Some Fundamental Issues in Reversible Logic

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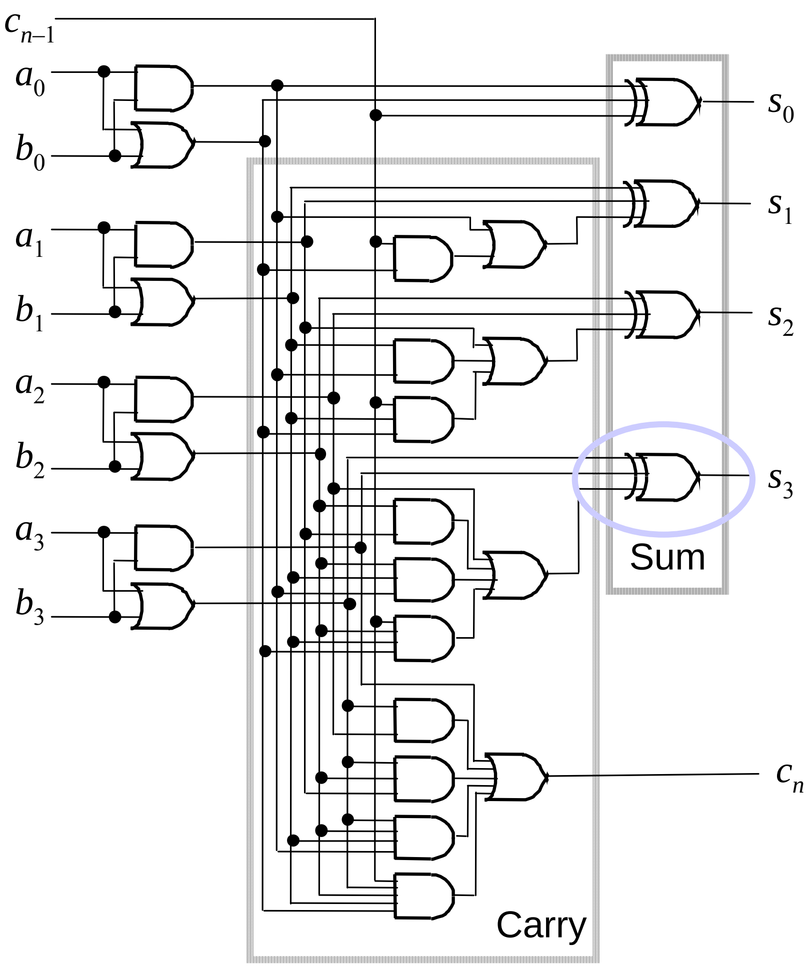
Lecture 3 continued

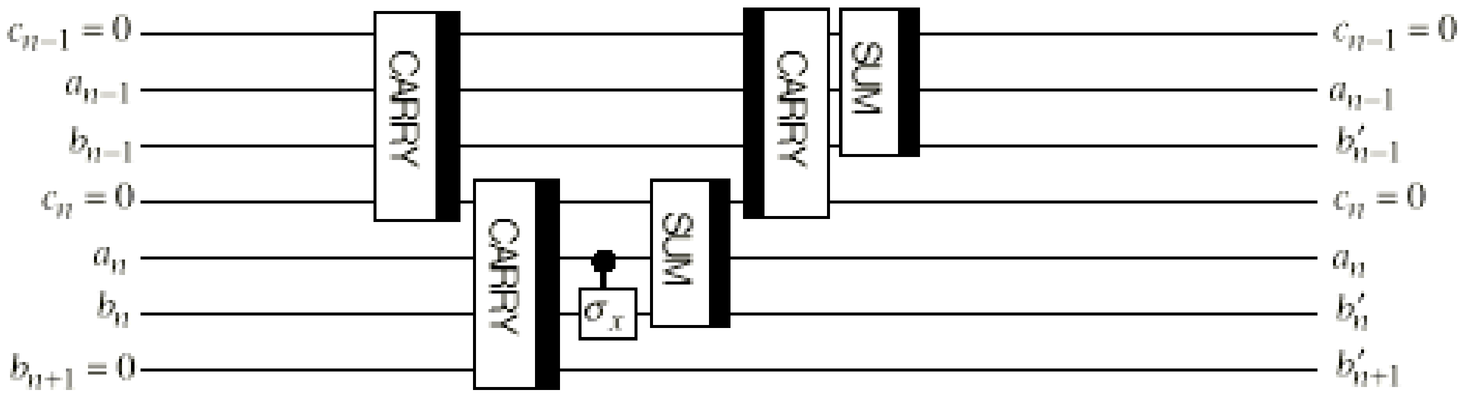
Classical vs. Quantum Circuits

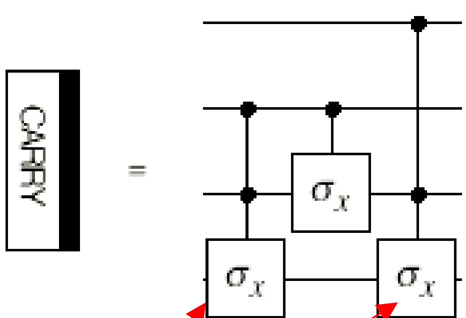
Classical adder

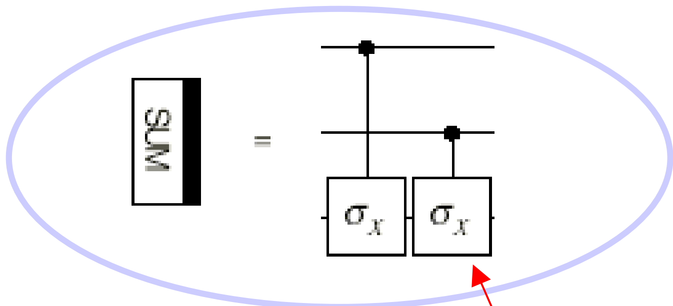


Classical vs. Quantum Circuits

Quantum adder

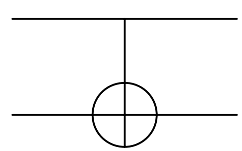






- Here we use Pauli rotations notation.
- Controlled σ_x is the same as controlled NOT

Controlled-controlled σ_x is the same as Toffoli

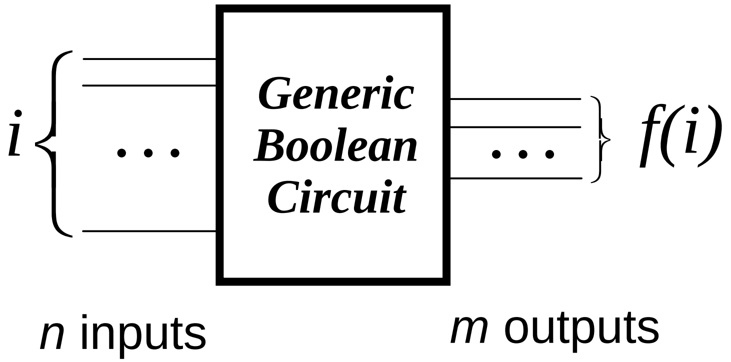


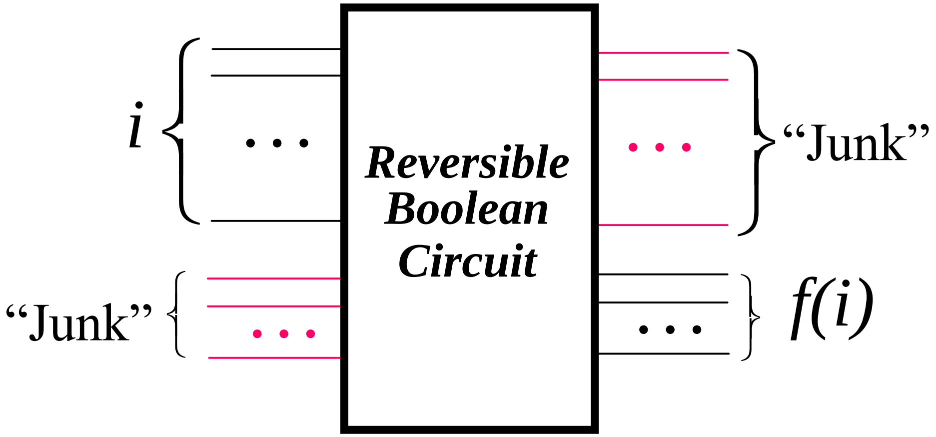
Controlled σ_x is the same as Feynman

Reversible Circuits

Reversible Circuits

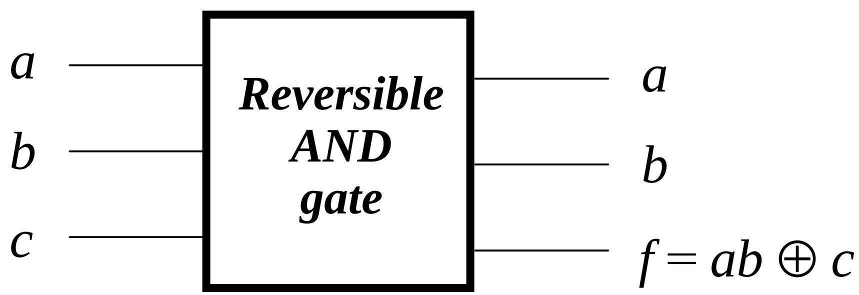
- Reversibility was studied around 1980 motivated by power minimization considerations
- Bennett, Toffoli et al. showed that any classical logic circuit *C* can be made reversible with modest overhead





Reversible Circuits

- How to make a given *f* reversible
  - Suppose $f: i \rightarrow f(i)$ has *n* inputs *m* outputs
  - Introduce *n* extra outputs and *m* extra inputs
  - Replace *f* by $f_{\text{rev}}: i, j \rightarrow i, f(i) \oplus j$ where $\oplus$ is XOR
- Example 1: $f(a,b) = \text{AND}(a,b)$

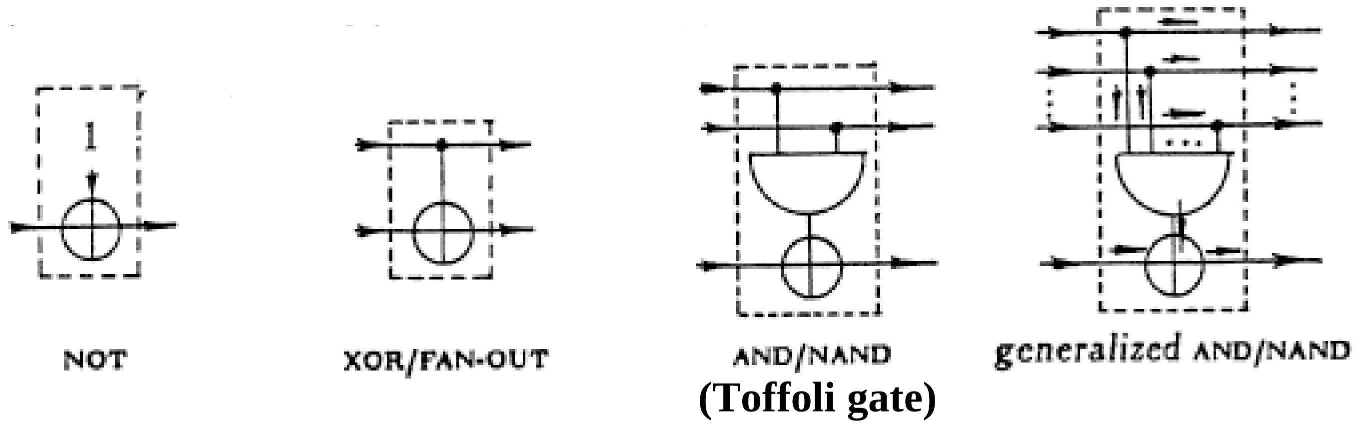


a	*b*	*c*	*a*	*b*	*f*
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	0	1
1	1	0	1	1	1
1	1	1	1	1	0

- This is the well-known Toffoli gate, which realizes AND when $c = 0$, and NAND when $c = 1$.

Reversible Circuits

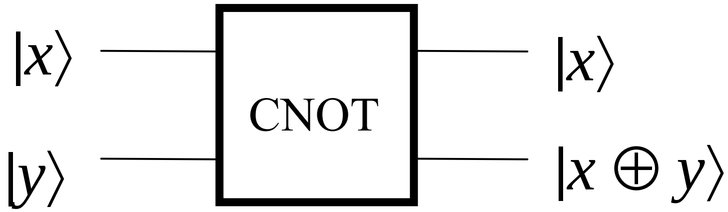
- Reversible gate family [Toffoli 1980]



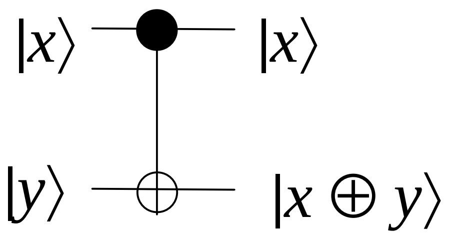
- Every Boolean function has a reversible implementation using Toffoli gates.
- There is no universal reversible gate with fewer than three inputs

Permutation Quantum Gates are reversible

- **Two-Input Gate:** Controlled NOT (CNOT)



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

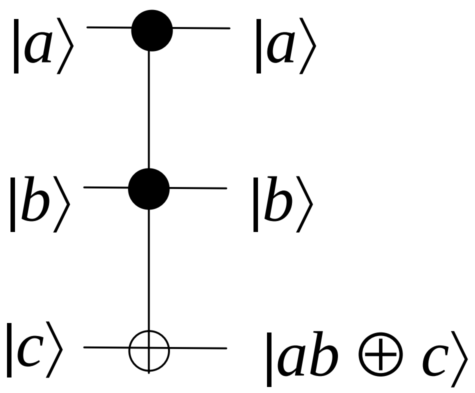


  – CNOT maps $|x\rangle|0\rangle \rightarrow |x\rangle||x\rangle$ and $|x\rangle|1\rangle \rightarrow |x\rangle||\text{NOT } x\rangle$

Quantum Gates

- **3-Input gate:** Controlled CNOT (C^2NOT or Toffoli gate)

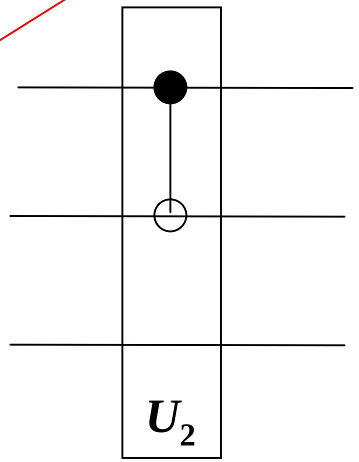
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$



Example

We calculate the Unitary Matrix U_2 of the second block from left.

$$U_2 = \quad CNOT(x_1, x_2) \otimes I_1$$



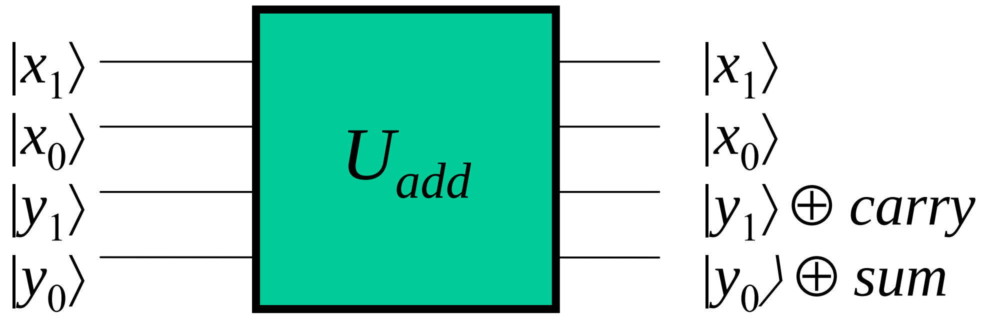
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Unitary matrix of CNOT or Feynman gate with EXOR down

As we can check in the schematics, the Unitary Matrices U_2 and U_4 are the same

Reversible Circuits

- **Implementing a Half Adder**
  - *Problem:* Implement the classical functions $sum = x_1 \oplus x_0$ and $carry = x_1 x_0$
- Generic design:



Reversible Circuits

- From equations we can find the following truth table

x_1 x_0 y_1 y_0 x_1 $x0$ $y1 \oplus x1x0$

$y_0 \oplus x_1 \oplus x_0$

0000	0000
0001	0001
0010	0010
0011	0011
0100	0101
0101	0100
0110	0111
0111	0110
1000	1001
1001	1000
1010	1011
1011	1010
1100	1110
1101	1111
1110	1100
1111	1101

- This truth table can be rewritten to the following unitary matrix (remember that rows and columns of unitary matrix are enumerated starting from 0 on top an on left). NEXT SLIDE

Quantum Circuits

- **Half Adder**: Generic design (contd.)

$$U_{ADD} = \begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
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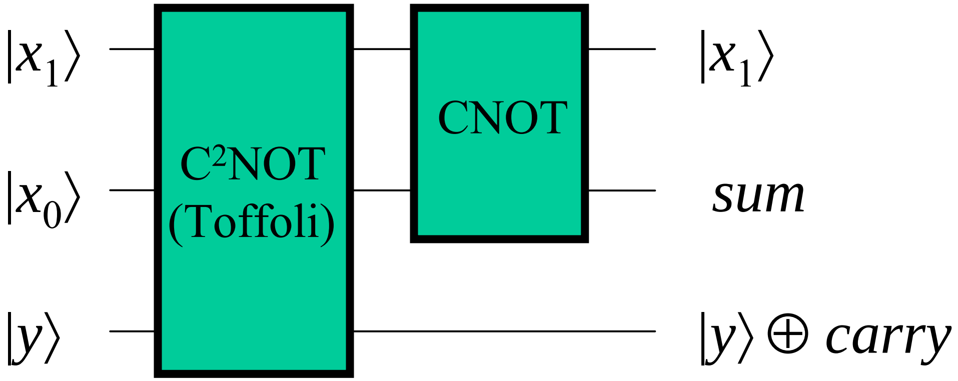
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0
\end{pmatrix}$$

Reversible Circuits

- **Half Adder**: Specific (reduced) design

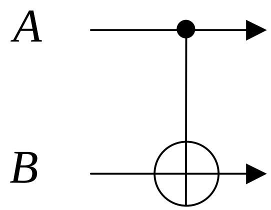


Agenda

- Introduction and history
- Reversible Logic and Reversible Gates
- Genetic algorithms
- The Model
- Simulation
- Conclusion

Reversible gates…

Feynman, Toffoli, Fredkin, …

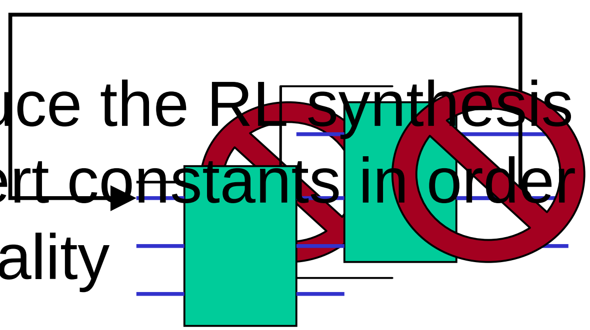


$P = A$

$Q = P \oplus B$

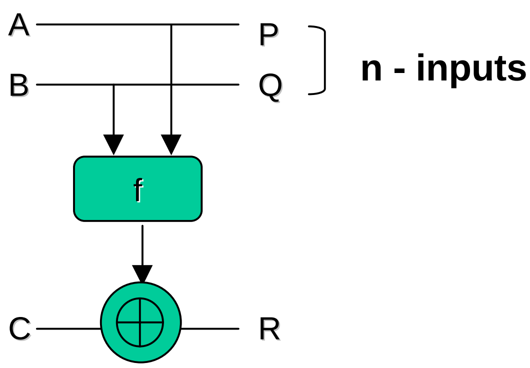
Mapping of I/O allows unique (P,Q)$\Rightarrow$(A,B)

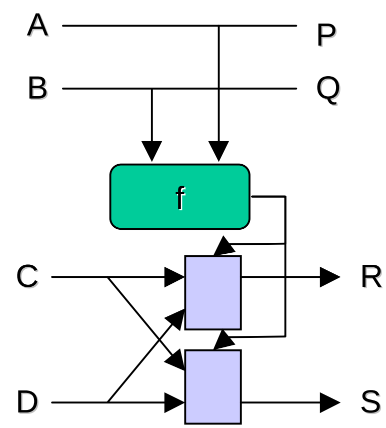
and Reversible Circuits

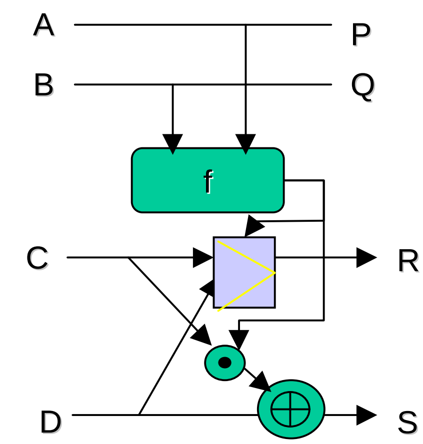


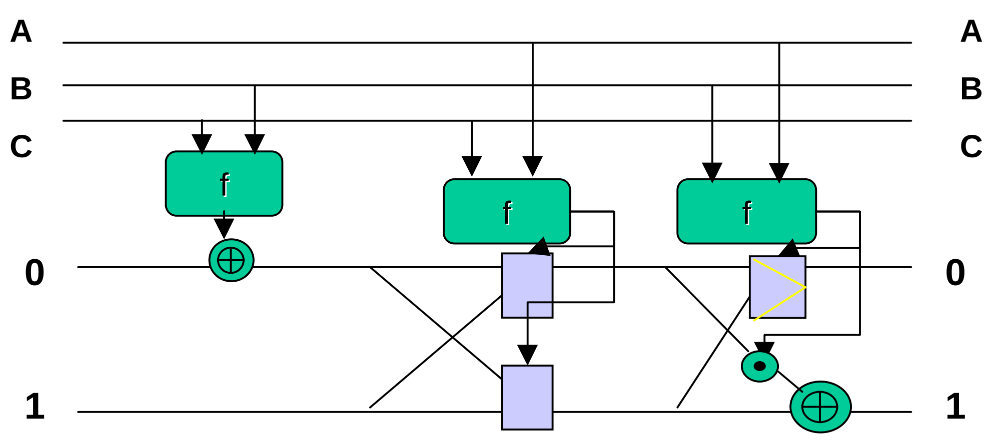
•To reduce the RL synthesis limitations one can insert constants in order to modify the functionality

Generalized Reversible Gates

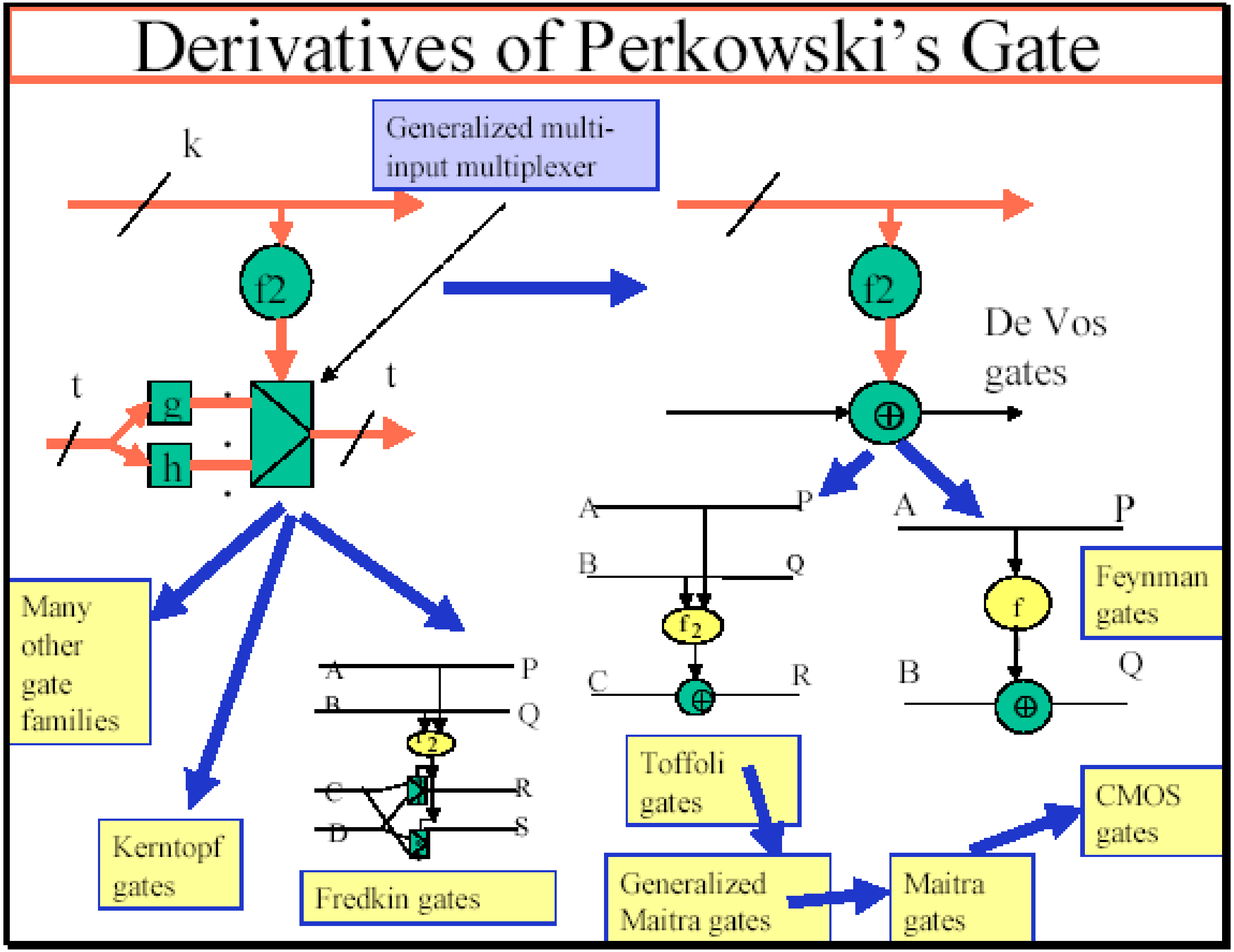








Perkowski gates family



Cascades

- ***Mixed data/control inputs (generalized complex control gates)***
- ***All :***
  - ESOP
  - Factorized-ESOP
  - MV Complex Terms
  - XOR family

Last slide of lecture 3

- Example: