Postulates of Quantum Mechanics

SOURCES

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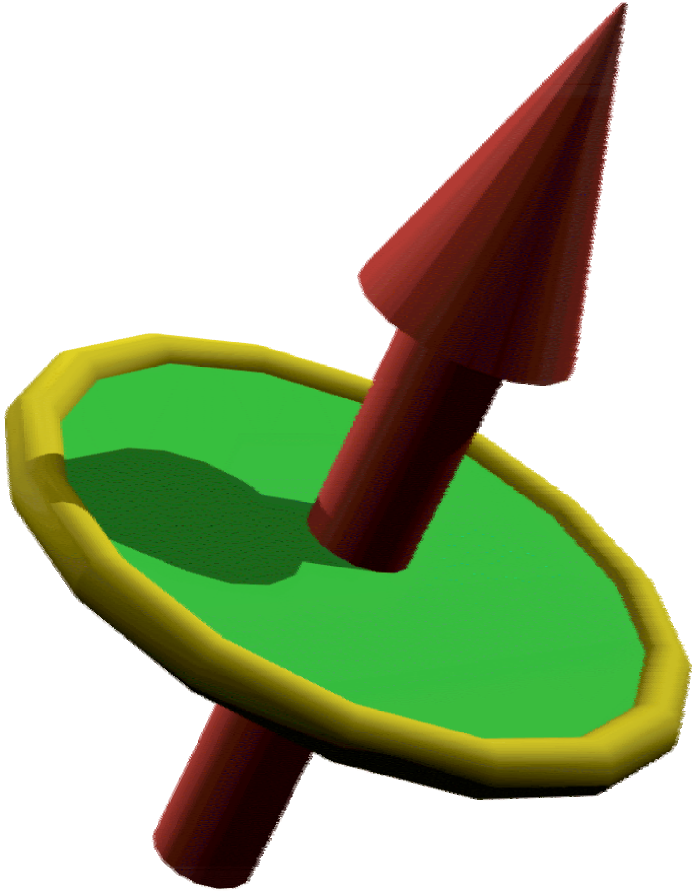


Administrivia/Overview

- Readings & and homeworks are now posted
- Basic quantum theory
  - Background concepts: Systems & states
  - Distinguishable states, state vectors, Hilbert spaces
  - Ket Notation, measurement, wavefunctions
  - Operators, observables, entanglement
  - Unitary transformations & time-evolution

Linear Operators

- V,W: Vector spaces.
- A linear operator A from V to W is a linear function $A{:}V\rightarrow W$. An operator *on* V is an operator from V to itself.
- Given bases for V and W, we can represent linear operators as matrices.
- An operator A on V is *Hermitian* **iff** it is self-adjoint ($A=A^\dagger$). Its diagonal elements are real.

Eigenvalues & Eigenvectors

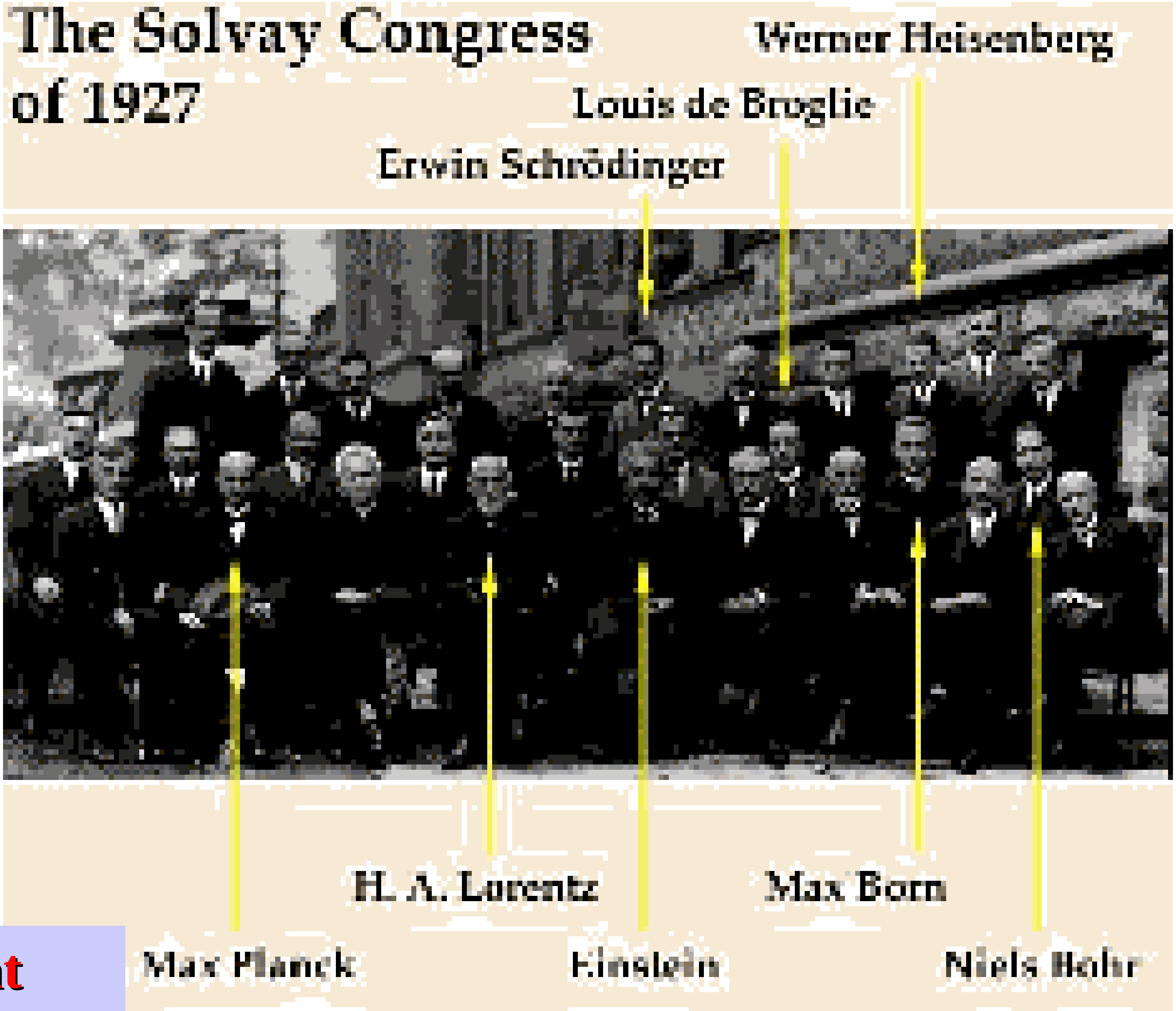
- $\boldsymbol{v}$ is called an *eigenvector* of linear operator $\boldsymbol{A}$ iff $\boldsymbol{A}$ just multiplies $\boldsymbol{v}$ by a scalar x, *i.e.* $\boldsymbol{A}\boldsymbol{v}=x\boldsymbol{v}$
  - "eigen" (German) = "characteristic".
- x, the *eigenvalue* corresponding to eigenvector $\boldsymbol{v}$, is just the scalar that $\boldsymbol{A}$ multiplies $\boldsymbol{v}$ by.
- x is *degenerate* if it is shared by 2 eigenvectors that are not scalar multiples of each other.
- Any Hermitian operator has all real-valued eigenvectors, which are orthogonal (for distinct eigenvalues).

Unitary Transformations

- A matrix (or linear operator) U is *unitary* iff its inverse equals its adjoint: $U^{-1} = U^{\dagger}$
- Some properties of unitary transformations:
  - Invertible, bijective, one-to-one.
  - The set of row vectors is orthonormal.
  - Ditto for the set of column vectors.
  - Preserves vector length: $|U\Psi| = |\Psi|$
    - Therefore also preserves total probability over all states:

$$|\Psi|^2 \equiv \sum_i |\Psi(s_i)|^2$$

  - Corresponds to a **change of basis**, from one orthonormal basis to another.
  - Or, a generalized rotation of Ψ in Hilbert space



A great breakthrough

Postulates of Quantum Mechanics
Lecture objectives

- Why are postulates important?
  - … they provide the **connections** between the physical, real, world <u>**and**</u> the quantum mechanics mathematics *used to model these systems*
- **Lecture Objectives**
  - Description of connections
  - Introduce the postulates
  - Learn <u>how to use</u> them
  - …and <u>*when to use*</u> them

Physical Systems - Quantum Mechanics Connections

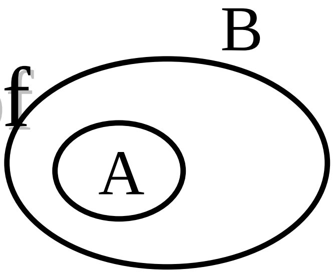
Postulate 1	Isolated physical system	⇄	Hilbert Space
Postulate 2	Evolution of a physical system	⇄	Unitary transformation
Postulate 3	Measurements of a physical system	⇄	Measurement operators
Postulate 4	Composite physical system	⇄	Tensor product of components

Postulate 1:

State Space

Systems and Subsystems

- Intuitively speaking, a *physical system* consists of a <u>region of spacetime</u> **&** all the entities (*e.g.* particles & fields) contained within it.
  - The universe (over all time) is a physical system
  - Transistors, computers, people: also physical systems.
- One physical system A is a *subsystem* of another system B (write $A \subseteq B$) iff A is completely contained within B.



- Later, we may try to make these definitions more formal & precise.

Closed vs. Open Systems

- A subsystem is *closed* to the extent that no particles, information, energy, or entropy enter or leave the system.
  - The universe is (presumably) a closed system.
  - Subsystems of the universe may be *almost* closed
- Often in physics we consider statements about closed systems.
  - These statements may often be *perfectly* true only in a *perfectly* closed system.
  - However, they will often also be *approximately* true in any *nearly* closed system (in a well-defined way)

Concrete vs. Abstract Systems

- Usually, when reasoning about or interacting with a system, an entity (*e.g*. a physicist) has in mind a ***description* of the system**.
- A description that contains ***every*** property of the system is an *exact or concrete* description.
  - That system (to the entity) is a *concrete* system.
- Other descriptions are ***abstract*** descriptions.
  - The system (as considered by that entity) is an *abstract* system, to some degree.
- We ***nearly always*** deal with **abstract systems**!
  - Based on the descriptions that are available to us.

States & State Spaces

- A *possible state S* of an abstract system ***A*** (described by a description ***D***) is any concrete system ***C*** that is consistent with ***D***.
  - *I.e.*, it is possible that the system in question could be completely described by the description of ***C***.
- The ***state space*** of ***A*** is the set of all possible states of ***A***.
- Most of the class, the concepts we've discussed can be applied to ***either*** classical or quantum physics
  - Now, let's get to the **uniquely quantum** stuff…

Distinguishability of States

- Classical <u>and</u> quantum mechanics <u>differ</u> regarding the ***distinguishability* of states**.
- In **classical mechanics**, there is no issue:
  - Any two states ***s*, *t*** are either the same (***s* = *t***), or different (***s* ≠ *t***), and that's all there is to it.
- In **quantum mechanics** (*i.e.* ***in reality***):
  - There are pairs of states ***s* ≠ *t*** that are *mathematically distinct*, but not 100% *physically distinguishable*.
  - Such states <u>cannot be reliably distinguished</u> by *any number of measurements, no matter how precise.*
    - But you *can* know the real state (with high probability), *if* you prepared the system to be in a certain state.

Postulate 1: State Space

- Postulate 1 **defines "the setting" in which Quantum Mechanics takes place, which is the Hilbert space (inner product space which satisfies the condition of completeness)**

- **Postulate1: Any isolated physical space is associated with a complex vector space with inner product called the *State Space* of the system.**
  - **The system is completely described by a *state vector*, a unit vector, pertaining to the state space.**
  - **The state space describes all possible states the system can be in.**
  - **Postulate 1 does NOT tell us either what the state space or state vector is.**

An example of a state space

A Qubit: The Simplest State Space

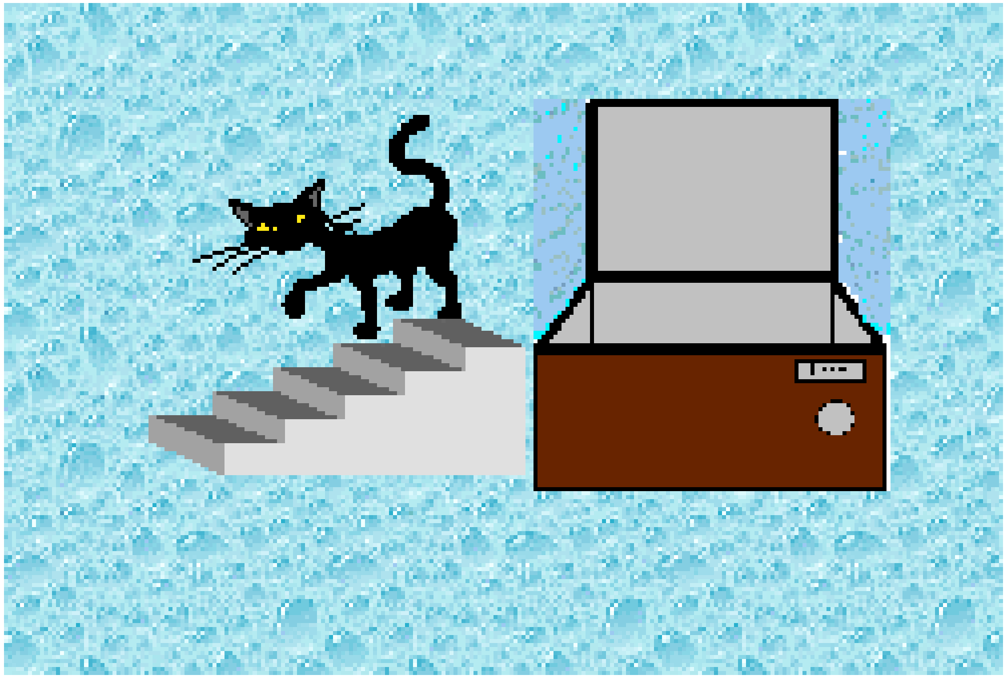
The simplest quantum system is a state space with 2 dimensions -- there are two possible states the system can be in! $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$ ➡ a qubit!

Recall: state vector is a unit vector, so

$$\langle\psi|\psi\rangle = 1 \Rightarrow |\alpha_0|^2 + |\alpha_1|^2 = 1$$ (normalization condition)

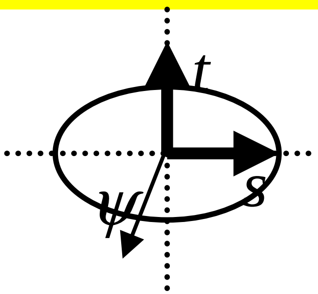
A linear combination of states is called a *superposition* of states ➡ qualitatively new feature: a qubit can be a mixture of two classical bits!

Schroedinger's Cat and Explanation of Qubits



<u>Postulate 1:</u> An isolated physical system is described by a unit vector (*state vector*) in a Hilbert space (*state space*)

Distinguishability of States, more precisely

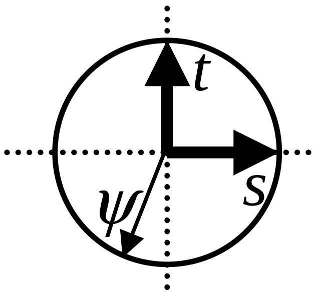


- Two state vectors $\boldsymbol{s}$ and $\boldsymbol{t}$ are *(perfectly) distinguishable* or *orthogonal* (write $\boldsymbol{s}\perp\boldsymbol{t}$) iff $s^\dagger t = 0$. (Their inner product is zero.)
- State vectors $\boldsymbol{s}$ and $\boldsymbol{t}$ are *perfectly indistinguishable* or *identical* (write $\boldsymbol{s}=\boldsymbol{t}$) iff $s^\dagger t = 1$. (Their inner product is one.)
- Otherwise, $\boldsymbol{s}$ and $\boldsymbol{t}$ are both *non-orthogonal*, and *non-identical*. ***Not perfectly distinguishable***.
- We say, "the *amplitude* of state $\boldsymbol{s}$, given state $\boldsymbol{t}$, is $\boldsymbol{s}^\dagger\boldsymbol{t}$". *Note:* amplitudes are complex numbers.

State Vectors & Hilbert Space

- Let ***S*** be any maximal set of distinguishable possible states ***s*, *t*, …** of an abstract system ***A***.
- Identify the elements of ***S*** with unit-length, mutually-orthogonal (basis) vectors in an abstract complex vector space ***H***.
  - The "Hilbert space"

- **Postulate 1:** The possible states ψ of A can be *identified* with the unit vectors of H.



Postulate 2:

Evolution

Postulate 2: Evolution

- Evolution of an isolated system can be expressed as:

$$|v(t_2)\rangle = U(t_1, t_2)|v(t_1)\rangle$$

where **t_1, t_2** are moments in time and **$U(t_1, t_2)$** is a unitary operator.

- **U may vary with time. Hence, the corresponding segment of time is explicitly specified:**

$$\mathbf{U(t_1, t_2)}$$

- **the process is in a sense Markovian (*history doesn't matter*) and reversible, since**

$$\boxed{U^\dagger U|v\rangle = |v\rangle}$$

Unitary operations preserve inner product

Example of evolution

Example: Hadamard Gate

Hadamard Gate: $$\left.\begin{aligned} H|0\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \\ H|1\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \end{aligned}\right\} \Rightarrow H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow H|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow H|1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

(in this case the unitary matrix H has trivial time dependence)

Time Evolution

- **Recall the Postulate:** (Closed) systems evolve (change state) over time via unitary transformations.

$$\Psi_{t2} = U_{t1\rightarrow t2}\Psi_{t1}$$

- Note that since *U* is linear, a small-factor change in amplitude of a particular state at t1 leads to a correspondingly small change in the amplitude of the corresponding state at t2.
  – Chaos (sensitivity to initial conditions) requires an ensemble of initial states that are different enough to be distinguishable (in the sense we defined)
    - Indistinguishable initial states never beget distinguishable outcomes - “analog” computing is infeasible?

Wavefunctions

- Given any set ***S*** of system states (mutually distinguishable, or not),
- A quantum state vector can also be translated to a ***wavefunction*** $\Psi : S \rightarrow \mathbf{C}$, giving, for each state $s \in S$, **the amplitude $\Psi(s)$** of that state.
  - When $\boldsymbol{s}$ is another state vector, and the real state is $\boldsymbol{t}$, then $\Psi(\boldsymbol{s})$ is just $\boldsymbol{s}^{\dagger}\boldsymbol{t}$.
  - Ψ is called a *wavefunction* because its time evolution obeys an equation (Schrödinger's equation) which has the form of a ***wave equation*** when ***S*** ranges over a space of *positional* states.

Schrödinger's Wave Equation

We have a system with states given by $(\boldsymbol{x},t)$ where:

- $\boldsymbol{t}$ is a global time coordinate, and
- $\boldsymbol{x}$ describes $N/3$ particles $(p_1,\ldots,p_{N/3})$ with masses $(m_1,\ldots,m_{N/3})$ in a 3-D Euclidean space,
- where each p_i is located at coordinates $(x_{3i}, x_{3i+1}, x_{3i+2})$, and
- where particles interact with potential energy function $V(\boldsymbol{x},t)$,

• the wavefunction $\Psi(\boldsymbol{x},t)$ obeys the following (2nd-order, linear, partial) differential equation:

$$-\frac{\hbar}{2}\left(\sum_{j=0}^{N-1}\frac{1}{m_{\lfloor j/3\rfloor}}\frac{\partial^2}{\partial x_j^2}\Psi(\boldsymbol{x},t)\right)+V(\boldsymbol{x},t)=i\hbar\frac{\partial}{\partial t}\Psi(\boldsymbol{x},t)$$

Features of the wave equation

- Particles' *momentum* state ***p*** is encoded implicitly by the particle's wavelength λ: $p=h/\lambda$
- The *energy* of any state is given by the frequency ν of rotation of the wavefunction in the complex plane: $\boldsymbol{E=h\nu}$.
- By simulating this simple equation, one can observe basic quantum phenomena such as:
  - Interference fringes
  - Tunneling of wave packets through potential barriers

Heisenberg and Schroedinger views of Postulate 2

The evolution of a closed system is described by a unitary transformation.

$$|\psi(t_2)\rangle = \mathbf{U}(t_2,t_1)|\psi(t_1)\rangle$$

This is Heisenberg picture

$$\longleftrightarrow \quad \mathbf{U}(t,t_1) = \exp\left[-\frac{i}{\hbar}\mathbf{H}(t-t_1)\right]$$

$$\longleftrightarrow \quad i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \mathbf{H}|\psi(t)\rangle$$

This is Schroedinger picture

Planck's constant (set to unity)

Hamiltonian (must be input from physical considerations)

$\mathbf{H}^{\tau} = \mathbf{H}$ ⟹ Hamiltonian has a spectral decomposition

$$\Longrightarrow \quad \mathbf{H} = \sum_E E|E\rangle\langle E| \quad \Longrightarrow \quad |E\rangle = e^{-iEt/\hbar}|E\rangle$$

Energy eignvalues

Stationary states

..in this class we are interested in Heisenberg's view…..

Postulate 3:

Quantum Measurement

Probability and Measurement

- A *yes/no measurement* is an **interaction** designed to determine whether a given system is in a certain state ***s***.
- The amplitude of state ***s***, given the actual state ***t*** of the system determines the ***probability*** of getting a "yes" from the measurement.
- **Important:** For a system prepared in state ***t***, *any* measurement that asks "is it in state **s**?" will return "yes" with probability $\Pr[\boldsymbol{s}|\boldsymbol{t}] = |\boldsymbol{s}^\dagger \boldsymbol{t}|^2$
  - After the measurement, the state is changed, in a way we will define later.

A Simple Example of distinguishable, non-distinguishable states and measurements

- Suppose abstract system *S* has a set of only 4 distinguishable possible states, which we'll call s_0, s_1, s_2, and s_3, with corresponding ket vectors $|s_0\rangle$, $|s_1\rangle$, $|s_2\rangle$, and $|s_3\rangle$.
- Another possible state is then the vector

$$\frac{1}{\sqrt{2}}|s_0\rangle - \frac{i}{\sqrt{2}}|s_3\rangle$$

- Which is equal to the column matrix:

$$\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ -i/\sqrt{2} \end{bmatrix}$$

- If measured to see if it is in state s_0, we have a 50% chance of getting a "yes".

Observables

- Hermitian operator A on V is called an ***observable*** if there is an orthonormal (all unit-length, and mutually orthogonal) subset of its eigenvectors that forms a ***basis of V***.
- **Postulate 3:** Every measurable physical property of a system is described by a corresponding operator A. Measurement outcomes correspond to eigenvalues.
- **Postulate 3a:** The probability of an outcome is given by the squared absolute amplitude of the corresponding eigenvector(s), given the state.

Towards QM Postulate 3 on measurement and general formulas

A measurement is described by an **Hermitian** operator (*observable*)

$$M = \sum_m m\, P_m$$

- P_m is the projector onto the eigenspace of M with eigenvalue m
- After the measurement the state will be $\frac{P_m|\psi\rangle}{\sqrt{p(m)}}$ **with probability $p(m) = \langle\psi|P_m|\psi\rangle$.**
- e.g. measurement of a qubit in the computational basis
  - measuring $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ gives:
    - $|0\rangle$ with probability $\langle\psi|0\rangle\langle 0|\psi\rangle = |\langle 0|\psi\rangle|^2 = |\alpha|^2$
    - $|1\rangle$ with probability $\langle\psi|1\rangle\langle 1|\psi\rangle = |\langle 1|\psi\rangle|^2 = |\beta|^2$

Postulate 3: Quantum Measurement

The measurement of a closed system is described by a collection of operators $\mathbf{M}_m$ which act on the state space such that

1) $p(m) = \langle \psi | \mathbf{M}_m^{\tau} \mathbf{M}_m | \psi \rangle$ describes the probability the measurement outcome m occurred

2) $|\psi'\rangle = \dfrac{\mathbf{M}_m |\psi\rangle}{\sqrt{\langle \psi | \mathbf{M}_m^{\tau} \mathbf{M}_m | \psi \rangle}}$ is the state of the system after measurement outcome m occurred

3) $\sum_m \mathbf{M}_m^{\tau} \mathbf{M}_m = \mathbf{I} \Leftrightarrow \sum_m p(m) = 1$ Completeness relation

Notes: *Measurement* is an external observation of a system and so disturbs its unitary evolution

Now we use this notation for an Example of Qubit Measurement

There are two possible outcomes in the measurement of a qubit: $|0\rangle$ and $|1\rangle$

$$\Rightarrow \mathbf{M}_0 = |0\rangle\langle 0| \qquad \mathbf{M}_1 = |1\rangle\langle 1| \qquad (\mathbf{M}_0 + \mathbf{M}_1 = \mathbf{I})$$

So the probability that $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$ is in the state $|0\rangle$ is

$$p(0) = \langle\psi|\mathbf{M}_0^{\tau}\mathbf{M}_0|\psi\rangle = \left(\alpha_0^*\langle 0| + \alpha_1^*\langle 1|\right)\left(|0\rangle\langle 0|\right)^{\tau}\left(|0\rangle\langle 0|\right)\left(\alpha_0|0\rangle + \alpha_1|1\rangle\right)$$

$$= |\alpha_0|^2 |\langle 0|0\rangle|^2 = |\alpha_0|^2$$

And the state vector changes: $|\psi\rangle \rightarrow \dfrac{\mathbf{M}_0}{|\alpha_0|}|\psi\rangle = \dfrac{\alpha_0}{|\alpha_0|}|0\rangle$

What happens to a system after a Measurement?

- After a system or subsystem is measured from outside, its state appears to *collapse* <u>to exactly match the measured outcome</u>
  - the amplitudes of all states perfectly distinguishable from states consistent with that outcome drop to zero
  - states consistent with measured outcome can be considered “renormalized” so their probabilities sum to 1
- This “collapse” *seems* nonunitary (& nonlocal)
  - However, this behavior is now explicable as the expected consensus phenomenon that would be experienced even by entities within a closed, perfectly unitarily-evolving world (Everett, Zurek).

Distinguishability

Only orthogonal states can be distinguished in a measurement!

Why? Suppose $|\psi_1\rangle$ and $|\psi_2\rangle$ are not orthogonal, but can be distinguished by two measurement operators, $\mathbf{E}_1$ and $\mathbf{E}_2$ where $\mathbf{E}_1 + \mathbf{E}_2 = \mathbf{I}$, and $\mathbf{E}_i = \mathbf{M}_i^\dagger \mathbf{M}_i$ We must have

$$\langle \psi_1 | \mathbf{E}_1 | \psi_1 \rangle = 1 \quad \text{and} \quad \langle \psi_2 | \mathbf{E}_2 | \psi_2 \rangle = 1$$

since by assumption the states can be distinguished. However we can also write $|\psi_2\rangle = \alpha|\psi_1\rangle + \beta|\varphi\rangle$ where $\langle \psi | \varphi \rangle = 0$ and $|\alpha|^2 + |\beta|^2 = 1$

Since $\mathbf{E}_1 + \mathbf{E}_2 = \mathbf{I}$ we have $\langle \psi_1 | \mathbf{E}_2 | \psi_1 \rangle = 0$. But then

$$\langle \psi_2 | \mathbf{E}_2 | \psi_2 \rangle = |\beta|^2 \langle \varphi | \mathbf{E}_2 | \varphi \rangle \le |\beta|^2 < 1 \quad \otimes$$

(Unless $\alpha = 0$ in which case states are orthogonal)

On the other hand

Thus we have contradiction, states can be distinguished unless they are orthogonal

Projective Measurements

Observable: A Hermitian operator on the state space.

Can write: $\mathbf{M} = \sum_m m\mathbf{P}_m \Rightarrow p(m) = \langle \psi | \mathbf{P}_m^\dagger \mathbf{P}_m | \psi \rangle = \langle \psi | \mathbf{P}_m | \psi \rangle$

(each measurement operator is a projector!)

Average value of a measurement:

$$\mathrm{E}(\mathbf{M}) = \sum_m m p(m) = \sum_m m \langle \psi | \mathbf{P}_m | \psi \rangle = \langle \psi | \sum_m m\mathbf{P}_m | \psi \rangle = \langle \psi | \mathbf{M} | \psi \rangle$$

(expectation value)

Standard deviation of a measurement:

$$\Delta(\mathbf{M}) = \sqrt{\left\langle \left(\mathbf{M} - \langle \mathbf{M} \rangle\right)^2 \right\rangle} = \sqrt{\langle \mathbf{M}^2 \rangle - \langle \mathbf{M} \rangle^2} \quad \text{where } \langle \mathbf{M} \rangle = \langle \psi | \mathbf{M} | \psi \rangle$$

Uncertainty Principle

$$\Delta(\mathbf{p})\Delta(\mathbf{q}) \geq \frac{\left|\langle\psi|[\mathbf{p},\mathbf{q}]\psi\rangle\right|}{2}$$

Why? $\langle\psi|\mathbf{A}^2|\psi\rangle\langle\psi|\mathbf{B}^2|\psi\rangle \geq \left|\langle\psi|\mathbf{AB}|\psi\rangle\right|^2 = \frac{1}{4}\left|\langle\psi|[\mathbf{A},\mathbf{B}]+\{\mathbf{A},\mathbf{B}\}\psi\rangle\right|^2$

$$\frac{1}{4}\left(\left|\langle\psi|[\mathbf{A},\mathbf{B}]\psi\rangle\right|^2 + \left|\langle\psi|\{\mathbf{A},\mathbf{B}\}\psi\rangle\right|^2\right) \geq \frac{1}{4}\left|\langle\psi|[\mathbf{A},\mathbf{B}]\psi\rangle\right|^2$$

Set $\mathbf{A} = \mathbf{p} - \langle\mathbf{p}\rangle, \mathbf{B} = \mathbf{q} - \langle\mathbf{q}\rangle$ and the result follows!

Measurement errors are not arbitrarily reducible!

Positive Operator-Valued Measurements (POVM)

POVM: Any complete set $\{\mathbf{E}_m\}$ of positive operators $\left(\sum_m \mathbf{E}_m = 1\right)$

Recall $|\psi_1\rangle$ and $|\psi_2\rangle = \alpha|\psi_1\rangle + \beta|\varphi\rangle$. Write

$$\left.\begin{aligned} \mathbf{E}_1 &= |\varphi\rangle\langle\varphi| \\ \mathbf{E}_2 &= \frac{\left(\beta^*|\psi_1\rangle - \alpha^*|\varphi\rangle\right)\left(\beta\langle\psi_1| - \alpha\langle\varphi|\right)}{|\alpha|^2 + |\beta|^2} \\ \mathbf{E}_3 &= I - \mathbf{E}_1 - \mathbf{E}_2 \end{aligned}\right\}$$

If you observe $\mathbf{E}_1$ then you know the state is $|\psi_2\rangle$. If you observe $\mathbf{E}_2$ then you know the state is $|\psi_1\rangle$. If you observe $\mathbf{E}_3$ then you don't know the state

POVMs:

Advantage: can never mis-identify a state

Disadvantage: sometimes you get no information

Phase

Global Phase: $e^{i\theta}|\psi\rangle$ is physically indistinguishable from $|\psi\rangle$

$$\langle\psi|e^{-i\theta}\mathbf{M}_m^{\tau}\mathbf{M}_m e^{i\theta}|\psi\rangle = \langle\psi|\mathbf{M}_m^{\tau}\mathbf{M}_m|\psi\rangle$$

Relative Phase: $|\psi\rangle = |\chi\rangle + |\varphi\rangle$ can be physically distinguished from $|\psi\rangle = |\chi\rangle + e^{i\theta}|\varphi\rangle$

➡ a basis-dependent concept

Local Phase: If the phase is a function of position and/or time we say that it is local

$$\theta = \theta(\vec{x}, t)$$

(not relevant (yet) for quantum computing)

Density Operators

- For a given state $|\psi\rangle$, the **probabilities of all the basis states *s_i*** are determined by an Hermitian operator or matrix ρ (the *density* matrix):

$$\rho = [\rho_{i,j}] = |\psi\rangle\langle\psi| = [c_j^* c_i] = \begin{bmatrix} c_1^* c_1 & \cdots & c_n^* c_1 \\ \vdots & \ddots & \vdots \\ c_1^* c_n & \cdots & c_n^* c_n \end{bmatrix}$$

- The diagonal elements $\rho_{i,i}$ are the probabilities of the basis states.
  - The off-diagonal elements are **"coherences".**
- The density matrix describes the state exactly.

Mixed States

- Suppose one only knows of a system that it is in one of a ***statistical ensemble*** of state vectors $\boldsymbol{v}_i$ ("pure" states), each with density matrix ρ_i and probability P_i. This is called a ***mixed state***.
- This is ensemble is *completely described*, *for all physical purposes*, by the expectation value (weighted average) of density matrices:

$$\rho = \sum P_i \rho_i$$

  – Note: even if there were *uncountably* many $\boldsymbol{v}_i$, the state remains fully described by $< n^2$ complex numbers, where n is the number of basis states!

Postulate 4:

Composite Systems

Compound Systems

- Let **C=AB** be a system composed of two separate subsystems A,B each with vector spaces *A*, *B* with bases $|a_i\rangle$, $|b_j\rangle$.
- The state space of C is a vector space ***C=A⊗B*** given by the ***tensor product*** of spaces *A* and *B*, with basis states labeled as $|a_i b_j\rangle$.

Postulate 4: Composite Systems

The state space of a composite system is the tensor product of the state spaces of its components.

System A: $|\chi\rangle$
System B: $|\varphi\rangle$

System AB: $|\chi\rangle \otimes |\varphi\rangle$

Common usage:

$$|\chi\rangle \otimes |\varphi\rangle$$

Physical system (call it V)

Ancilla system (corresponds to measurement outcomes -- call it M)

Unitary Dynamics + Projective Measurements
= General Measurement

Composition example

The state space of a composite physical system is the ***tensor product of the state spaces of the components***

- n qubits represented by a 2^n-dimensional Hilbert space
- composite state is $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \ldots \otimes |\psi_n\rangle$
- e.g. 2 qubits:

  $|\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$

  $|\psi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$

  $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle = \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle$
- entanglement

  2 qubits are ***entangled*** if $|\psi\rangle \neq |\psi_1\rangle \otimes |\psi_2\rangle$ for any $|\psi_1\rangle, |\psi_2\rangle$

  e.g. $|\psi\rangle = \alpha|00\rangle + \beta|11\rangle$

Entanglement

- If the state of compound system C ***can*** be expressed as a tensor product of states of two independent subsystems A and B,

  $\psi_c = \psi_a \otimes \psi_b$,

- then, we say that A and B are ***not entangled***, and they have individual states.
  - E.g. $|00\rangle+|01\rangle+|10\rangle+|11\rangle=(|0\rangle+|1\rangle)\otimes(|0\rangle+|1\rangle)$
- **Otherwise, A and B are *entangled*** (basically correlated); their states are not independent.
  - E.g. $|00\rangle+|11\rangle$

Size of Compound State Spaces

- Note that a system composed of many separate subsystems has a ***very large* state space**.
- Say it is composed of N subsystems, each with k basis states:
  - The compound system has k^N basis states!
  - There are states of the compound system having nonzero amplitude in *all* these k^N basis states!
  - In such states, all the distinguishable basis states are (simultaneously) possible outcomes (each with some corresponding probability)
  - Illustrates the "many worlds" nature of quantum mechanics.

General Measurements in compound spaces

Let $\mathbf{U}: V \otimes M \to V \otimes M$ be defined so that

$$\mathbf{U}|\psi\rangle|0\rangle \equiv \sum_m \mathbf{M}_m|\psi\rangle|m\rangle$$

for a fixed state $|0\rangle$ in M and a general state $|\psi\rangle$ in V

$$\Rightarrow \left(\mathbf{U}|\varphi\rangle|0\rangle\right)^{\tau}\left(\mathbf{U}|\psi\rangle|0\rangle\right) \equiv \langle\varphi|\langle 0|\mathbf{U}^{\tau}\mathbf{U}|\psi\rangle|0\rangle = \sum_{m,m'}\langle\varphi|\mathbf{M}^{\tau}_m\mathbf{M}_{m'}|\psi\rangle\langle m|m'\rangle$$

$$= \sum_m \langle\varphi|\mathbf{M}^{\tau}_m\mathbf{M}_m|\psi\rangle = \langle\varphi|\psi\rangle \Rightarrow$$

$\mathbf{U}$ can be defined on entire space $V \otimes M$

Now set $\mathbf{P}_m = \mathbf{I}_V \otimes |m\rangle\langle m|$

$$\Rightarrow p(m) = \langle\psi|\langle 0|\mathbf{U}^{\tau}\mathbf{P}_m\mathbf{U}|\psi\rangle|0\rangle = \sum_{m',m''}\langle\psi|\mathbf{M}^{\tau}_{m'}\langle m'|\left(I_V \otimes |m\rangle\langle m|\right)\mathbf{M}_{m''}|\psi\rangle|m''\rangle$$

$$= \sum_m \langle\psi|\mathbf{M}^{\tau}_m\mathbf{M}_m|\psi\rangle \Rightarrow$$

General Measurement!

Superdense Coding

Idea: exploit entanglement to send more information from point A to point B that would classically be allowed

Problem: Alice has 2 classical bits of information she wants to send to Bob, but is only allowed to send Bob a single qubit. Can she do it?

Solution: Yes! Put Alice's qubit in an entangled state with Bob's! Alice acts on her qubit and then sends it to Bob -- this allows Bob to uniquely deduce Alice's 2 classical bits!

Key Points to Remember:

- An abstractly-specified system may have many possible states; only some are *distinguishable.*
- A quantum state/vector/wavefunction Ψ assigns a complex-valued *amplitude* $\Psi(s_i)$ to each distinguishable state s_i (out of some basis set)
- The probability of state s_i is $|\Psi(s_i)|^2$, the square of $\Psi(s_i)$'s length in the complex plane.
- States evolve over time via unitary (invertible, length-preserving) transformations.
- Statistical mixtures of states are represented by weighted sums *of density matrices* $\rho=|\Psi\rangle\langle\Psi|$.

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