of the universe may be almost closed

• Often in physics we consider statements about closed systems.
  – These statements may often be perfectly true only in a perfectly closed system.
  – However, they will often also be approximately true in any nearly closed system (in a well-defined way)
Concrete vs. Abstract Systems

• Usually, when reasoning about or interacting with a system, an entity (e.g. a physicist) has in mind a **description** of the system.

• A description that contains **every** property of the system is an **exact or concrete** description.
  – That system (to the entity) is a **concrete** system.

• Other descriptions are **abstract** descriptions.
  – The system (as considered by that entity) is an **abstract** system, to some degree.

• We **nearly always** deal with **abstract systems**!
  – Based on the descriptions that are available to us.
A *possible state* \( S \) of an abstract system \( A \) (described by a description \( D \)) is any concrete system \( C \) that is consistent with \( D \).

– *I.e.*, it is possible that the system in question could be completely described by the description of \( C \).

The *state space* of \( A \) is the set of all possible states of \( A \).

Most of the class, the concepts we’ve discussed can be applied to *either* classical or quantum physics

– Now, let’s get to the *uniquely quantum* stuff…
Distinguishability of States

- Classical and quantum mechanics differ regarding the *distinguishability of states*.

- In classical mechanics, there is no issue:
  - Any two states $s, t$ are either the same ($s = t$), or different ($s \neq t$), and that’s all there is to it.

- In quantum mechanics (i.e. in reality):
  - There are pairs of states $s \neq t$ that are *mathematically distinct*, but *not 100% physically distinguishable*.
  - Such states cannot be reliably distinguished by *any number of measurements, no matter how precise*.
    - But you *can know* the real state (with high probability), *if you prepared* the system to be in a certain state.
**Postulate 1: State Space**

- Postulate 1 defines “the setting” in which Quantum Mechanics takes place, which is the **Hilbert space** (inner product space which satisfies the condition of completeness).

- **Postulate 1:** Any isolated physical space is associated with a **complex vector space with inner product** called the **State Space** of the system.
  - The system is completely described by a **state vector**, a unit vector, pertaining to the state space.
  - The state space describes all possible states the system can be in.
  - Postulate 1 does NOT tell us either what the state space or state vector is.
An example of a state space

A Qubit: The Simplest State Space

The simplest quantum system is a state space with 2 dimensions -- there are two possible states the system can be in!

$$|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$$ → a qubit!

Recall: state vector is a unit vector, so

$$\langle \psi | \psi \rangle = 1 \Rightarrow |\alpha_0|^2 + |\alpha_1|^2 = 1$$ (normalization condition)

A linear combination of states is called a superposition of states⇒ qualitatively new feature: a qubit can be a mixture of two classical bits!
Schroedinger’s Cat and Explanation of Qubits

Postulate 1: An isolated physical system is described by a unit vector (state vector) in a Hilbert space (state space)
Distinguishability of States, more precisely

- Two state vectors \( s \) and \( t \) are *(perfectly) distinguishable* or *orthogonal* (write \( s \perp t \)) iff \( s^\dagger t = 0 \). (Their inner product is zero.)

- State vectors \( s \) and \( t \) are *perfectly indistinguishable* or *identical* (write \( s = t \)) iff \( s^\dagger t = 1 \). (Their inner product is one.)

- Otherwise, \( s \) and \( t \) are both *non-orthogonal*, and *non-identical*. *Not perfectly distinguishable.*

- We say, “the *amplitude* of state \( s \), given state \( t \), is \( s^\dagger t \)”. *Note:* amplitudes are *complex numbers.*
State Vectors & Hilbert Space

- Let $S$ be any maximal set of distinguishable possible states $s, t, \ldots$ of an abstract system $A$.

- Identify the elements of $S$ with unit-length, mutually-orthogonal (basis) vectors in an abstract complex vector space $H$.
  - The “Hilbert space”

- **Postulate 1:** The possible states $\psi$ of $A$ can be identified with the unit vectors of $H$. 
Postulate 2: Evolution
Postulate 2: Evolution

- Evolution of an isolated system can be expressed as:
  \[ |v(t_2)\rangle = U(t_1, t_2) |v(t_1)\rangle \]

  where \( t_1, t_2 \) are moments in time and \( U(t_1, t_2) \) is a unitary operator.
  - \( U \) may vary with time. Hence, the corresponding segment of time is explicitly specified:
    \[ U(t_1, t_2) \]
  - the process is in a sense Markovian (history doesn’t matter) and reversible, since
    \[ U^\dagger U |v\rangle = |v\rangle \]

Unitary operations preserve inner product
Example of evolution

Example: Hadamard Gate

Hadamard Gate: \[ H|0\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \]
\[ H|1\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \]
\[ \Rightarrow H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \]

\[ |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow H|0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \]

\[ |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow H|1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \]

(in this case the unitary matrix H has trivial time dependence)
Recall the Postulate: (Closed) systems evolve (change state) over time via unitary transformations.

\[ \Psi_{t_2} = U_{t_1 \rightarrow t_2} \Psi_{t_1} \]

- Note that since \( U \) is linear, a small-factor change in amplitude of a particular state at \( t_1 \) leads to a correspondingly small change in the amplitude of the corresponding state at \( t_2 \).

  - Chaos (sensitivity to initial conditions) requires an ensemble of initial states that are different enough to be distinguishable (in the sense we defined)
    - Indistinguishable initial states never beget distinguishable outcomes - “analog” computing is infeasible?
Given any set $S$ of system states (mutually distinguishable, or not),

A quantum state vector can also be translated to a wavefunction $\Psi : S \rightarrow \mathbb{C}$, giving, for each state $s \in S$, the amplitude $\Psi(s)$ of that state.

- When $s$ is another state vector, and the real state is $t$, then $\Psi(s)$ is just $s^\dagger t$.
- $\Psi$ is called a wavefunction because its time evolution obeys an equation (Schrödinger’s equation) which has the form of a wave equation when $S$ ranges over a space of positional states.
We have a system with states given by \((x,t)\) where:
- \(t\) is a **global time** coordinate, and
- \(x\) describes \(N/3\) particles \((p_1, \ldots, p_{N/3})\) with masses \((m_1, \ldots, m_{N/3})\) in a 3-D Euclidean space,
- where each \(p_i\) is located at coordinates \((x_{3i}, x_{3i+1}, x_{3i+2})\), and
- where particles interact with potential energy function \(V(x,t)\),

**the wavefunction** \(\Psi(x,t)\) **obeys** the following (2\(^{nd}\)-order, linear, partial) differential equation:

\[
- \frac{\hbar}{2} \left( \sum_{j=0}^{N-1} \frac{1}{m_{\lfloor j/3 \rfloor}} \frac{\partial^2}{\partial x_j^2} \Psi(x,t) \right) + V(x,t) = \frac{\hbar}{\partial t} \Psi(x,t)
\]
Features of the wave equation

- Particles’ *momentum* state $p$ is encoded implicitly by the particle’s wavelength $\lambda$: $p=h/\lambda$
- The *energy* of any state is given by the frequency $\nu$ of rotation of the wavefunction in the complex plane: $E=h\nu$.
- By simulating this simple equation, one can observe basic quantum phenomena such as:
  - Interference fringes
  - Tunneling of wave packets through potential barriers
Heisenberg and Schroedinger views of Postulate 2

The evolution of a closed system is described by a unitary transformation.

\[ |\psi(t_2)\rangle = U(t_2, t_1) |\psi(t_1)\rangle \]

This is Heisenberg picture

\[ U(t, t_1) = \exp \left[-\frac{i}{\hbar} H(t - t_1) \right] \]

This is Schroedinger picture

\[ i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H |\psi(t)\rangle \]

Planck's constant
(set to unity)

Hamiltonian (must be input from physical considerations)

\[ H^c = H \rightarrow \text{Hamiltonian has a spectral decomposition} \]

\[ H = \sum E |E\rangle \langle E| \rightarrow |E\rangle = e^{-iE t / \hbar} |E\rangle \]

Energy eigenvalues
Stationary states

..in this class we are interested in Heisenberg’s view.....
Postulate 3: Quantum Measurement
A *yes/no measurement* is an *interaction* designed to determine whether a given system is in a certain state $s$. The amplitude of state $s$, given the actual state $t$ of the system determines the *probability* of getting a “yes” from the measurement.

**Important:** For a system prepared in state $t$, *any* measurement that asks “is it in state $s$?” will return “yes” with probability $\Pr[s|t] = |s^\dagger t|^2$

- After the measurement, the state is changed, in a way we will define later.
A Simple Example of distinguishable, non-distinguishable states and measurements

• Suppose abstract system $S$ has a set of only 4 distinguishable possible states, which we’ll call $s_0$, $s_1$, $s_2$, and $s_3$, with corresponding ket vectors $|s_0\rangle$, $|s_1\rangle$, $|s_2\rangle$, and $|s_3\rangle$.

• Another possible state is then the vector $\frac{1}{\sqrt{2}}|s_0\rangle - \frac{i}{\sqrt{2}}|s_3\rangle$

• Which is equal to the column matrix:

$$\begin{bmatrix}
\frac{1}{\sqrt{2}} \\
0 \\
0 \\
-\frac{i}{\sqrt{2}}
\end{bmatrix}$$

• If measured to see if it is in state $s_0$, we have a 50% chance of getting a “yes”.
Hermitian operator $A$ on $V$ is called an observable if there is an orthonormal (all unit-length, and mutually orthogonal) subset of its eigenvectors that forms a basis of $V$.

**Postulate 3:** Every measurable physical property of a system is described by a corresponding operator $A$. Measurement outcomes correspond to eigenvalues.

**Postulate 3a:** The probability of an outcome is given by the squared absolute amplitude of the corresponding eigenvector(s), given the state.
Towards QM Postulate 3 on measurement and general formulas

A measurement is described by an Hermitian operator \( \text{observable} \)

\[
M = \sum m P_m
\]

– \( P_m \) is the projector onto the eigenspace of \( M \) with eigenvalue \( m \)

– After the measurement the state will be \( \frac{P_m |\psi\rangle}{\sqrt{p(m)}} \) with probability \( p(m) = \langle \psi | P_m |\psi\rangle \).

– e.g. measurement of a qubit in the computational basis
  • measuring \( |\psi\rangle = \alpha |0\rangle + \beta |1\rangle \) gives:
    • \( |0\rangle \) with probability \( \langle \psi | 0 \rangle \langle 0 | \psi \rangle = |\langle 0 | \psi \rangle|^2 = |\alpha|^2 \)
    • \( |1\rangle \) with probability \( \langle \psi | 1 \rangle \langle 1 | \psi \rangle = |\langle 1 | \psi \rangle|^2 = |\beta|^2 \)
Postulate 3: Quantum Measurement

The measurement of a closed system is described by a collection of operators $M_m$ which act on the state space such that

1) $p(m) = \langle \psi | M_m^c M_m | \psi \rangle$ describes the probability the measurement outcome $m$ occurred

2) $|\psi'\rangle = \frac{M_m |\psi\rangle}{\sqrt{\langle \psi | M_m^c M_m | \psi \rangle}}$ is the state of the system after measurement outcome $m$ occurred

3) $\sum_m M_m^c M_m = I \iff \sum_m p(m) = 1$ Completeness relation

Notes: Measurement is an external observation of a system and so disturbs its unitary evolution
Now we use this notation for an Example of Qubit Measurement

There are two possible outcomes in the measurement of a qubit: $|0\rangle$ and $|1\rangle$

$$M_0 = |0\rangle\langle 0| \quad M_1 = |1\rangle\langle 1| \quad (M_0 + M_1 = I)$$

So the probability that $|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$ is in the state $|0\rangle$ is

$$p(0) = \langle \psi | M_0^* M_0 | \psi \rangle = (\alpha_0^* \langle 0| + \alpha_1^* \langle 1|)(0 \langle 0|)(\alpha_0 |0\rangle + \alpha_1 |1\rangle)$$

$$= |\alpha_0|^2 \langle 0|0\rangle^2 = |\alpha_0|^2$$

And the state vector changes: $|\psi\rangle \rightarrow \frac{M_0}{|\alpha_0|} |\psi\rangle = \frac{\alpha_0}{|\alpha_0|} |0\rangle$
After a system or subsystem is measured from outside, its state appears to **collapse** to exactly match the measured outcome:
- the amplitudes of all states perfectly distinguishable from states consistent with that outcome **drop to zero**
- states consistent with measured outcome can be considered “renormalized” so their probabilities **sum to 1**

This “collapse” **seems nonunitary** (& nonlocal):
- However, this behavior is now explicable as the expected consensus phenomenon that would be experienced even by entities within a closed, perfectly unitarily-evolving world (Everett, Zurek).
Only orthogonal states can be distinguished in a measurement!

Why? Suppose $|\psi_1\rangle$ and $|\psi_2\rangle$ are not orthogonal, but can be distinguished by two measurement operators, $E_1$ and $E_2$ where $E_1 + E_2 = I$, and $E_i = M_i^* M_i$. We must have
\[
\langle \psi_1 | E_1 | \psi_1 \rangle = 1 \quad \text{and} \quad \langle \psi_2 | E_2 | \psi_2 \rangle = 1
\]
since by assumption the states can be distinguished. However we can also write $|\psi_2\rangle = \alpha |\psi_1\rangle + \beta |\varphi\rangle$ where $\langle \psi | \varphi \rangle = 0$ and $|\alpha|^2 + |\beta|^2 = 1$.

Since $E_1 + E_2 = I$ we have $\langle \psi_1 | E_2 | \psi_1 \rangle = 0$. But then
\[
\langle \psi_2 | E_2 | \psi_2 \rangle = |\beta|^2 \langle \varphi | E_2 | \varphi \rangle \leq |\beta|^2 < 1
\]
(Unless $\alpha = 0$ in which case states are orthogonal)

On the other hand

Thus we have contradiction, states can be distinguished unless they are orthogonal.
Observable: A Hermitian operator on the state space.

Can write: \[ M = \sum_m mP_m \Rightarrow p(m) = \langle \psi | P_m^\dagger P_m | \psi \rangle = \langle \psi | P_m | \psi \rangle \] (each measurement operator is a projector!)

Average value of a measurement:
\[ E(M) = \sum_m mp(m) = \sum_m m \langle \psi | P_m | \psi \rangle = \langle \psi | \sum_m mP_m | \psi \rangle = \langle \psi | M | \psi \rangle \] (expectation value)

Standard deviation of a measurement:
\[ \Delta(M) = \sqrt{\langle (M - \langle M \rangle)^2 \rangle} = \sqrt{\langle M^2 \rangle - \langle M \rangle^2} \] where \( \langle M \rangle = \langle \psi | M | \psi \rangle \)
Uncertainty Principle

\[ \Delta(p) \Delta(q) \geq \frac{\left| \langle \psi | [p, q] \psi \rangle \right|}{2} \]

Why?

\[ \langle \psi | A^2 \psi \rangle \langle \psi | B^2 \psi \rangle \geq \langle \psi | AB \psi \rangle^2 = \frac{1}{4} \left| \langle \psi | [A, B] + \{A, B\} \psi \rangle \right|^2 \]

\[ \frac{1}{4} \left( \langle \psi | [A, B] \psi \rangle^2 + \langle \psi | \{A, B\} \psi \rangle^2 \right) \geq \frac{1}{4} \left| \langle \psi | [A, B] \psi \rangle \right|^2 \]

Set \( A = p - \langle p \rangle, B = q - \langle q \rangle \) and the result follows!

Measurement errors are not arbitrarily reducible!
Positive Operator-Valued Measurements (POVM)

POVM: Any complete set \( \{E_m\} \) of positive operators \( \sum_m E_m = 1 \).

Recall \( |\psi_1\rangle \) and \( |\psi_2\rangle = \alpha |\psi_1\rangle + \beta |\varphi\rangle \). Write

\[
E_1 = |\varphi\rangle \langle \varphi | \\
E_2 = \frac{\langle \beta^* | \psi_1 \rangle - \alpha^* | \varphi \rangle \langle \beta | \psi_1 \rangle - \alpha \langle \varphi | \rangle}{|\alpha|^2 + |\beta|^2} \\
E_3 = I - E_1 - E_2
\]

If you observe \( E_1 \) then you know the state is \( |\psi_2\rangle \).

If you observe \( E_2 \) then you know the state is \( |\psi_1\rangle \). If you observe \( E_3 \) then you don’t know the state.

POVMs:

- Advantage: can never mis-identify a state
- Disadvantage: sometimes you get no information
Phase

Global Phase: \( e^{i\theta} \ket{\psi} \) is physically indistinguishable from \( \ket{\psi} \)

\[
\langle \psi \left| e^{-i\theta} M_m^\tau M_m e^{i\theta} \right| \psi \rangle = \langle \psi \left| M_m^\tau M_m \right| \psi \rangle
\]

Relative Phase: \( \ket{\psi} = \ket{\chi} + \ket{\varphi} \) can be physically distinguished from \( \ket{\psi} = \ket{\chi} + e^{i\theta} \ket{\varphi} \)

\[\rightarrow \text{a basis-dependent concept}\]

Local Phase: If the phase is a function of position and/or time we say that it is local

\[ \theta = \theta(\vec{x}, t) \]

(not relevant (yet) for quantum computing)
Density Operators

• For a given state $|\psi\rangle$, the probabilities of all the basis states $s_i$ are determined by an Hermitian operator or matrix $\rho$ (the density matrix):

$$\rho = [\rho_{i,j}] = |\psi\rangle\langle\psi| = [c_j^* c_i] =
\begin{bmatrix}
  c_1^* c_1 & \cdots & c_n^* c_1 \\
  \vdots & \ddots & \vdots \\
  c_1^* c_n & \cdots & c_n^* c_n
\end{bmatrix}
$$

• The diagonal elements $\rho_{i,i}$ are the probabilities of the basis states.
  – The off-diagonal elements are “coherences”.

• The density matrix describes the state exactly.
Mixed States

• Suppose one only knows of a system that it is in one of a statistical ensemble of state vectors \( \psi_i \) (“pure” states), each with density matrix \( \rho_i \) and probability \( P_i \). This is called a mixed state.

• This ensemble is completely described, for all physical purposes, by the expectation value (weighted average) of density matrices:

\[
\rho = \sum P_i \rho_i
\]

– Note: even if there were uncountably many \( \psi_i \), the state remains fully described by \( <n^2 \) complex numbers, where \( n \) is the number of basis states!
Postulate 4:

Composite Systems
Compound Systems

Let $C = AB$ be a system composed of two separate subsystems $A, B$ each with vector spaces $A, B$ with bases $|a_i\rangle, |b_j\rangle$.

The state space of $C$ is a vector space $C = A \otimes B$ given by the tensor product of spaces $A$ and $B$, with basis states labeled as $|a_i b_j\rangle$. 


Postulate 4: Composite Systems

The state space of a composite system is the tensor product of the state spaces of its components.

System A: \( |\chi\rangle \)
System B: \( |\varphi\rangle \)

System AB: \( |\chi\rangle \otimes |\varphi\rangle \)

Common usage:

- Physical system (call it V)
- Ancilla system (corresponds to measurement outcomes -- call it M)

Unitary Dynamics + Projective Measurements = General Measurement
Composition example

The state space of a composite physical system is the \textit{tensor product of the state spaces of the components}.

- \(n\) qubits represented by a \(2^n\)-dimensional Hilbert space
- composite state is \(|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \ldots \otimes |\psi_n\rangle\)
- e.g. 2 qubits:
  \[|\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle\]
  \[|\psi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle\]
  \[|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle = \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle\]
- entanglement

2 qubits are \textit{entangled} if \(|\psi\rangle \neq |\psi_1\rangle \otimes |\psi_2\rangle\) for any \(|\psi_1\rangle, |\psi_2\rangle\)
- e.g. \(|\psi\rangle = \alpha|00\rangle + \beta|11\rangle\)
Entanglement

• If the state of compound system $C$ can be expressed as a tensor product of states of two independent subsystems $A$ and $B$,
  \[ \psi_c = \psi_a \otimes \psi_b, \]
  then, we say that $A$ and $B$ are not entangled, and they have individual states.
  – E.g. $|00\rangle + |01\rangle + |10\rangle + |11\rangle = (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)$

• Otherwise, $A$ and $B$ are entangled (basically correlated); their states are not independent.
  – E.g. $|00\rangle + |11\rangle$
Size of Compound State Spaces

- Note that a system composed of many separate subsystems has a very large state space.
- Say it is composed of $N$ subsystems, each with $k$ basis states:
  - The compound system has $k^N$ basis states!
  - There are states of the compound system having nonzero amplitude in all these $k^N$ basis states!
  - In such states, all the distinguishable basis states are (simultaneously) possible outcomes (each with some corresponding probability)
  - Illustrates the “many worlds” nature of quantum mechanics.
General Measurements in compound spaces

Let \( U : V \otimes M \rightarrow V \otimes M \) be defined so that
\[
U|\psi\rangle|0\rangle \equiv \sum_m M_m |\psi\rangle|m\rangle
\]
for a fixed state \( |0\rangle \) in \( M \) and a general state \( |\psi\rangle \) in \( V \)
\[
\Rightarrow (U|\varphi\rangle|0\rangle)^{\dagger}U|\psi\rangle|0\rangle = \sum_{m,m'} \langle \varphi | M^*_m M_m |\psi\rangle<m|m'\rangle
\]
\[
= \sum_m \langle \varphi | M^*_m M_m |\psi\rangle = \langle \varphi |\psi\rangle \Rightarrow U \text{ can be defined on entire space } V \otimes M
\]

Now set \( P_m = I_V \otimes |m\rangle\langle m| \)
\[
\Rightarrow p(m) = \langle \psi | 0 |U^*P_m U|\psi\rangle|0\rangle = \sum_{m',m''} \langle \psi | M^*_{m'} |m'\rangle \langle I_V \otimes |m\rangle\langle m| M_{m''} |\psi\rangle|m''\rangle
\]
\[
= \sum_m \langle \psi | M^*_{m} M_m |\psi\rangle \Rightarrow \text{General Measurement!}
\]
Superdense Coding

Idea: exploit entanglement to send more information from point A to point B that would classically be allowed.

Problem: Alice has 2 classical bits of information she wants to send to Bob, but is only allowed to send Bob a single qubit. Can she do it?

Solution: Yes! Put Alice's qubit in an entangled state with Bob's! Alice acts on her qubit and then sends it to Bob -- this allows Bob to uniquely deduce Alice's 2 classical bits!
Key Points to Remember:

- An abstractly-specified system may have many possible states; only some are distinguishable.

- A quantum state/vector/wavefunction $\Psi$ assigns a complex-valued amplitude $\Psi(s_i)$ to each distinguishable state $s_i$ (out of some basis set).

- The probability of state $s_i$ is $|\Psi(s_i)|^2$, the square of $\Psi(s_i)$’s length in the complex plane.

- States evolve over time via unitary (invertible, length-preserving) transformations.

- Statistical mixtures of states are represented by weighted sums of density matrices $\rho=|\Psi\rangle\langle\Psi|$. 
Bibliography & acknowledgements

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